Math 7320 @ LSU
Spring, 2021
Final Exam

This exam is due by 5:00 PM on Monday, May 3.

No collaboration or communication with any human being is allowed regarding any of these problems, except that you may ask me questions for clarification. You must cite all references that you utilize in devising your solution.

1. Let the matrix A be given by

$$A = \begin{bmatrix} -3 & -5 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

a. Find the matrix exponential e^{tA}.

b. Find the general solution of the ODE

$$\frac{dx}{dt}(t) = A\,x(t)$$

in $\mathbb{R}^4$ as a linear combination of the four *modes* of the system.

c. Find all initial conditions $x(0)$ for which the solution $x(t)$ is a bounded function defined for $t \in [0, \infty)$.

2. Let a potential $q \in L^2[0,1]$ be given. Recall that solutions $u(x)$ to $-u'' + (q(x) - \lambda)u = 0$ with fixed initial conditions at $x = 0$ or $x = 1$ are entire in λ for each fixed x, for example, the solution $c(x;\lambda)$ with $c(0;\lambda) = 1$ and $c'(0;\lambda) = 0$ and the other special solutions $\tilde{c}$, s, and $\tilde{s}$ that we defined in class.
Consider the boundary-value problem

$$\begin{aligned} &-u'' + (q(x) - \lambda)u = f \\ &u'(0) = 0 \\ &u(1) = 0 \end{aligned} \tag{1}$$

for $f \in L^2[0,1]$. A complex number λ is a Neumann-Dirichlet (N-D) eigenvalue of $-d^2/dx^2 + q(x)$ on $[0,1]$ if there exists a nonzero solution to the homogeneous boundary-value problem (1) with $f = 0$.

a. For general $q \in L^2[0,1]$, find the Green function (solution kernel) $G(x,y;\lambda)$ for Problem 1, and prove that it has the form

$$G(x,y;\lambda) = \frac{1}{W(\lambda)}\tilde{G}(x,y;\lambda),$$

in which $W(\lambda)$ is entire and $\tilde{G}(x,y;\lambda)$ is entire for each fixed pair (x,y). Write it as explicitly as possible using the special solutions to the homogeneous problem.

b. Prove that $\lambda \in \mathbb{C}$ satisfies $W(\lambda) = 0$ if and only if λ is a N-D eigenvalue.

c. Prove that, if q is real valued, then the set of N-D eigenvalues is a discrete set $\{\lambda_n\}_{n=1}^{\infty}$ of real numbers.

d. Prove that, for each positive integer n,

$$\frac{dW}{d\lambda}(\lambda_n) \neq 0$$

by proving a result analogous to the one in the exercise on p. 10 of my online notes "Schrödinger ODE 2".

3. By using the Poincaré-Bendixson Theorem or one of its cousins, one can prove that the Van der Pol oscillator

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$$

with $\mu > 0$ has a limit cycle. (I'm not asking you to do this.)

a. For a general system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

prove that for any trajectory γ,

$$\int_\gamma \big(g(x,y)dx - f(x,y)dy\big) = 0.$$

b. Use this result to prove that, for the Van der Pol oscillator, any limit cycle must contain a point (x,y) with $x = 1$ or $x = -1$.