Some notes on <u>1D systems</u> and reductions

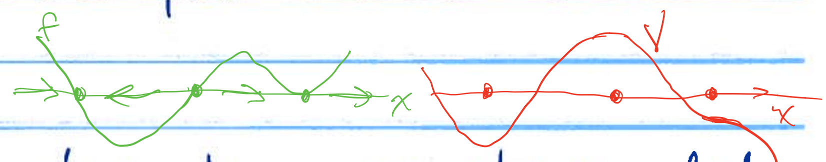
Ref: Steven Strogatz, Nonlinear Dynamics and Chaos

For an ODE $\dot{x} = F(x)$ in $\mathbb{R}^n$, if $F(x) = -\nabla V(x)$ for some scalar potential function V, then

$$\frac{d}{dt}V(x(t)) = \nabla V \cdot \dot{x} = -|\nabla V|^2 \leq 0$$

so V is decreasing along flows unless the flow is a fixed point.

In 1D, such a potential $V(x)$ always exists. This makes it analytically easy to prove that periodic solutions (besides fixed points) do not exist.

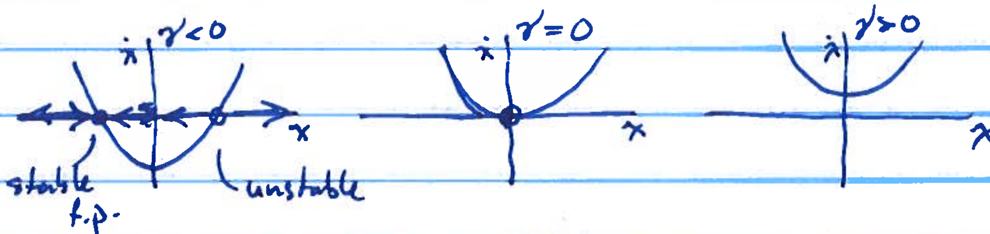


1D systems become more interesting when a parameter is involved. As the parameter varies, the fixed points of the system and their stability changes, resulting in <u>bifurcations</u>.

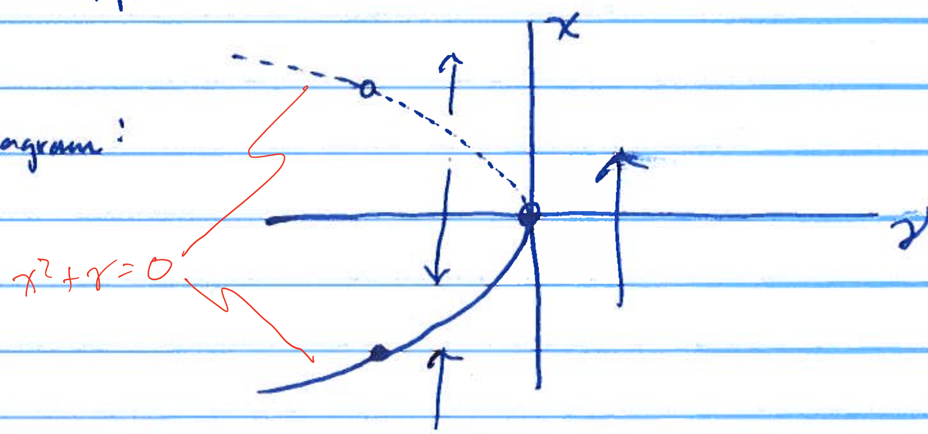
- saddle-node bifurcation, or blue-sky bifurcation.

  typical form: $\dot{x} = x^2 + r$

phase-line analysis:



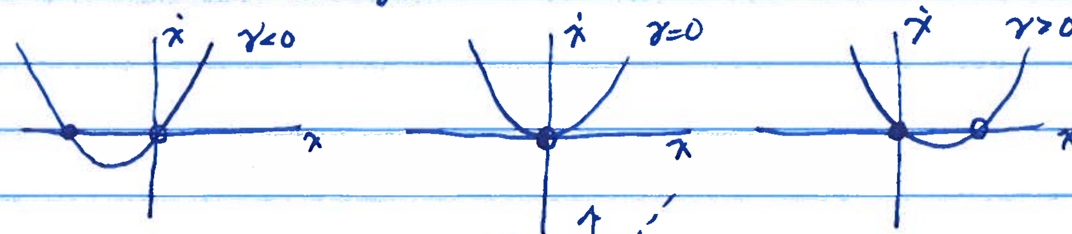
bifurcation diagram:



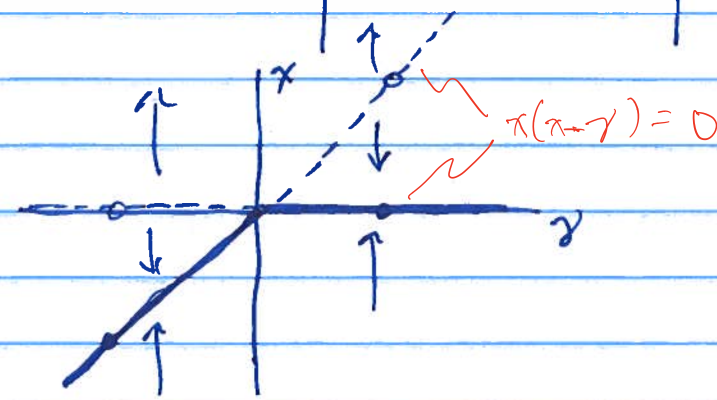
- transcritical bifurcation

  typical form : $\dot{x} = x(x - \gamma)$

  phase-line analysis :



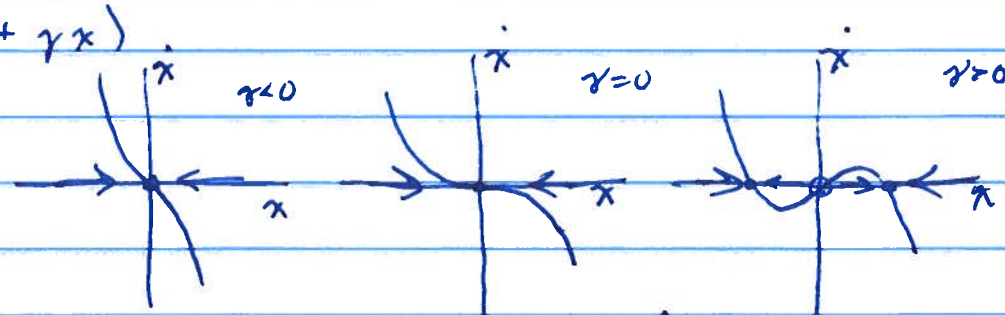
  bifurcation diagram :



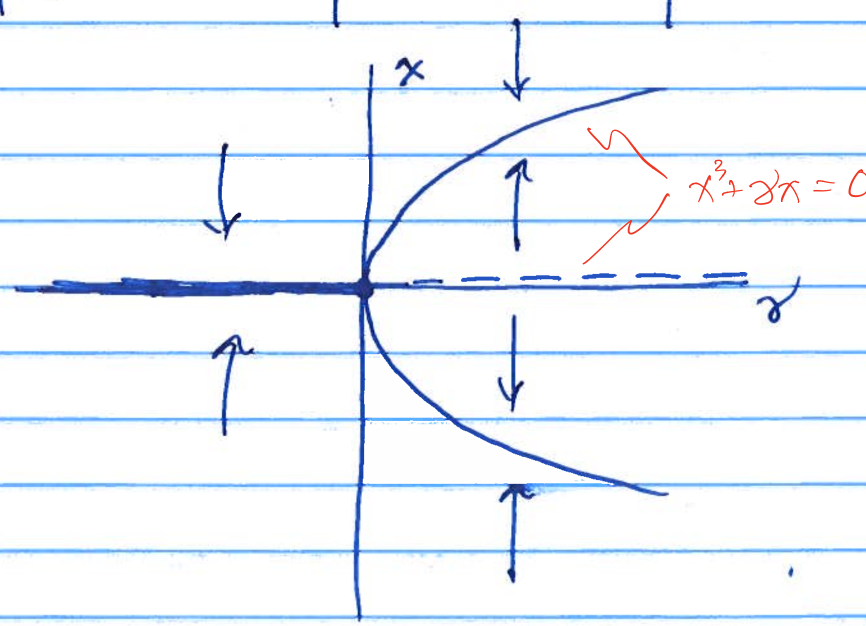
- pitchfork bifurcation

  $\dot{x} = -(x^3 + \gamma x)$

  phase-line analysis :



