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A DIFFUSE INTERFACE MODEL FOR ELECTROWETTING WITH MOVING CONTACT LINES

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We introduce a diffuse interface model for the phenomenon of electrowetting on dielectric and present an analysis of the arising system of equations. Moreover, we study discretization techniques for the problem. The model takes into account different material parameters on each phase and incorporates the most important physical processes, such as incompressibility, electrostatics and dynamic contact lines; necessary to properly reflect the relevant phenomena. The arising nonlinear system couples the variable density incompressible Navier–Stokes equations for velocity and pressure with a Cahn–Hilliard type equation for the phase variable and chemical potential, a convection diffusion equation for the electric charges and a Poisson equation for the electric potential. Numerical

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experiments are presented, which illustrate the wide range of effects the model is able to capture, such as splitting and coalescence of droplets.

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1. Introduction

The term electrowetting on dielectric refers to the local modification of the surface tension between two immiscible fluids via electric actuation. This allows for change of shape and wetting behavior of a two-fluid system and, thus, for its manipulation and control.

The existence of such a phenomenon was originally discovered by Lippmann,[47] more than a century ago (see also Refs. 7, 52, 11 and 66). However, only recently has electrowetting found a wide spectrum of applications, specially in the realm of micro-fluidics.[22,23,35] One can mention, for example, reprogrammable lab-on-chip systems,[45,61] auto-focus cell phone lenses,[12] colored oil pixels and video speed smart paper.[40,59,60] In Ref. 44, the reverse electrowetting process has been proposed as an approach to energy harvesting.

From the examples presented above, it becomes clear that it is very important for applications to have a better understanding of this phenomenon and it is necessary to obtain reliable computational tools for the simulation and control of these effects. The computational models must be complete enough, so that they can reproduce the most important physical effects, yet sufficiently simple that it is possible to extract from them meaningful information in a reasonable amount of computing time. Several works have been concerned with the modeling of electrowetting. The approaches include experimental relations and scaling laws,[42,72] empirical models,[49] studies concerning the dependence of the contact angle[31,64] or the shape of the droplet[51,24] on the applied voltage, lattice Boltzmann methods[5,4] and others. Of relevance to our present discussion are the works Refs. 74, 73, 27 and 30. To the best of our knowledge, Refs. 74 and 73 are the first papers where the contact line pinning was included in an electrowetting model. On the other hand, the models of Refs. 27 and 30 are the only ones that are intrinsically three-dimensional and do not assume any special geometric configuration. They have the limitation, however, that they assume the density of the two fluids to be constant and they apply a no-slip boundary condition to the fluid-solid interface, thus limiting the movement of the droplet.

The purpose of this work is to propose and analyze an electrowetting model that is intrinsically three-dimensional; it takes into account that all material parameters are different in each of the fluids; and it is derived (as long as this is possible) from physical principles. To do so, we extend the diffuse interface model of Ref. 27. The main additions are the fact that we allow the fluids to have different densities — thus leading to a variable density Cahn–Hilliard Navier–Stokes system — and that we treat the contact line movement in a thermodynamically consistent way, namely

using the so-called generalized Navier boundary condition (see Refs. 58 and 57). We must mention however, that to be able to analyze the model, we must introduce a simplification on the generalized Navier boundary condition, which basically amounts to an additional regularization, see Sec. 2.3 for details. In addition, we propose a (phenomenological) approach to contact line pinning and study stability and convergence of discretization techniques. In this respect, our work also differs from Refs. 27 and 30, since our approach deals with a practical fully discrete scheme, for which we derive *a priori* estimates and convergence results. We extend the numerical scheme developed and analyzed in Ref. 62 to our electrowetting model and show (under certain conditions) its stability. As this reference shows, the method possesses excellent mass conservation properties, so that our approach to electrowetting does so as well. Finally, under the classical assumption that the mesh size h and the regularization parameter δ satisfy $h = \mathcal{O}(\delta)$, we present several numerical simulations.

