Math 7320 @ LSU
Spring, 2021
Problem Set 2

1. Prove the following statement on continuation of bounded solutions of an ODE.

Let K and W be subsets of $\mathbb{R}^n$ with K compact and nonempty and W open and $K \subset W$; and let $f : W \to \mathbb{R}^n$ be Lipschitz. Let $x : (a, b) \to K$ be a solution of $\dot{x} = f(x)$, with $a < b$. Then there are numbers c and d with $c < a < b < d$ such that this solution is extended to a solution $x : (c, d) \to W$.

2. Prove that the definition of global stable manifold ($W^{\mathrm{s}}(\bar{x}, \mathcal{U})$ in the notes) actually does not depend on $\mathcal{U}$ even though $\mathcal{U}$ is used in the definition.

3. Exercise 1.0.3 in Guckenheimer/Holmes. Using the Lyapunov function $V = (x^2 + \sigma y^2 + \sigma z^2)/2$, obtain conditions on $\sigma > 0$, ρ, and $\beta > 0$ sufficient for global asymptotic stability of the origin $(x, y, z) = (0, 0, 0)$ in the Lorenz system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = \rho x - y - xz, \qquad \dot{z} = -\beta z + xy$$

Are these conditions also necessary?

4. Let $g(x, y)$ and $h(x, y)$ be continuously differentiable real-valued functions of $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Prove that the function g is constant on solutions of the following ODE in $\mathbb{R}^{2d}$:

$$(\dot{x}, \dot{y}) \;=\; h(x, y)\big(D_y g(x, y),\, -D_x g(x, y)\big).$$

Prove that, if $d = 1$ and g has an isolated critical point corresponding to a strict local minimum or maximum, then this system has periodic trajectories.

5. [From exercise 1.3.2 in G/H] For the following ODEs

(i) $\ddot{x} + \varepsilon \dot{x}^2 + \sin x \;=\; 0$
(ii) $\dot{x} = -x + x^2, \;\; \dot{y} = x + y + \varepsilon y^2$
(iii) $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$

do the following for $\varepsilon = 0$ and for small $\varepsilon < 0$ and $\varepsilon > 0$:

(A) Determine all fixed points and make a careful sketch of the flow about each fixed point.

(B) Make the sketches as accurate as possible by finding eigenvalues and eigenvectors of the linearizations. Make sure the sketches depict all essential interesting features. For (i), prove that, for $\varepsilon = 0$, there are periodic orbits about certain fixed points; this is straightforward using the idea of the previous problem. But also for $\varepsilon \neq 0$, there are periodic orbits, and this information will let you know what the stable and unstable manifolds are. See if you can prove this. If you can't, then you may still use the result in doing part (C) for (i).

(C) Determine the local stable and unstable manifolds.