Math 7320 @ LSU
Spring, 2021
Problem Set 2

1. Prove the following statement on continuation of bounded solutions of an ODE.

Let K and W be subsets of $\mathbb{R}^n$ with K compact and nonempty and W open and $K \subset W$; and let $f : W \to \mathbb{R}^n$ be Lipschitz. Let $x : (a, b) \to K$ be a solution of $\dot{x} = f(x)$, with $a < b$. Then there are numbers c and d with $c < a < b < d$ such that this solution is extended to a solution $x : (c, d) \to W$.

2. Prove that the definition of global stable manifold ($W^{\mathrm{s}}(\bar{x}, \mathcal{U})$ in the notes) actually does not depend on $\mathcal{U}$ even though $\mathcal{U}$ is used in the definition.

3. Exercise 1.0.3 in Guckenheimer/Holmes. Using the Lyapunov function $V = (x^2 + \sigma y^2 + \sigma z^2)/2$, obtain conditions on $\sigma > 0$, ρ, and $\beta > 0$ sufficient for global asymptotic stability of the origin $(x, y, z) = (0, 0, 0)$ in the Lorenz system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = \rho x - y - xz, \qquad \dot{z} = -\beta z + xy$$

Are these conditions also necessary?

4. Let $g(x, y)$ and $h(x, y)$ be continuously differentiable real-valued functions of $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Prove that the function g is constant on solutions of the following ODE in $\mathbb{R}^{2d}$:

$$(\dot{x}, \dot{y}) \;=\; h(x, y)\big(D_y g(x, y),\, -D_x g(x, y)\big).$$

Prove that, if $d = 1$ and g has an isolated critical point corresponding to a strict local minimum or maximum, then this system has periodic trajectories.

5. [From exercise 1.3.2 in G/H] For the following ODEs

(i) $\ddot{x} + \varepsilon \dot{x}^2 + \sin x \;=\; 0$
(ii) $\dot{x} = -x + x^2, \;\; \dot{y} = x + y + \varepsilon y^2$
(iii) $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$

do the following for $\varepsilon = 0$ and for small $\varepsilon < 0$ and $\varepsilon > 0$:

(A) Determine all fixed points and make a careful sketch of the flow about each fixed point.

(B) Make the sketches as accurate as possible by finding eigenvalues and eigenvectors of the linearizations. Make sure the sketches depict all essential interesting features. For (i), prove that, for $\varepsilon = 0$, there are periodic orbits about certain fixed points; this is straightforward using the idea of the previous problem. But also for $\varepsilon \neq 0$, there are periodic orbits, and this information will let you know what the stable and unstable manifolds are. See if you can prove this. If you can't, then you may still use the result in doing part (C) for (i).

(C) Determine the local stable and unstable manifolds.

① Let t_0 be a point in (a,b). Since f is continuous, the solution $x:(a,b)\to W$ of $\dot{x}=f(x)$ satisfies

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s))\,ds \qquad \forall\ t\in(a,b).$$

Since f is Lipschitz on the bounded set K and $x(t)\in K$ $\forall\ t\in(a,b)$, the function $s\mapsto f(x(s))$ is bounded on (a,b), and therefore we obtain the following limit:

$$\lim_{t\to b} x(t) = x(t_0) + \int_{t_0}^{b} f(x(s))\,ds,$$

which is in K since K is compact. Let us extend x to $x:(a,b]$ by continuity. We then can deduce

$$\lim_{t\to b}\dot{x}(t) = \lim_{t\to b} f(x(t)) = f(x(b)).$$

Therefore the equation $\dot{x}(t)=f(x(t))$ is valid on $(a,b]$. The existence theorem provides a solution to $\dot{y}=f(y)$ with $y(b)=x(b)$ on some interval (a',d) with $a<a'<b<d$. Notice that the function

$$z(t) = \begin{cases} x(t) & \text{for } a'<t\le b \\ y(t) & \text{for } b\le t<d \end{cases}$$

also satisfies $\dot{z}=f(z)$ and $z(b)=y(b)$. The uniqueness theorem implies that $z=x$, so that y does extend

the solution x in W to a larger interval (a,d).
An analogous argument allows one to extend the left endpoint of the solution's domain.