Through private communication we have become aware of the following recent contributions: discretization schemes for the model proposed in Ref. 27 are studied in Ref. 43; the models of Refs. 27 and 30 have been extended, using the techniques of Ref. 1, in Refs. 26 and 36 where discretization issues are also discussed.

This work is organized as follows. In Sec. 1.1 we introduce the notation and some preliminary assumptions necessary for our discussion. Section 2 describes the model that we shall be concerned with and its physical derivation. A formal energy estimate and a formal weak formulation of our problem is shown in Sec. 3. The energy estimate shown in this section serves as a basis for the precise definition of our notion of solution and the proof of its existence. The details of this are accounted for in Sec. 4. In Sec. 5 we discuss discretization techniques for our problem and present some numerical experiments aimed at showing the capabilities of our model: droplet splitting and coalescence as well as contact line movement. Finally, in Sec. 6, we briefly discuss convergence of the discrete solutions to solutions of a semi-discrete problem.

1.1. *Notation and preliminaries*

Figure 1 shows the basic configuration for the electrowetting on dielectric problem.[22,23] We use the symbol Ω to denote the domain occupied by the fluid and $\Omega^\star$ for the fluid and dielectric plates, thus, $\Omega \subset \Omega^\star$. In this manner, we assume that Ω and $\Omega^\star$ are convex, bounded connected domains in $\mathbb{R}^d$, for $d = 2$ or 3, with $\mathcal{C}^{0,1}$ boundaries. The boundary of Ω is denoted by Γ and $\partial^\star\Omega^\star = \partial\Omega^\star\backslash\Gamma$, $\mathbf{n}$ stands for the outer unit normal to Γ. We denote by $[0, T]$ with $0 < T < \infty$ the time interval of interest. For any vector valued function $\mathbf{w} : \Omega \to \mathbb{R}^d$ that is smooth enough so as to have a trace on Γ, we define the tangential component of $\mathbf{w}$ as

$$\mathbf{w}_{\boldsymbol{\tau}}|_\Gamma := \mathbf{w}|_\Gamma - (\mathbf{w}|_\Gamma \cdot \mathbf{n})\mathbf{n}, \tag{1.1}$$

and, for any scalar function $f, \partial_{\boldsymbol{\tau}} f := (\nabla f)_{\boldsymbol{\tau}}$.

Fig. 1.   The basic configuration of an electrowetting on dielectric device. The solid black region depicts the dielectric plates and the white region denotes a droplet of one fluid (say water), which is surrounded by another (air). We denote by Ω the fluid domain, by Γ its boundary, by $\Omega^\star$ the region occupied by the fluids and the plates and by $\partial^\star\Omega^\star := \partial\Omega^\star\backslash\Gamma$.

We will use standard notation for spaces of Lebesgue integrable functions $L^p(\Omega)$, $1 \leq p \leq \infty$ and Sobolev spaces $W_p^m(\Omega)$ $1 \leq p \leq \infty$, $m \in \mathbb{N}_0$.[2] Vector valued functions and spaces of vector valued functions will be denoted by boldface characters. For $S \subset \mathbb{R}^d$, by $\langle \cdot, \cdot \rangle_S$ we denote, indistinctly, the $L^2(S)$- or $\mathbf{L}^2(S)$-inner product. If no subscript is given, we assume that the domain is Ω. If $S \subset \mathbb{R}^{d-1}$, then the inner product is denoted by $[\cdot, \cdot]_S$ and if no subindex is given, the domain must be understood to be Γ. We define the following spaces:

$$H_\star^1(\Omega^\star) := \{v \in H^1(\Omega^\star) : v|_{\partial^\star\Omega^\star} = 0\}, \tag{1.2}$$

normed by

$$\|v\|_{H_\star^1} := \|\nabla v\|_{\mathbf{L}^2(\Omega^\star)}$$

and

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0\}, \tag{1.3}$$

which we endow with the norm

$$\|\mathbf{v}\|_{\mathbf{V}}^2 := \|\nabla\mathbf{v}\|_{\mathbf{L}^2}^2 + \|\mathbf{v}_\tau\|_{\mathbf{L}^2(\Gamma)}^2.$$

Clearly, for these norms, they are Hilbert spaces.

