Math 7320 @ LSU
Spring, 2021
Problem Set 1

1. Prove that ~~$e^{A+B} = e^A e^B$ if $AB = BA$; and prove that~~

$$e^{t(A+B)} = e^{tA}\, e^{tB} \qquad \text{for all } t \in \mathbb{R}$$

if and only if $AB = BA$.

2. Putzer's theorem provides an algorithm for computing the solution of a linear constant-coefficient homogeneous system $\dot{x} = Ax$ without first computing the Jordan normal form for the generator A. Let A be a linear operator in a finite-dimensional complex vector space of dimension n, and let $\{\lambda_i\}_{i=1}^n$ denotes its eigenvalues. For $k = 1, \ldots, n$, define the operators

$$A_k = \prod_{j=1}^{k} (A - \lambda_j E), \quad \text{and } A_0 = E$$

and the scalar functions $r_j(t)$ as the solution of the system

$$\begin{aligned} dr_1/dt &= \lambda_1 r_1\,, & r_1(0) &= 1 \\ dr_j/dt &= \lambda_j r_j + r_{j-1}\,, & r_j(0) &= 0 \quad \text{for } j \geq 2. \end{aligned}$$

Putzer's theorem states that

$$e^{tA} = \sum_{k=0}^{n-1} r_{k+1}(t) A_k.$$

Prove Putzer's theorem.

3. Let $\mathcal{A} := \{A(t) : t \in (a,b)\}$ be a commuting invertible family of $n \times n$ matrices with complex entries that ~~is~~ are differentiable with respect to t. Define

$$\begin{aligned} \mathcal{A}^{-1} &:= \{A(t)^{-1} : t \in (a,b)\} \\ \dot{\mathcal{A}} &:= \{\dot{A}(t) : t \in (a,b)\}, \end{aligned}$$

in which the dot refers to differentiation with respect to t. Prove that the union $\mathcal{A} \cup \mathcal{A}^{-1} \cup \dot{\mathcal{A}}$ is a commuting family of matrices.

4. Let J be a finite interval in the real line, and let B be the ball $B = \big\{ y \in \mathbb{R}^n \;\big|\; |y - x_0| \leq \rho \big\}$, where $x_0 \in \mathbb{R}^n$ and $\rho > 0$ are fixed. Prove that the set of functions

$$X = \big\{ x : J \to B \;\big|\; x \text{ is continuous} \big\}$$

endowed with the supremum norm

$$\|x_1 - x_2\|_{\sup} = \sup_{t \in J} |x_1(t) - x_2(t)|$$

is a complete metric space.

5. Let X be defined as in problem (4), let t_0 be in J, and let $F : B \to \mathbb{R}^n$ be of Lipschitz class. Prove that, for each function $x \in X$, the following two conditions are equivalent.

(IVP) x is differentiable at each point of J, $\dot{x} = F(x)$, and $x(t_0) = x_0$.

(IE) $x(t) = x_0 + \int_{t_0}^{t} F(x(s))\, ds$.

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(1) $e^{t(A+B)} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k (A+B)^k$; the t^2 coefficient is $\frac{1}{2}(A^2+B^2+AB+BA)$

$$e^{tA} e^{tB} = \left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k\right)\left(\sum_{l=0}^{\infty} \frac{1}{l!} t^l B^l\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{k!\,l!} t^{k+l} A^k B^l$$

$$= \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} \frac{1}{(n-m)!\,m!} A^{n-m} B^m = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \sum_{m=0}^{n} \binom{n}{m} A^{n-m} B^m .$$

the t^2 coefficient is $\frac{1}{2}(A^2+B^2+2AB)$

If $e^{t(A+B)} = e^{tA}e^{tB}$ for all t, then all coefficients as power series in t are equal. For the t^2 coefficient, we obtain $AB+BA=2AB$, i.e., $AB=BA$. If $AB=BA$, then, for all k, $(A+B)^k = \sum_{l=0}^{k}\binom{k}{l} A^{k-l} B^l$, and therefore all the t-coefficients of $e^{t(A+B)}$ and $e^{tA}e^{tB}$ coincide. Thus $e^{t(A+B)} = e^{tA}e^{tB}$. Particularly, for $t=1$, we obtain $e^{A+B} = e^A e^B$.

