§1.5 in G-H

The periodically perturbed pendulum, or the parametrically forced pendulum.
The downward force on the pendulum can be periodically perturbed by moving the whole pendulum up and down periodically. The acceleration of the periodic movement imparts a force that is added to that of gravity:

$$\ddot{\phi} + (\alpha^2 + \beta \cos t)\sin\phi = 0$$

This can be cast in third-order autonomous form as

$$(\dagger) \qquad \begin{aligned} \dot{\phi} &= v \\ \dot{v} &= -(\alpha^2 + \beta\cos\theta)\sin\phi \\ \dot{\theta} &= 1 \end{aligned}$$

and the state (ϕ, v, θ) lies in $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$, where

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

The two fixed points $(\phi, v) = (0,0)$ and $(\phi, v) = (\pi, 0)$ of the non-autonomous system become two periodic orbits of $(\dagger)$:

$$(0) \qquad (\phi, v, \theta) = (0, 0, t \,(\mathrm{mod}\, 2\pi))$$

$$(\pi) \qquad (\phi, v, \theta) = (\pi, 0, t\,(\mathrm{mod}\, 2\pi))$$

We want to analyze how the trajectories near these stationary ones behave.
To do this, let us consider their linear approximations about the periodic trajectories (0) and (π). Recall that we derived the <u>derivative of the trajectory</u> of an ODE with respect to the initial condition.
It is the solution of the linear ODE with generator matrix equal to the derivative (Jacobian matrix) of the vector field of the ODE evaluated along the trajectory and initial condition

equal to the deviation of the initial state (ϕ, v, θ) from the trajectory. Let us apply this to the trajectories (0) and (π).

The Jacobian matrix is

$$J_{(\phi,v,\theta)} = \begin{bmatrix} 0 & 1 & 0 \\ -(\alpha^2+\beta\cos\theta)\cos\phi & 0 & -(\alpha^2-\beta\sin\theta)\sin\phi \\ 0 & 0 & 0 \end{bmatrix}$$

At (0) and (π), we obtain

$$J_0 = \begin{bmatrix} 0 & 1 & 0 \\ -(\alpha^2+\beta\cos t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_\pi = \begin{bmatrix} 0 & 1 & 0 \\ \alpha^2+\beta\cos t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the linearized systems are

$$(\tilde{0}) \quad \begin{aligned} \dot{\tilde{\phi}} &= \tilde{v} \\ \dot{\tilde{v}} &= -(\alpha^2+\beta\cos t)\tilde{\phi} \\ \dot{\tilde{\theta}} &= 0 \end{aligned} \qquad\qquad (\tilde{\pi}) \quad \begin{aligned} \dot{\tilde{\phi}} &= \tilde{v} \\ \dot{\tilde{v}} &= (\alpha^2+\beta\cos t)\tilde{\phi} \\ \dot{\tilde{\theta}} &= 0 \end{aligned}$$

We can throw away $\tilde{\theta}$, and let us reuse ϕ and v to obtain the periodic linear systems

$$(0) \quad \begin{aligned} \dot{\phi} &= v \\ \dot{v} &= -(\alpha^2+\beta\cos t)\phi \end{aligned} \qquad\qquad (\pi) \quad \begin{aligned} \dot{\phi} &= v \\ \dot{v} &= (\alpha^2+\beta\cos t)\phi \end{aligned}$$

If these are rendered non-autonomous, with θ replacing t in the vector field and with the additional equation $\dot{\theta} = 1$, then the natural Poincaré map uses the 2D hypersurface $\mathbb{T}\times\mathbb{R}$ within $\mathbb{T}\times\mathbb{R}\times\mathbb{T}$, and because of the periodicity of θ, i.e., $\theta(t) = t \pmod{2\pi}$

the Poincaré map is trivially guaranteed. The return time is always equal to 2π. In fact, we are in the situation of the Floquet theory. Let us write (υ) or (π) as

$$\dot{u} = A(t)u,$$

with $u(t) = (\phi(t), v(t))$. Let $u_1 = (\phi_1, v_1)$ and $u_2 = (\phi_2, v_2)$ be solutions with

$$u_1(0) = (1, 0)$$
$$u_2(0) = (0, 1),$$

and set $U(t)$ equal to the standard fundamental matrix solution

$$U(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ v_1(t) & v_2(t) \end{bmatrix}.$$

Then the Poincaré map is

$$\begin{pmatrix} \phi_0 \\ v_0 \end{pmatrix} \longmapsto U(2\pi) \begin{pmatrix} u_0 \\ \phi_0 \end{pmatrix}.$$