To take into account the fact that our problem will be time-dependent we introduce the following notation. Let E be a normed space with norm $\|\cdot\|_E$. The space of functions $\varphi : [0, T] \rightarrow E$ such that the map $(0, T) \ni t \mapsto \|\varphi(t)\|_E \in \mathbb{R}$ is L^p-integrable is denoted by $L^p(0, T, E)$ or $L^p(E)$. To discuss the time discretization of our problem, we introduce a time-step $\Delta t > 0$ (for simplicity assumed constant) and let $t_n = n\Delta t$ for $0 \leq n \leq N := \lceil T/\Delta t \rceil$. For any time-dependent function, φ, we denote $\varphi^n := \varphi(t_n)$ and the sequence of values $\{\varphi^n\}_{n=0}^N$ is denoted by $\varphi_{\Delta t}$. For any sequence $\varphi_{\Delta t}$ we define the time-increment operator $\mathfrak{d}$ by

$$\mathfrak{d}\varphi^n := \varphi^n - \varphi^{n-1} \tag{1.4}$$

and the time average operator $\overline{(\cdot)}$ by

$$\overline{\varphi^n} := \frac{1}{2}(\varphi^n + \varphi^{n-1}). \tag{1.5}$$

On sequences $\varphi_{\Delta t} \subset E$ we define the norms

$$\|\varphi_{\Delta t}\|_{\ell^2(E)}^2 := \Delta t \sum_{n=0}^{N} \|\varphi^n\|_E^2,$$

$$\|\varphi_{\Delta t}\|_{\ell^\infty(E)} := \max_{0 \leq n \leq N} \{\|\varphi^n\|_E\},$$

$$\|\varphi_{\Delta t}\|_{\mathfrak{h}^{1/2}(E)}^2 := \sum_{n=1}^{N} \|\mathfrak{d}\varphi^n\|_E^2,$$

which are, respectively, discrete analogues of the $L^2(E)$, $L^\infty(E)$ and $H^{1/2}(E)$ norms. When dealing with energy estimates of time discrete problems, we will make, without explicit mention, repeated use of the following elementary identity

$$2a(a - b) = a^2 - b^2 + (a - b)^2. \tag{1.6}$$

2. Model Derivation

In this section we briefly describe the derivation of our model. The procedure used to obtain it is quite similar to the arguments used in Refs. 27, 58 and 1 and it fits into the general ideological framework of so-called phase-field models.[6,41] In phase-field methods, sharp interfaces are replaced by thin transitional layers where the interfacial forces are now smoothly distributed and, thus, there is no need to explicitly track interfaces.

2.1. *Diffuse interface model*

To develop a phase-field model, we begin by introducing a so-called phase field variable ϕ and an interface thickness δ. The phase field variable acts as a marker that will be almost constant (in our case ± 1) in the bulk regions, and will smoothly transition between these values in an interfacial region of thickness δ. Having introduced the phase field, all the material properties that depend on the phase are slave variables and defined as

$$\Psi(\phi) = \frac{\Psi_1 - \Psi_2}{2} \arctan\left(\frac{\phi}{\delta}\right) + \frac{\Psi_1 + \Psi_2}{2}, \tag{2.1}$$

where Ψ_i are the values on each of the phases.

Remark 2.1. (Material properties) Relation (2.1) is not the only possible definition of the phase dependent quantities. For instance, Ref. 69 proposes to use a linear average between the bulk values. This approach has the advantage that the derivative of a phase-dependent field with respect to the phase (expressions that contain such quantities appear repeatedly) is constant, which greatly simplifies the calculations. However, this definition cannot be guaranteed to stay in the physical range of values which might lead to, say, a vanishing density or viscosity. On the

other hand, Ref. 48 proposes to use a harmonic average which guarantees that positive quantities stay bounded away from zero. In this work, we will assume that, with the exception of the permittivity ε, (2.1) is the way the slave variables are defined, which has the advantage that guarantees that the field stays within the physical bounds. Any other definition with this property is equally suitable for our purposes.