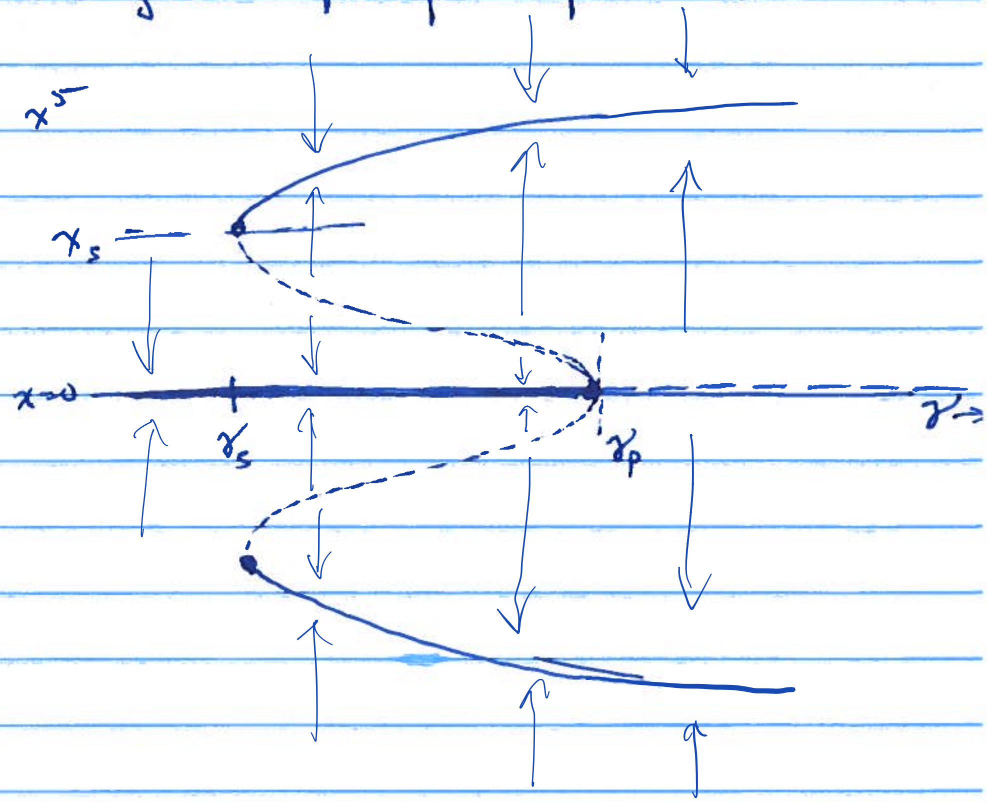
  bifurcation diagram :



