Periodic linear ODEs [See Yakubovich/Starzhinskii 1975]

Consider the linear non-autonomous ODE $\dot{x} = A(t)x$ and its matrix IVP $\dot{X} = A(t)X$, $X(0) = E$. It is in general not true that the solution is

(*) $$X(t) = \exp \int_0^t A(s)\,ds,$$

but if $\{A(t),\ t \in (a,b)\}$ is a commuting family, then $X(t)$ does satisfy the ODE in (a,b).

Theorem If $A : (a,b) \to M(n, \mathbb{C})$ is continuous and $A(s)A(t) = A(t)A(s)$ for all $s,t \in (a,b)$, then (*) satisfies $\dot{X}(t) = A(t)X(t)\ \forall\ t \in (a,b)$.

Proof Let $t \in (a,b)$ be given, and let h denote a real variable such that $t+h \in (a,b)$. We have

(†) $$X(t+h) = \exp\left[\int_0^t A(s)\,ds + \int_t^{t+h} A(s)\,ds\right] = \left(\exp\int_0^t A(s)\,ds\right)\left(\exp\int_t^{t+h} A(s)\,ds\right)$$

because $\int_0^t A(s)\,ds$ and $\int_t^{t+h} A(s)\,ds$ commute, as is seen from

$$\int_0^t A(s)ds \int_t^{t+h} A(r)dr = \int_t^{t+h}\left[\int_0^t A(s)ds\right]A(r)dr = \int_t^{t+h}\int_0^t A(s)A(r)\,ds\,dr$$
$$= \int_t^{t+h}\int_0^t A(r)A(s)\,ds\,dr = \int_0^t\int_t^{t+h} A(r)A(s)\,dr\,ds = \int_0^t\left[\int_t^{t+h} A(r)dr\right]A(s)ds$$
$$= \int_t^{t+h} A(r)dr \int_0^t A(s)ds .$$

Next, $\int_t^{t+h} A(s)\,ds = \int_t^{t+h} A(t)\,ds + \int_t^{t+h} (A(s) - A(t))\,ds = hA(t) + h\mathcal{E}(h)$, where $|\mathcal{E}(h)| \le \max_{|s-t|\le h} |A(s)-A(t)| \to 0$ as $h \to 0$ by continuity of A. Thus

$$\exp\int_t^{t+h} A(s)\,ds = \exp\left[hA(t) + h\mathcal{E}(h)\right]$$

$$= E + hA(t) + h\tilde{\mathcal{E}}(h)$$

where $\tilde{\mathcal{E}}(h) \to 0$ as $h \to 0$. Going back to (+), we obtain

$$X(t+h) = X(t)\left(E + hA(t) + h\tilde{\mathcal{E}}(h)\right),$$

or $\quad X(t+h) - X(t) = hA(t)X(t) + h\tilde{\mathcal{E}}(h)X(t)$

Since $\tilde{\mathcal{E}}(h)X(t) \to 0$ as $h \to 0$, we obtain

$$\frac{d}{dt}X(t) = A(t)X(t). \qquad \blacksquare$$

Theorem Let Y be an $n \times n$ matrix with complex entries such that $\det Y \ne 0$. Then $\exists$ a matrix R such that $Y = e^R$.

Proof Define a matrix-valued function of a complex variable λ by $L(\lambda) = (1-\lambda)E + \lambda Y$. Since $\det L(\lambda)$ is an n^{th}-degree polynomial in λ, it has no more than n roots; and neither 0 nor 1 is a root. Therefore, let $\mu: [0,1] \to \mathbb{C}$ be a differentiable path such that $\forall\, t \in [0,1]$, $\mu(t)$ is not a root of $\det L(\lambda)$ and $\mu(0) = 0$ and $\mu(1) = 1$. Thus $t \to L(\mu(t))^{-1}$ is a differentiable function on $[0,1]$, with $L(\mu(0)) = E$ and $L(\mu(1)) = Y$. Differential calculus yields

$$\frac{d}{dt} L(\mu(t)) = \left[(Y-E)\frac{d\mu}{dt}(t)\, L(\mu(t))^{-1} \right] L(\mu(t)) .$$

$\{B(t) : t \in \mathbb{R}\}$ is a commuting family of matrices?

~~Since $\{L(\mu(t)) : t \in [0,1]\}$ is a commuting family~~

Set $B(t) = (Y-E)\frac{d\mu}{dt}(t)\, L(\mu(t))^{-1}$. By the previous theorem,

$$L(\mu(t)) = \exp \int_0^t B(s)\,ds$$

Thus $Y = L(\mu(1)) = \exp \int_0^1 B(s)\,ds$ ∎

<u>Lemma</u> Let $X(t)$ be a solution of $\dot{X} = A(t)X$, with $X(0) = E$. Then $\det X(t) \neq 0$ for all t.