We model the droplet and surrounding medium as an incompressible Newtonian viscous two-phase fluid, so that its behavior is governed by the variable density incompressible Navier–Stokes equations. The equation of conservation of momentum can be written in several forms. We chose the one proposed by Guermond and Quartapelle (Ref. 37, see also Refs. 67 and 69) because its nonlinear term possesses a skew symmetry property similar to the constant density Navier–Stokes equations,

$$\sigma(\sigma\mathsf{u})_t + (\rho\mathsf{u} \cdot \nabla\mathsf{u}) + \frac{1}{2}\nabla \cdot (\rho\mathsf{u})\mathsf{u} - \nabla \cdot (\eta\mathbf{S}(\mathsf{u})) + \nabla\mathsf{p} = \mathbf{F}, \qquad (2.2a)$$

$$\nabla \cdot \mathsf{u} = 0, \qquad (2.2b)$$

where $\sigma = \sqrt{\rho}$ and ρ is the density of the fluid and depends on the phase field; u is the velocity of the fluid; p is the pressure; η is the viscosity of the fluid and depends on ϕ; $\mathbf{S}(\mathsf{u}) = \frac{1}{2}(\nabla\mathsf{u} + \nabla\mathsf{u}^{\mathsf{T}})$ is the symmetric part of the gradient and $\mathbf{F}$ are the external forces acting on the fluid.

Remark 2.2. (Convective terms) The reader might wonder what is the purpose of writing the material derivative as in (2.2a). The advantages are twofold. First, as noticed in Ref. 37, we recover the fundamental skew-symmetry of the convective terms, i.e.

$$\int_\Omega \left(\rho\mathbf{v} \cdot \nabla\mathbf{w} + \frac{1}{2}\nabla \cdot (\rho\mathbf{v})\mathbf{w} \right) \mathbf{w} = 0, \qquad (2.3)$$

for all vector fields $\mathbf{v}, \mathbf{w}$ such that $\mathbf{v} \cdot \mathbf{n} = 0$, irrespective of the fact that the density ρ is not constant. In other words, we reproduce a key property of homogeneous flows. This is important for the analysis of the system. Second, notice that this identity holds even for nonsolenoidal fields. This is important with regard to discretization, where we do not have solenoidal fields. The interested reader is referred to Ref. 17 where, for the standard Cahn–Hilliard Navier–Stokes problem, a numerical simulation illustrates the loss of stability that might occur if this modification is not accounted for.

The phase field can be thought of as a scalar that is convected by the flow. Hence its motion is described by

$$\phi_t + \nabla \cdot (\phi\mathsf{u}) = -\nabla \cdot \mathbf{J}_\phi, \qquad (2.4)$$

for some flux field $\mathbf{J}_\phi$ which will be found later.

To model the interaction between the applied voltage and the fluid, we introduce the charge density q. Another possibility, not explored here, is to introduce ion concentrations, thus leading to a Nernst–Planck Poisson-like system, see Refs. 30, 65, 55 and 54. In this respect the reader is referred also to Ref. 26, where the authors show that if, in such a system, we consider only three species carrying charges $0, +1$ and -1, respectively, we can obtain the equations for charge density that we derive below. The electric displacement field $\mathbf{D}$ is defined in $\Omega^\star$. The evolution of these two quantities is governed by Maxwell's equations, i.e.