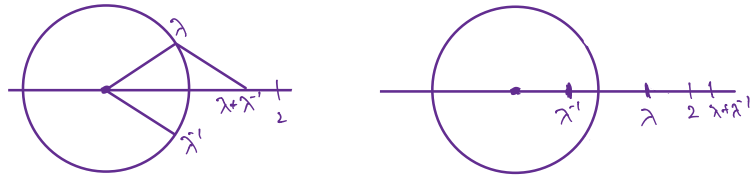
Since $\operatorname{tr} A(t) = 0$, we obtain

$$\det U(t) = 1 \qquad \forall\, t \in \mathbb{R}.$$

Therefore the eigenvalues λ of $U(2\pi)$ satisfy

$$\lambda + \lambda^{-1} = \operatorname{tr} U(2\pi) = \phi_1(2\pi) + v_2(2\pi),$$

so the eigenvalues are reciprocals of one another, ie, if λ is an eigenvalue, then so is λ^{-1}.



$\operatorname{tr} U(2\pi) < 2$ $\qquad\qquad$ $\operatorname{tr} U(2\pi) > 2$

When $\operatorname{tr} U(2\pi) < 0$, the eigenvalues are complex conjugates of modulus 1, and when $\operatorname{tr} U(2\pi) > 0$, they are real reciprocals of one another, say $|\lambda| > 1$ and $|\lambda^{-1}| < 1$.

When $\beta = 0$, we can find $\operatorname{tr} U(2\pi)$ because $A(t)$ is in fact independent of t:

$$A(t) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{bmatrix} & (0) \\[2ex] \begin{bmatrix} 0 & 1 \\ \alpha^2 & 0 \end{bmatrix} & (\pi) \end{cases} \qquad (\beta = 0)$$

We obtain

$$U(t) = e^{tA} = \begin{cases} \begin{bmatrix} \cos t\alpha & \alpha^{-1}\sin t\alpha \\ -\alpha \sin t\alpha & \cos t\alpha \end{bmatrix} & (0) \\[2ex] \begin{bmatrix} \cosh t\alpha & \alpha^{-1}\sinh t\alpha \\ \alpha \sinh t\alpha & \cosh t\alpha \end{bmatrix} & (2\pi) \end{cases} \qquad (\beta = 0).$$

The eigenvalues of these matrices at $t = 2\pi$ are

$$\begin{cases} \lambda = e^{\pm 2\pi i \alpha} & (0) \\[1ex] \lambda = e^{\pm 2\pi \alpha} & (\pi) \end{cases} \qquad (\beta = 0).$$

Since the ei values are continuous functions of β, we see that the two eigenvalues λ_1 and λ_2 satisfy

$$(0) \qquad |\lambda_1| = |\lambda_2| = 1 \qquad (\cos 2\pi\alpha \neq 1,\ 0 < \beta \ll 1) \quad [\ll \text{ depends on } \alpha]$$

$$(\pi) \qquad 0 < |\lambda_1| < 1 < |\lambda_2| \qquad (0 < \beta \ll 1)$$

The fixed point $(\pi, 0)$ for the linear system is unstable, and by the Hartman-Grobman Theorem, the original nonlinear system is also unstable. For the fixed point $(\pi, 0)$, the linear system is (neutrally) Lyapunov stable as long as $\alpha \notin \frac{1}{2}\mathbb{Z}$. The nonlinear system is not necessarily stable. When $\alpha \in \frac{1}{2}\mathbb{Z}$, the system is "in resonance". The linear system is not stable because $U(2\pi)$ has a double eigenvalue of modulus 1, and therefore all solutions except for the eigenfunction mode experiences linear growth of its Poincaré map.

More on ODEs in the plane

Bendixson's Criterion is a connection between the divergence of a vector field and the existence of closed orbits.

Let $\langle f(x,y), g(x,y)\rangle$ be a C^1 vector field, and let $\gamma(t) = (x(t), y(t))$ be a solution of $\{\dot{x} = f(x,y),\ \dot{y} = g(x,y)\}$. Then $\langle \dot{x}, \dot{y}\rangle \cdot \langle g, -f\rangle \equiv 0$. Thus, if γ is a closed trajectory that is the boundary of a domain D, then

$$0 = \int_\gamma \big(g(x(t), y(t))\,\dot{x}(t) - f(x(t), y(t))\,\dot{y}(t) \big)\,dt$$

$$= \int_\gamma \big(g(x,y)\,dx - f(x,y)\,dy \big) = \iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dx\,dy$$

Thus $\nabla \cdot \langle f, g\rangle$ cannot be of one sign in D. Bendixson's criterion states that, if $\nabla \cdot \langle f, g\rangle$ is of one sign in a simply connected region, then the ODE has no periodic orbits there.