(2) $W^s_{loc}(\bar{x}, \mathcal{U}) = \{x \in \mathcal{U} : \phi_t(x) \in \mathcal{U} \ \forall\ t \geq 0 \text{ and } \phi_t(x) \to \bar{x} \text{ as } t \to \infty\}$

$W^s(\bar{x}, \mathcal{U}) = \bigcup_{t \in \mathbb{R}} \phi_t(W^s_{loc}(\bar{x}, \mathcal{U}))$

Let $\mathcal{U}$ and $\mathcal{V}$ be open sets about $\bar{x}$.

Let $x \in W^s(\bar{x}, \mathcal{U})$ be given. So $\exists\ t_1 \in \mathbb{R}$ and $y \in W^s_{loc}(\bar{x}, \mathcal{U})$ s.th. $x = \phi_{t_1}(y)$. Since $\phi_s(y) \to \bar{x}$, $\exists\ s_0 : s > s_0 \Rightarrow \phi_s(y) \in \mathcal{V}$.

Thus, $s > s_0 - t_1 \Rightarrow \phi_s(x) = \phi_s(\phi_{t_1}(y)) = \phi_{s+t_1}(y) \in \mathcal{V}$.

Let $s_1 > s_0 - t_1$ be given, and put $z = \phi_{s_1}(x)$. Then $z \in \mathcal{V}$ and $\forall\ t \geq 0$, $s_1 + t \geq s_0 - t_1$, so $\phi_t(z) = \phi_{t+s_1}(x) \in \mathcal{V}$.

Furthermore, $\lim_{t\to\infty} \phi_t(z) = \lim_{t\to\infty} \phi_t(x) = \bar{x}$. We've shown that $z \in W^s_{loc}(\bar{x}, \mathcal{V})$. It follows that $x = \phi_{-s_1}(z) \in W(\bar{x}, \mathcal{V})$

Thus, $W^s(\bar{x}, \mathcal{U}) \subset W^s(\bar{x}, \mathcal{V})$, and, reciprocally, one obtains $\supset$. ▨

(3) $\dot{x} = \sigma(y-x), \qquad \dot{y} = \rho x - y - xz, \qquad \dot{z} = -\beta z + xy \qquad \sigma, \beta > 0$

Set $V(x,y,z) = \frac{1}{2}(\alpha x^2 + \sigma y^2 + \sigma z^2)$, with $\alpha > 0$.

Then
$$\dot{V}(x,y,z) = \nabla V \cdot \langle \dot{x}, \dot{y}, \dot{z} \rangle = \langle \alpha x, \sigma y, \sigma z \rangle \cdot \langle \dot{x}, \dot{y}, \dot{z} \rangle$$
$$= \sigma \alpha x(y-x) + \sigma y(\rho x - y - xz) + \sigma z(-\beta z + xy)$$
$$= -\sigma\alpha x^2 - \sigma y^2 - \sigma\beta z^2 + \sigma(\rho+\alpha)xy$$
$$= -\sigma\beta z^2 - \sigma [x\ y] \begin{bmatrix} \alpha & \frac{1}{2}(\rho+\alpha) \\ \frac{1}{2}(\rho+\alpha) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This is negative for all $(x,y,z) \neq (0,0,0)$ whenever the determinant is positive, i.e., whenever

$$4\alpha - (\rho+\alpha)^2 > 0,$$

or whenever $-\alpha - 2\sqrt{\alpha} < \rho < -\alpha + 2\sqrt{\alpha}$. The value $\alpha = 1$ was proposed in the problem, and the sufficient condition for global stability is therefore $-3 < \rho < 1$. It is not necessary because, by allowing α to run over all positive numbers, and taking the union over all ρ-intervals $(-\alpha-2\sqrt{\alpha}, -\alpha+2\sqrt{\alpha})$ one obtains global asymptotic stability for all $\rho \in (-\infty, 1)$.