$$\nabla \cdot \mathbf{D} = q, \quad \mathbf{D}_t + q\mathsf{u} + \mathbf{J_D} = 0, \tag{2.5}$$

for some flux $\mathbf{J_D}$. Notice that we assume the magnitude of the velocity of the fluid is negligible in comparison with the speed of light, and that the frequency of voltage actuation is sufficiently small, so that magnetic effects can be ignored. Taking the time derivative of the first equation and substituting in the second we obtain

$$q_t + \nabla \cdot (q\mathsf{u}) = -\nabla \cdot \mathbf{J_D}. \tag{2.6}$$

To close the system, we must prescribe boundary conditions, determine the force $\mathbf{F}$ exerted on the fluid, and find constitutive relations for the fluxes $\mathbf{J}_\phi$ and $\mathbf{J_D}$. We are assuming the solid walls are impermeable, therefore if $\mathbf{n}$ is the normal to Γ, $\mathsf{u} \cdot \mathbf{n} = 0$ on Γ and $\mathbf{J}_\flat \cdot \mathbf{n} = 0$ for any flux $\mathbf{J}_\flat$. To find the rest of the boundary conditions, $\mathbf{F}$ and relations for the fluxes, we denote the surface tension between the two phases by γ and define the Ginzburg–Landau double well potential by

$$\mathcal{W}(\xi) = \begin{cases} (\xi + 1)^2, & \xi < -1, \\ \frac{1}{4}(1 - \xi^2)^2, & |\xi| \leq 1, \\ (\xi - 1)^2, & \xi > 1. \end{cases}$$

Remark 2.3. (The Ginzburg–Landau potential) The original definition, given by Cahn and Hilliard, of the potential is logarithmic. See, for instance, Ref. 34. This way, the potential becomes infinite if the phase field variable is out of the range $[-1, 1]$, thus guaranteeing that the phase field variable ϕ stays within that range. This is difficult to treat both in the analysis and numerics and hence practitioners have used the Ginzburg–Landau potential $c(1 - \xi^2)^2$, for some $c > 0$. We go one step further and restrict the growth of the potential to quadratic away from the range of interest. With this restriction Caffarelli and Müller,[20] have shown uniform L^∞-bounds on the solutions of the Cahn–Hilliard equations (which as we will see below the phase field must satisfy). This has also proved useful in the numerical discretization of the Cahn–Hilliard and Cahn–Hilliard Navier–Stokes equations, see Refs. 68, 67 and 62.

Finally, we introduce the interface energy density function, which describes the energy due to the fluid-solid interaction. Let θ_s be the contact angle that, at equilibrium, the interface between the two fluids makes with respect to the solid walls

(see Refs. 58, 32 and 62) and define

$$\Theta_{fs}(\phi) = \frac{\cos\theta_s}{2}\sin\left(\frac{\pi\phi}{2}\right).$$

Then, up to a constant, the interfacial energy density equals $\gamma\Theta_{fs}(\phi)$.

Let us write the free energy of the system

$$\mathfrak{E} = \gamma\int_\Omega\left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi)\right) + \gamma\int_\Gamma\Theta_{fs}(\phi) + \frac{1}{2}\int_\Omega\frac{1}{\varepsilon(\phi)}|\mathbf{D}|^2$$

$$+ \frac{1}{2}\int_\Omega\rho(\phi)|\mathbf{u}|^2 + \frac{\lambda}{2}\int_\Omega q^2, \tag{2.7}$$

where ε is the electric permittivity of the medium and $\lambda > 0$ is a regularization parameter. Computing the variation of the energy $\mathfrak{E}$ with respect to ϕ, while keeping all the other arguments fixed, we obtain that

$$\langle D_\phi\mathfrak{E}, \bar\phi\rangle = \int_\Omega\mu\bar\phi + \int_\Gamma L\bar\phi,$$

where μ is the so-called chemical potential which, in this situation, is given by

$$\mu = \gamma\left(\frac{1}{\delta}\mathcal{W}'(\phi) - \delta\Delta\phi\right) - \frac{\varepsilon'(\phi)}{2\varepsilon(\phi)^2}|\mathbf{D}|^2 + \frac{1}{2}\rho'(\phi)|\mathbf{u}|^2. \tag{2.8}$$

The quantity L is given by

$$L = \gamma(\Theta'_{fs}(\phi) + \delta\partial_{\mathbf{n}}\phi), \tag{2.9}$$

and can be regarded as a "chemical potential" on the boundary.