- **Combination of blue-sky and pitchfork bifurcations**

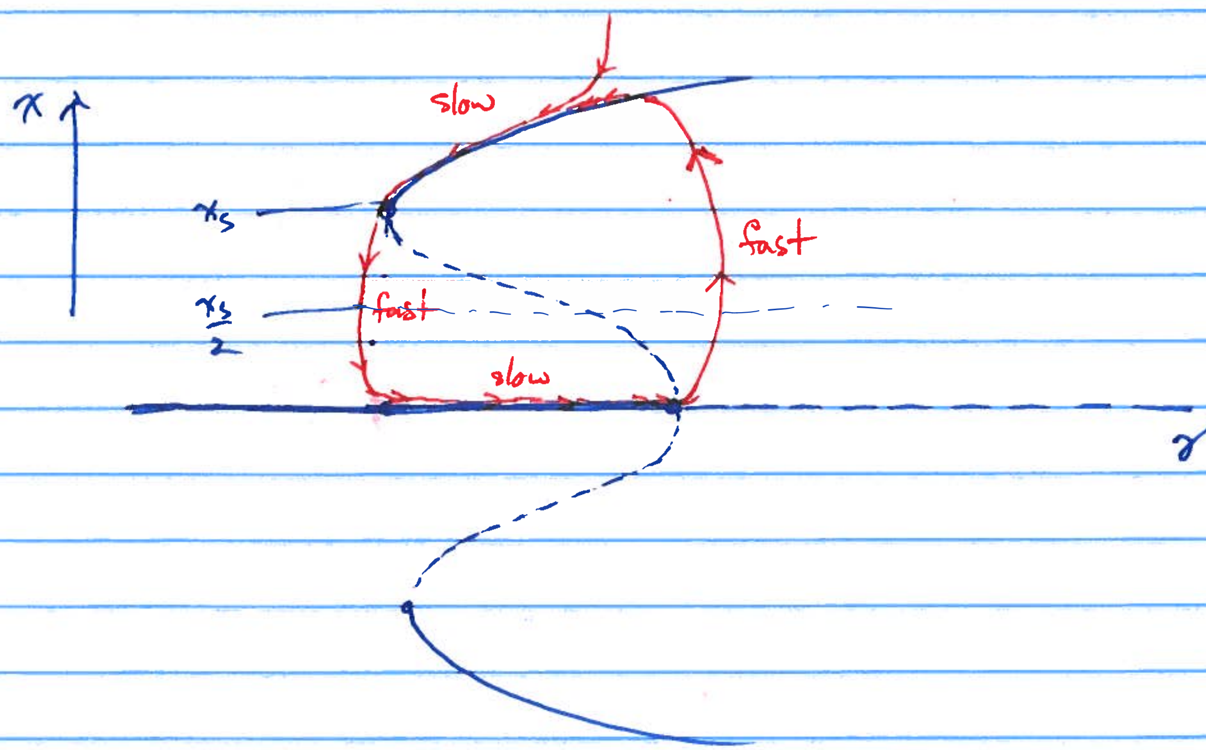
$$\dot{x} = \gamma x + x^3 - x^5$$

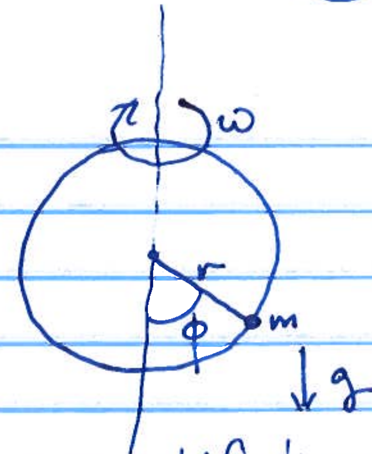
bifurcation diagram:



- **Hysteresis**

$$\dot{x} = \gamma x + x^3 - x^5$$
$$\dot{\gamma} = -\varepsilon(4x^2 - x_s^2)$$





- Overdamped bead on a rotating hoop.

Second-order eqn. for ϕ:

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$$

b: friction

Nondimensionalization:

- First divide by the force mg to make each term have units of 1/time and set $t = T\tau$ where T is a fixed unit of time and $\tau \in \mathbb{R}$.
  (so replace $\dot{\phi} = \frac{d\phi}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$, $\ddot{\phi} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$):

$$\left(\frac{r}{gT^2}\right)\frac{d^2\phi}{d\tau^2} = -\left(\frac{b}{mgT}\right)\frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi$$

- The "overdamped regime" is when $\frac{r}{gT^2}$ is very small compared with b/mgT.
  So let's set $b/mgT = 1$ so that $T = \frac{b}{mg}$
  and set $\varepsilon = \frac{r}{gT^2} = \frac{m^2gr}{b^2}$, $\gamma = \frac{r\omega^2}{g}$

$$\Rightarrow \quad \varepsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi$$

- Now brutally remove the second-order derivative to get a first-order system (RHS $=0$). This system experiences a pitchfork bifurcation: As γ increases (part 1) (say, you increase the rotation speed), the bead moves from $\phi = 0$ to $\phi = \phi_f$.

$\dot{\phi} = (\gamma\cos\phi - 1)\sin\phi$

heuristic justification → $\dot{\phi} = \psi$
$\dot{\psi} = \varepsilon^{-1}(-\psi - \sin\phi + \gamma \sin\phi \cos\phi)$

So ψ moves very fast unless (ϕ,ψ) is close to the relation $\psi = -\sin\phi + \gamma\sin\phi\cos\phi$

Two-dimensional systems and the Poincaré map.