4 $(\dot{x}, \dot{y}) = h(x,y)\big(D_y g(x,y), -D_x g(x,y)\big)$

$\dot{g}(x,y) = \nabla g \cdot \langle \dot{x}, \dot{y} \rangle = \langle D_x g, D_y g \rangle \cdot \langle h D_y g, -h D_x g \rangle = 0$.

Thus g is constant along trajectories.

In $\mathbb{R}^2$, let $\bar{x}$ be a strict local minimum for g.

Because g is C^1, its level sets are closed curves near $\bar{x}$, and these must therefore be the trajectories.

[This can be made rigorous ...]

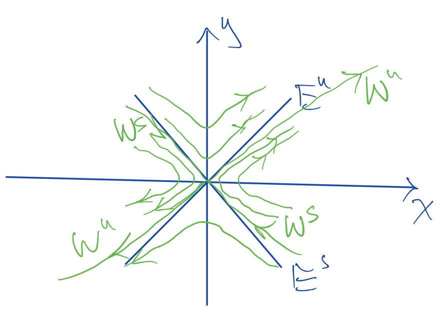
(5) (i) $\ddot{x} + \varepsilon\dot{x}^2 + \sin x = 0$
(ii) $\dot{x} = -x + x^2, \quad \dot{y} = x + y + \varepsilon y^2$
(iii) $\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$

(i) $\dot{x} = y$
$\dot{y} = -\varepsilon y^2 - \sin x$

The fixed points are $(x,y) = (\pi k, 0)$, $k \in \mathbb{Z}$.

$$J(\pi k, 0) = \begin{bmatrix} 0 & 1 \\ (-1)^{k+1} & 0 \end{bmatrix}$$

For $k \in 2\mathbb{Z}+1$, $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and the eigenvalues an e.vecs are



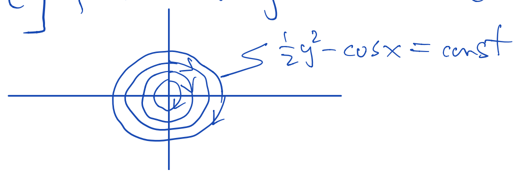
$\lambda_+ = 1$, $v_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_- = -1$, $v_- = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The local stable manifold are, by the SMT, smooth and tangent to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and the local unstable is tangent to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This is the approximate local picture for all ε. [The global picture for $\varepsilon \neq 0$ is qualitatively different from the global picture for $\varepsilon = 0$ because closed orbits for $\varepsilon = 0$ become oscillatory trajectories for $\varepsilon > 0$ that converge to the closest stable fixed point.]

For $k \in 2\mathbb{Z}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, with eigenvalues $\pm i$. This is independent of ε, but the SMT tells us nothing about W^s or W^u.

For $\varepsilon = 0$, the situation of problem 4 applies with $g(x,y) = \frac{1}{2}y^2 - \cos x$ and $h(x,y) = 1$, and thus trajectories are closed orbits. Since $\exp(tJ) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$, the trajectories are clockwise:



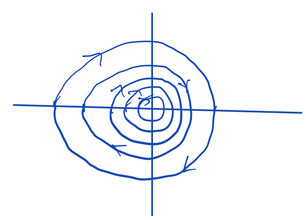
For $\varepsilon > 0$ ($\varepsilon < 0$ is similar).

In this case, we obtain $\frac{d}{dt} g(x(t), y(t)) = -\varepsilon y^3$. Using this, one can show, with some technical effort, that the trajectory starting at an initial condition $(k\pi - \delta, 0)$ with $\delta > 0$ small enough will reach a point $(k\pi + \delta', 0)$ with $0 < \delta' < \delta$ at some $t > 0$.