A few more definitions:

Given ODE $\dot{x} = f(x)$ in $\mathbb{R}^n$:

$S \subset \mathbb{R}^n$ is <u>invariant</u> if $\phi_t(S) \subset S \quad \forall\, t \in \mathbb{R}$

<u>+-invariant</u> if $\phi_t(S) \subset S \quad \forall\, t \geq 0$.

$D \subset \mathbb{R}^n$ is a <u>trapping region</u> if D is closed, connected, and +-invariant.

$A \subset \mathbb{R}^n$ is an <u>attracting set</u> if A is closed and invariant and A admits a +-invariant neighborhood U such that $\forall\, x \in U$, $\phi_t(x) \to A$ as $t \to \infty$.

An <u>attractor</u> is an attracting set that contains a dense trajectory.

A point p is <u>nonwandering</u> if for each neighborhood $\mathcal{U}$ of p, $\exists\ \{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$ as $n \to \infty$ such that $\phi_{t_n}(\mathcal{U}) \cap \mathcal{U} \neq \{\}$. The <u>nonwandering set</u> is the set of all nonwandering points.

See p. 45 in G/H for a theorem on nonwandering sets in $\mathbb{R}^2$.

And see p. 46 for some sketches of ω-limits.

<u>Example</u> $\begin{cases} \dot{x} = -x^4 \sin(\pi x^{-1}) \\ \dot{y} = -y \end{cases}$ in $\mathbb{R}^2_{(x,y)}$

The set $[-1,1] \times \{0\}$ is an attracting set but not an attractor.

It contains infinitudes of asymptotically stable and unstable fixed points and the Lyapunov but not asymptotically stable fixed point at 0.

The nonwandering set is the set of stable fixed points.

Structural Stability

Defn A vector field g is a (C^k, ε)-perturbation of a C^r vector field f $(k \le r)$ if f and g differ only on a compact set and

$$\left| \frac{\partial^i (f-g)}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \right| < \varepsilon$$

for all partial derivatives of order $i \le k$.

Defn Two C^r vector fields f and g are C^k-equivalent $(k \le r)$ if $\exists$ a C^k diffeomorphism that maps orbits of $\dot{x} = f(x)$ to orbits of $\dot{x} = g(x)$ while preserving orientation. They are C^k-conjugate if such a diffeomorphism also preserves time parameterization.

Defn A vector field f is structurally stable if $\forall\ \varepsilon > 0$, each (C^1, ε) perturbation of f is C^0-equivalent (topologically equivalent) to f.

Peixoto's Theorem A C^1 vector field on a compact 2D C^1 manifold is structurally stable if and only if the following three conditions hold.

(1) The set of fixed points and periodic orbits is finite, and each is hyperbolic

(2) There is no orbit that connects two hyperbolic saddle fixed points.

(3) The nonwandering set is a union of fixed points and periodic orbits.

This theorem can be applied to a trapping region in the plane, in which case (1) and (2) together imply (3).

The index of a closed curve C in the presence of a vector field $\langle f(x,y), g(x,y) \rangle$ is the increment of the angle$/2\pi$ of the vector around C. If $\bar{x}$ is a fixed point, the index of $\bar{x}$ is the index of a simple closed curve C that encircles $\bar{x}$ counterclockwise, with C enclosing no other fixed points.

$$k = \frac{1}{2\pi}\int_C d \arctan \frac{g(x,y)}{f(x,y)} = \frac{1}{2\pi}\int_C \frac{f\,dg - g\,df}{f^2+g^2}$$

Notice that, if C encloses a region in which the vector field does not vanish, then $\arctan \frac{g(x,y)}{f(x,y)}$ is a single-valued function, and thus $k=0$. Thus the values of k obtained from two different curves C_1 and C_2 are equal as long as $C_1 - C_2$ is the boundary of a region containing not points of vanishing of $\langle f, g\rangle$.

It is pretty easy to verify the following geometrically (G&H p. 51, Proposition 1.8.4).

(i) The index of a sink, source, or center is 1

(ii) The index of a hyperbolic saddle point ($\lambda_1 < 0$, $\lambda_2 > 0$) is -1

(iii) The index of a closed orbit is 1

(iv) The index of a closed curve is the sum of the indices of the fixed points it encircles. In particular, the index of a closed curve not encircling any fixed point is 0.

From this, you can deduce the nature of the set of fixed points enclosed by a periodic orbit.