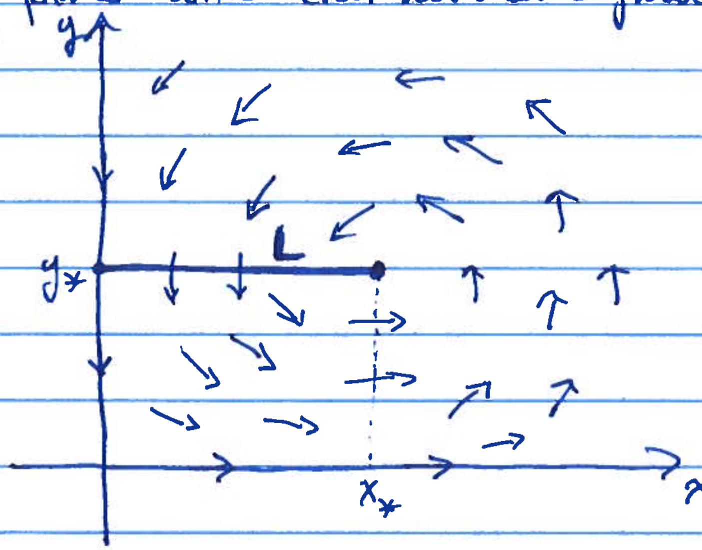
example The Lotka-Volterra model with a prey source (LV)

$$\dot{x} = kx - axy + sy$$
$$\dot{y} = -ly + bxy$$

The equilibrium points are $\begin{cases} (x,y) = (0,0) \\ (x,y) = \left(\frac{l}{b}, \frac{k}{a - s\frac{b}{l}}\right) \end{cases}$

More generally, consider a similar system with the coordinate axes as isoclines and with equilibrium points $(x,y) = (0,0)$ and $(x,y) = (x_*, y_*)$

Typical phase plane with cross-sectional segment $L = ((0,x_*), y_*)$



Stability of the LV model with prey source:

$$\frac{\partial(\dot{x},\dot{y})}{\partial(x,y)} = \begin{bmatrix} k-ay & -ax+s \\ by & -l+bx \end{bmatrix}$$

$$\left.\frac{\partial(\dot{x},\dot{y})}{\partial(x,y)}\right|_{(x_*,y_*)} = \begin{bmatrix} k\,\frac{sb}{sb-la} & \frac{1}{b}(sb-la) \\ k\,\frac{-lb}{sb-la} & 0 \end{bmatrix}$$

Notice that the real part of the eigenvalues, when $s>0$ and s is sufficiently small is negative (and the two evals are complex conjugates). But when $s=0$, the eigenvalues are purely imaginary. In the first case, the fixed point is stable. In the second case, the linearization does not tell us about stability (higher order derivatives of $F(x,y)=(\dot{x},\dot{y})$ would).

We can show that, for $s=0$, the system has a continuum of periodic trajectories as follows. The ODE system is

$$(\dot{x},\dot{y}) = F(x,y) = \big(x(k-ay),\ y(-l+bx)\big)$$

Consider the scaled rotated vector field

$$\frac{1}{xy}F^{\perp}(x,y) = \left(\frac{l-bx}{x},\ \frac{k-ay}{y}\right),$$

and observe that it is the gradient of

$$g(x,y) = \int^{x} \frac{l - bx'}{x'}\,dx' + \int^{y} \frac{k - ay'}{y'}\,dy'$$

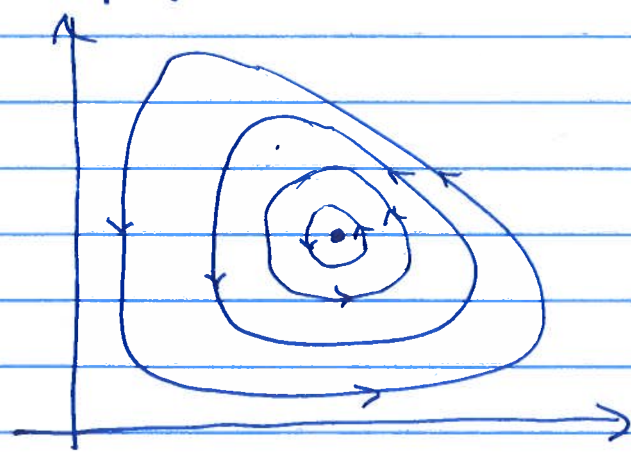
Thus an solution of $(\dot{x}, \dot{y}) = F(x,y)$ satisfies

$$\frac{d}{dt} g(x(t), y(t)) = \nabla g(x(t), y(t)) \cdot (\dot{x}(t), \dot{y}(t))$$

$$= \frac{1}{xy} F^{\perp}(x,y) \cdot F(x,y) = 0$$

Thus g is constant on the trajectories of the system, so the connected components of level sets of g are the trajectories

The level sets of g in this case are closed curves and the point (x_*, y_*):



For systems of this general ilk, one can ask the following questions:

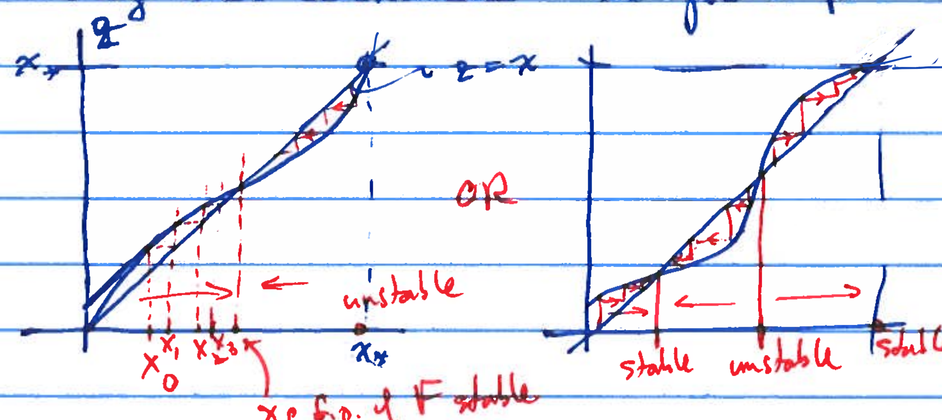
(1) Is it reasonable that, (for appropriate parameters) one could prove that any trajectory that begins on L will return to L.

(2) Given that (1) holds, let (x_0, y_*) be an initial condition, let (x_1, y_*) be the first return of the solution to L, (x_2, y_*) the second return, ..., (x_n, y_*) the n^{th} return. This generates a discrete dynamical system

$$x_{n+1} = F(x_n),$$

in which F is called the return map, or the Poincaré map (for this ODE and the segment L).

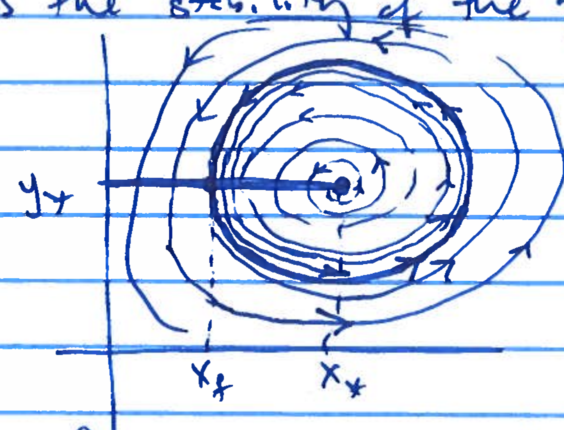
(3) One can prove (this is unique to 2D) that F is strictly increasing and continuous with a fixed point at x_*.



etc.

Thus, the Poincaré map is a discrete dynamical system that is easily visualized graphically by a "cobweb", or "Lamerey" diagram.

(4) Fixed points of F ($F(x_f) = x_f$) correspond to periodic orbits of the ODE. The stability of x_f matches the stability of the periodic orbit.



In this figure, x_* is unstable for F, so (x_*, y_*) is repelling → unstable for the ODE. x_f is stable for F, so the periodic orbit through (x_f, y_*) is stable, or attracting for the ODE.

Question: When does an ODE system in 2D admit a Poincaré map? Sometimes the Poincaré-Bendixson Theorem can be applied to address this question.

Defn (5) First we define the ω-limit (omega-limit) $\omega[\gamma]$ of an orbit γ [$\gamma(t) = (x(t), y(t))$ solution of $(\dot{x}, \dot{y}) = F(x,y)$].
$\omega[\gamma]$ is the set of all points $p = (x,y)$ such that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of increasing times tending to ∞ such that $\gamma(t_n) \to p$ as $n \to \infty$.

(k) The Poincaré–Bendixson Theorem

Let $\dot{x} = F(x)$ be an ODE in $\mathbb{R}^2$ with F continuously differentiable, and let γ be an orbit thereof. Then $\omega[\gamma]$ has one of the following properties:

- $\omega[\gamma]$ is empty
- $\omega[\gamma]$ contains infinitely many fixed points
- $\omega[\gamma]$ consists of a single fixed point
- $\omega[\gamma]$ is a periodic orbit
- $\omega[\gamma]$ consists of finitely many fixed points connected by homoclinic or heteroclinic orbits.