Then observe the following symmetry of the system:

Set $\xi(t) = x(-t)$ and $\eta(t) = -y(-t)$. Then $\dot{\xi}(t) = -\dot{x}(-t) = -y(-t) = \eta(t)$ and $\dot{\eta}(t) = \dot{y}(-t) = -\varepsilon y(-t)^2 - \sin x(-t) = -\varepsilon \eta(t)^2 - \sin \xi(t)$.

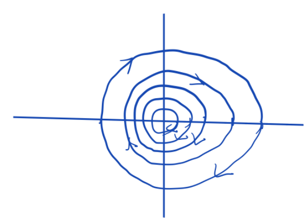
Thus $(\xi(t), \eta(t))$ is also a solution to the ODE. These solutions can be "pieced together" (you do the easy technical details) to obtain a closed trajectory about $(k\pi, 0)$. The approximate picture is like this:



In the case $\varepsilon < 0$ set $\xi(t) = -x(t)$ and $\eta(t) = -y(t)$, so that

$$\dot{\xi}(t) = -\dot{x}(t) = -y(t) = \eta(t) \text{ and}$$

$$\dot{\eta}(t) = -\dot{y}(t) = \varepsilon y(t)^2 + \sin x(t) = \varepsilon \eta(t)^2 - \sin \xi(t)$$



This rotation by $180°$ converts solutions of the system with parameter ε to the system with ε replaced by $-\varepsilon$. The picture is thus this

Directly from the definitions of W^s and W^u, we find that each consists of only the fixed point itself. This holds for all ε.

$$\frac{-2}{2 - 2\varepsilon} = \frac{-1}{1-\varepsilon} = -(1+\varepsilon)+$$

(ii) $\dot{x} = -x + x^2$
$\dot{y} = x + y + \varepsilon y^2$

fixed points: $(0,0)$

$(0, -\varepsilon^{-1})$ only for $\varepsilon \neq 0$ (far down)

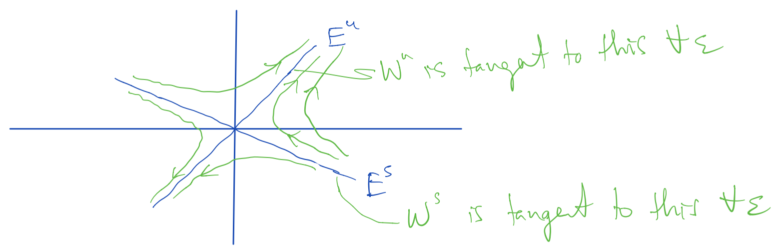
$(1, -2(\sqrt{1-4\varepsilon}+1)^{-1}) = (1, -1-\varepsilon + O(\varepsilon^2))$

$(1, \varepsilon^{-1} - \frac{1}{2\varepsilon}(\sqrt{1-4\varepsilon}+1)) = (1, -\varepsilon^{-1}+1+O(\varepsilon))$ only for $\varepsilon \neq 0$ (far down)

$$J = \begin{bmatrix} 2x-1 & 0 \\ 1 & 2\varepsilon y + 1 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$ e.vals ± 1 $\forall \varepsilon$

e.vecs $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $+1$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ for -1



$$J(0,-\varepsilon^{-1}) = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$ e.vals -1 $\forall\, \varepsilon \neq 0$

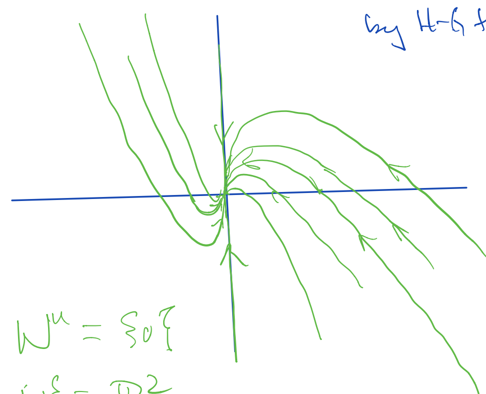
[Jordan chain: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, ie $(J+1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $(J+1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$]

$$\exp(tJ) = e^{-t}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

linear flow $\approx$ nonlinear flow by H-G thm.