(2) Set $r_0 \equiv 0$ and $A_0 = E$.

$$\frac{d}{dt}\left(\sum_{k=0}^{n-1} r_{k+1}(t) A_k\right) = \sum_{k=0}^{n-1} \frac{dr_{k+1}(t)}{dt} A_k = \sum_{k=0}^{n-1} (\lambda_{k+1} r_{k+1} + r_k) A_k = \sum_{k=0}^{n-1} \lambda_{k+1} r_{k+1} A_k + \sum_{k=0}^{n-1} r_k A_k$$

$$A\left(\sum_{k=0}^{n-1} r_{k+1} A_k\right) = \sum_{k=0}^{n-1} r_{k+1}(A-\lambda_{k+1}E) A_k + \sum_{k=0}^{n-1} \lambda_{k+1} r_{k+1} A_k = \sum_{k=0}^{n-1} r_{k+1} A_{k+1} + \sum_{k=0}^{n-1} \lambda_{k+1} r_{k+1} A_k$$

By the Cayley-Hamilton Theorem, the characteristic polynomial of A annihilates A, which is to say that $A_n = 0$. Using this and $r_0 = 0$, we obtain $\sum_{k=0}^{n-1} r_{k+1} A_{k+1} = \sum_{k=0}^{n-1} r_k A_k$ and from this,

$$\frac{d}{dt}\left(\sum_{k=0}^{n-1} r_{k+1}(t) A_k\right) = A\left(\sum_{k=0}^{n-1} r_{k+1}(t) A_k\right)$$

Also, $\sum_{k=0}^{n-1} r_{k+1}(0) A_k = E$ from the initial conditions for the functions r_k.

It follows that $\sum_{k=0}^{n-1} r_{k+1}(t) A_k = e^{tA}$, as e^{tA} is the unique solution to $\{\dot{X} = AX,\ X(0) = E\}$.

(3) Let $B = A(t_1)$ and $C = A(t_2)$ for some $t_1, t_2 \in (a,b)$. Multiplying both sides of $BC = CB$ by $C^{-1}B^{-1}$ on the left and by $B^{-1}C^{-1}$ on the right yields $B^{-1}C^{-1} = C^{-1}B^{-1}$, so that B^{-1} and C^{-1} commute. Thus $\mathcal{A}^{-1}$ is a commuting family.

Next, $BC^{-1} = C^{-1}CBC^{-1} = C^{-1}BCC^{-1} = C^{-1}B$, so all elements of $\mathcal{A}$ commute with all elements of $\mathcal{A}^{-1}$.

Set $D = \dot{A}(t_2)$ and let $F \in \{B, B^{-1}\}$. Then

$$BD = B \lim_{h\to 0} \frac{A(t_2+h) - A(t_2)}{h} = \lim_{h \to 0} \frac{BA(t_2+h) - BA(t_2)}{h}$$

$$= \lim_{h\to 0} \frac{A(t_2+h)B - A(t_2)B}{h} = \left(\lim_{h\to 0} \frac{A(t_2+h) - A(t_2)}{h}\right) B = DB$$

Thus all elements of $\dot{\mathcal{A}}$ commute with all elements of $\mathcal{A} \cup \mathcal{A}^{-1}$.

Lastly, we show that $\dot{\mathcal{A}}$ is a commuting family. Set $F = \dot{A}(t_1)$.

$$DF = \left(\lim_{h\to 0} \frac{A(t_2+h)-A(t_2)}{h}\right)\left(\lim_{h\to 0} \frac{A(t_1+h)-A(t_1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{A(t_2+h)-A(t_2)}{h} \frac{A(t_1+h)-A(t_1)}{h}\right) = \lim_{h\to 0}\left(\frac{A(t_1+h)-A(t_1)}{h} \frac{A(t_2+h)-A(t_2)}{h}\right)$$

$$= \left(\lim_{h\to 0} \frac{A(t_1+h)-A(t_1)}{h}\right)\left(\lim_{h\to 0} \frac{A(t_2+h)-A(t_2)}{h}\right) = FD.$$

We have shown that any two elements of $\mathcal{A} \cup \dot{\mathcal{A}} \cup \mathcal{A}^{-1}$ commute with each other.

(4) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence from X. You have to prove that $\exists\, x \in X$ s.th. $x_n \to x$ as $n \to \infty$ in X.

Briefly, these are the steps:

(1) $\forall\, t \in J$, $\{x_n(t)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}^n$ because X has the sup norm. Completeness of $\mathbb{R}^n$ produces a limit $x(t)$.

(2) Prove that the pointwise-in-J limit $x_n(t) \to x(t)$ is actually uniform, so that $x_n \to x$ in the sup norm.

(3) Prove that $x \in X$.

(a) "The uniform limit of continuous functions is continuous" is a standard result in UG advanced calculus

(b) The values $x(t)$ are in B because the sequence $\{x_n(t)\}$ is from B and B is closed.

(5) Let $x \in X$ be differentiable with $\dot{x} = F(x)$ and $x(t_0) = x_0$. Since x and F are continuous, so is $F \circ x$, so the FTC yields $x(t) = x_0 + \int_{t_0}^{t} F(x(s))\,ds$, which is IE.

Now let $x \in X$ satisfy IE. Since x and F are continuous, so is $F \circ x$, so $\int_{t_0}^{t} F(x(s))\,ds$ is differentiable (by the "other" FTC) with derivative $F(x(t))$ at each $t \in (a,b)$. Thus $\dot{x} = F(x)$, and IE directly gives $x(t_0) = x_0$.