<u>Proof</u> Set $D(t) = \det X(t)$. By Jacobi's formula,

$$\frac{d}{dt}D(t) = D(t)\, \mathrm{tr}\left(\dot{X}(t)\, X(t)^{-1}\right) = a(t)\, D(t)$$
$$D(0) = 1$$

Thus $D(t) = \exp \int_0^t a(s)\,ds \neq 0$. ∎

[to be proved later for $\dot{x} = f(x)$]

For now, let us assume uniqueness of the solution to the initial value problem $\dot{X} = A(t)X$, $X(0) = X_0$, with $A(t)$ continuous.

The following is the main theorem in Floquet theory for periodic linear ODEs.

Theorem Let $X : \mathbb{R} \to M(n, \mathbb{C})$ be a solution to the ODE $\dot{X} = A(t)X$, with $A : \mathbb{R} \to M(n, \mathbb{C})$ being continuous and having period T, that is $A(t+T) = A(t)$ for all $t \in \mathbb{R}$.

Then there is a matrix R such that $\forall\, t \in \mathbb{R}$,

(H) $$X(t+T) = X(t)e^{TR};$$

and the function $Z(t) = X(t)e^{-tR}$ has period T.

Note A function with the property (H) is called pseudo-periodic or quasi-periodic.

Proof Define $X_T(t) := X(t+T)$. Then

$\frac{d}{dt} X_T(t) = \dot{X}(t+T) = A(t+T)X(t+T) = A(t+T)X_T(t)$
$= A(t)X_T(t)$. By the previous lemma, $X(T)$ is invertible, and by the preceding theorem, there is a matrix R such that $X(T) = e^{TR}$. Thus $X_T(0) = e^{TR}$.
By the uniqueness of the solution to the IVP, $X_T(t) = X(t)e^{TR}$, which proves (H).

Finally, $Z(t+T) = X(t+T)e^{-(t+T)R} = X(t)e^{TR}e^{-(t+T)R}$
$= X(t)e^{-tR} = Z(t)$. $\blacksquare$

The quasi-periodic solution $X(t)$ is determined by its values for $t \in [0,T]$:

$$X(t) = X(t-nT)e^{nTR}, \qquad t \in [0,T] + nT.$$

The solution to the IVP $\dot{x} = A(t)x$; $x(0) = x_0$ can be described through the discrete system of states

$$x_0 \mapsto x_1 = e^{TR}x_0 \mapsto \dots \mapsto x_n = e^{nTR}x_0 \dots,$$

and completely by the sequence of functions on $[0,T]$

$$x_n(t) = X(t)x_n, \qquad t \in [0,T],$$

so that $x(t) = x_n(t-nT)$ for $t \in [0,T] + nT$.

Now make $\dot{x} = A(t)x$ autonomous with a new state variable $\theta \in \mathbb{R}/T\mathbb{Z}$ on the circle $S = \mathbb{R}/T\mathbb{Z}$

$$y \in (x,\theta) \in \mathbb{R}^n \times S =: \mathcal{X}$$

$$\begin{cases} \dot{y} = (\dot{x}, \dot{\theta}) = (A(\theta)x, 1) & [\text{here, } 1 \in \mathbb{R}, \text{ not in } S] \\ y(0) = (x(0), \bar{0}) & [\text{here, } \bar{0} \in S, \text{ not in } \mathbb{R}] \end{cases}$$

This system has a unique solution in phase space $\mathcal{X} = \mathbb{R}^n \times S$ (proofs later).

Now consider the following distinguished hypersurface in $\mathcal{X}$:

$$\Sigma = \mathbb{R}^n \times \{\bar{0}\} = \mathbb{R}^n \times \underbrace{\{T\mathbb{Z}\}}_{\text{a point in } S = \mathbb{R}/T\mathbb{Z}} \cong \mathbb{R}^n ,$$

and notice that $\{x_n\}_{n\in\mathbb{N}_0}$ defined above is the sequence of intersection points of the trajectory $y(t) = (x(t), \bar{t})$ with Σ. The discrete dynamical system

$$x_{n+1} = e^{TR} x_n$$

on Σ is the Poincaré map for the continuous dynamical system $\dot{y} = (A(\theta)x, 1)$ in $\mathcal{X}$, where x is the projection of $y \in \mathcal{X} = \mathbb{R}^n \times S$ onto $\mathbb{R}^n$.

We will return to Poincaré maps for more general nonlinear systems $\dot{x} = f(x)$.