$x_1 = a e^{-t}$
$x_2 = e^{-t}(at+b)$

$\Rightarrow x_2 = -x_1 \log \alpha x_1 + \beta x_1$



$W^u = \{0\}$
$W^s = \mathbb{R}^2$

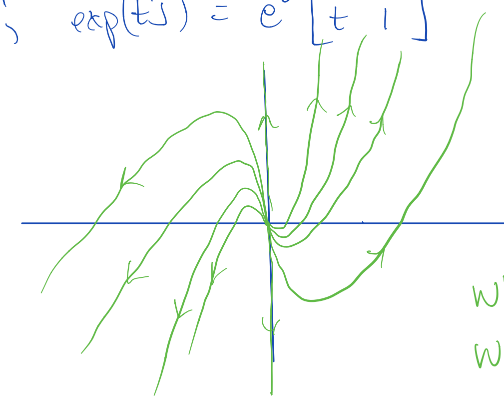
$$J\left(1, -2(1+\sqrt{1-4\varepsilon})^{-1}\right) = \begin{bmatrix} 1 & 0 \\ 1 & \sqrt{1-4\varepsilon} \end{bmatrix}$$ e.vals 1 and $\sqrt{1-4\varepsilon}$

$\varepsilon = 0$: $J = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$; $\exp(tJ) = e^t \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$

$x_1 = a e^t$

$x_2 = e^t(at+b)$

$\Rightarrow$ $x_2 = x_1 \log \alpha x_1 + \beta x_1$



$W^u = \mathbb{R}^2$

$W^s = \{0\}$

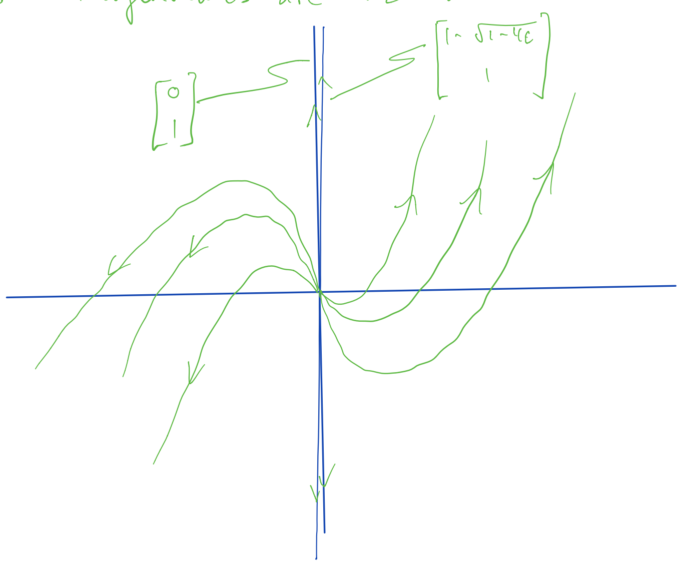
$\varepsilon \neq 0$: e.vec for 1 is $\begin{bmatrix} 1-\sqrt{1-4\varepsilon} \\ 1 \end{bmatrix} = \begin{bmatrix} 2\varepsilon + O(\varepsilon^2) \\ 1 \end{bmatrix}$

e.vec for $\sqrt{1-4\varepsilon} = 1 - 2\varepsilon + O(\varepsilon^2)$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x_1 = a\,[2\varepsilon + O(\varepsilon^2)]\,e^t$

$x_2 = a e^t + b e^{t(1-2\varepsilon+O(\varepsilon^2))} = e^t\left(a + b e^{t(-2\varepsilon + O(\varepsilon^2))}\right)$

The eigenvectors and eigenvalues are 2ε-close to each other



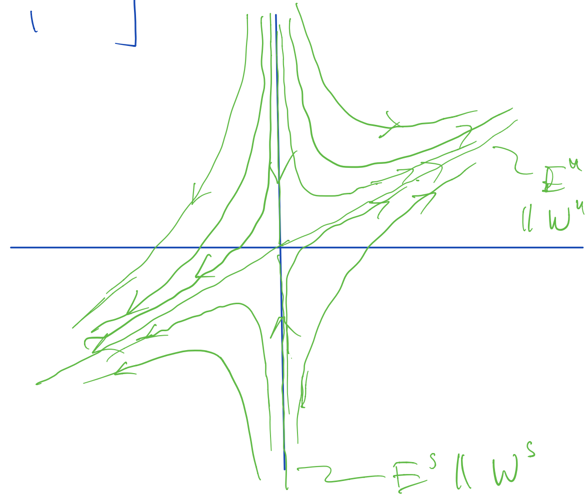
$W^u = \mathbb{R}^2$

$W^s = \{0\}$

$$J\left(1, \varepsilon^{-1}\tfrac{1}{2}\left(1+\sqrt{1-4\varepsilon}\right)\right) = \begin{bmatrix} 1 & 0 \\ 1 & -\sqrt{1-4\varepsilon} \end{bmatrix} \quad \text{e.vals } 1 \text{ and } -\sqrt{1-4\varepsilon}$$

$$\text{e.vec for } 1: \quad \begin{bmatrix} 1+\sqrt{1-4\varepsilon} \\ 1 \end{bmatrix} = \begin{bmatrix} 2-2\varepsilon+O(\varepsilon^2) \\ 1 \end{bmatrix}$$

$$\text{e.vec for } -\sqrt{1-4\varepsilon}: \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



(iii) $\dot{x} = y$
$\dot{y} = -x + \varepsilon(1-x^2)$

$$J = \begin{bmatrix} 0 & 1 \\ -1-2\varepsilon x & 0 \end{bmatrix}$$

fixed points: $(x_0, 0)$, $(x_\infty, 0)$

$$x_0 = \frac{2\varepsilon}{1+\sqrt{1+4\varepsilon^2}} = \frac{\varepsilon}{1+\varepsilon^2+O(\varepsilon^4)} = \varepsilon(1-\varepsilon^2+O(\varepsilon^4))$$

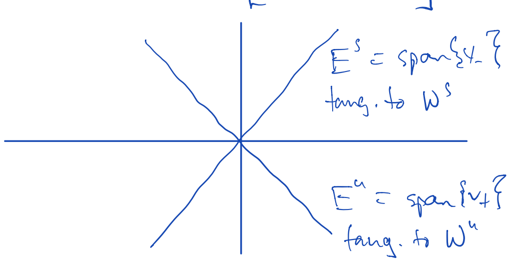
$$x_\infty = -\frac{1+\sqrt{1+4\varepsilon^2}}{2\varepsilon} = -\varepsilon^{-1}(1+\varepsilon^2+O(\varepsilon^4))$$

$E(x,y) = \frac{1}{2}(y^2+x^2) + \varepsilon\left(\frac{x^3}{3} - x\right)$ is constant on the trajectories.

$$J(x_0, 0) = \begin{bmatrix} 0 & 1 \\ -\sqrt{1+4\varepsilon^2} & 0 \end{bmatrix}$$; purely imag e.vals and closed trajectories $\forall \varepsilon$
$\Rightarrow W^s$ & W^u are $\{0\}$.

$$J(x_\infty, 0) = \begin{bmatrix} 0 & 1 \\ \sqrt{1+4\varepsilon^2} & 0 \end{bmatrix}$$; evals satisfy $\lambda^2 = \sqrt{1+4\varepsilon^2}$

so $\lambda_\pm = \pm\sqrt[4]{1+4\varepsilon^2} = \pm\left(1+\varepsilon^2+O(\varepsilon^4)\right)$



e.vectors: $v_\pm = \begin{bmatrix} 1 \\ \mp\sqrt[4]{1+4\varepsilon^2} \end{bmatrix}$