Remark 2.4. (Chemical potential) From the definition of the chemical potential μ we see that the product $\mu\nabla\phi$ includes the usual terms that define the surface tension, i.e.

$$\gamma\left(\frac{1}{\delta}\mathcal{W}'(\phi) - \delta\Delta\phi\right)\nabla\phi.$$

Additionally, it has the term

$$-\frac{\varepsilon'(\phi)}{2\varepsilon(\phi)^2}|\mathbf{D}|^2\nabla\phi,$$

which, in some sense, can be thought of as coming from the Maxwell stress tensor.

With this notation, let us take the time derivative of the free energy:

$$\frac{d\mathfrak{E}}{dt} = \int_\Omega\mu\phi_t + \int_\Gamma L\phi_t + \int_\Omega\mathbf{E}\cdot\mathbf{D}_t + \int_\Omega\rho(\phi)\mathbf{u}\cdot\mathbf{u}_t + \lambda\int_\Omega qq_t,$$

where $\mathbf{E}$ is the electric field, defined as $\mathbf{E} := \varepsilon^{-1}\mathbf{D}$. Let us rewrite each of the terms in this expression. Using (2.4) and the impermeability conditions,

$$\int_\Omega\mu\phi_t = -\int_\Omega\mu\nabla\cdot(\phi\mathbf{u} + \mathbf{J}_\phi) = \int_\Omega\nabla\mu\cdot(\phi\mathbf{u} + \mathbf{J}_\phi).$$

Using (2.5)

$$\int_\Omega \mathbf{E} \cdot \mathbf{D}_t = -\int_\Omega \mathbf{E} \cdot (q\mathbf{u} + \mathbf{J_D}).$$

For the boundary term, we introduce the material derivative at the boundary $\dot{\phi} = \phi_t + \mathbf{u_\tau}\partial_\tau\phi$ and rewrite

$$\int_\Gamma L\phi_t = \int_\Gamma L(\dot{\phi} - \mathbf{u_\tau}\partial_\tau\phi).$$

Notice that $\sigma(\sigma\mathbf{u})_t = \rho\mathbf{u}_t + \frac{1}{2}\rho_t\mathbf{u}$, so that using (2.2), and integrating by parts, we obtain

$$\int_\Omega \rho(\phi)\mathbf{u} \cdot \mathbf{u}_t = \int_\Omega \mathbf{F} \cdot \mathbf{u} - \frac{1}{2}\int_\Omega \rho(\phi)_t|\mathbf{u}|^2 + \int_\Gamma \eta(\mathbf{S}(\mathbf{u}) \cdot \mathbf{n}) \cdot \mathbf{u_\tau} - \int_\Omega \eta|\mathbf{S}(\mathbf{u})|^2.$$

Finally, using (2.6) and the impermeability condition $(q\mathbf{u} + \mathbf{J_D}) \cdot \mathbf{n}|_\Gamma = 0$,

$$\lambda \int_\Omega qq_t = -\lambda \int_\Omega q\nabla \cdot (q\mathbf{u} + \mathbf{J_D}) = \lambda \int_\Omega \nabla q \cdot (q\mathbf{u} + \mathbf{J_D}).$$

With the help of these calculations, we find that the time-derivative of the free energy can be rewritten as

$$\dot{\mathfrak{E}} = -\int_\Omega \mu\nabla\phi \cdot \mathbf{u} + \int_\Omega \mathbf{J}_\phi \cdot \nabla\mu + \int_\Gamma L(\dot{\phi} - \mathbf{u_\tau}\partial_\tau\phi) - \int_\Omega \mathbf{E} \cdot (q\mathbf{u} + \mathbf{J_D})$$

$$+ \int_\Omega \mathbf{F} \cdot \mathbf{u} - \frac{1}{2}\int_\Omega \rho'(\phi)\phi_t|\mathbf{u}|^2 + \int_\Gamma \eta(\mathbf{S}(\mathbf{u}) \cdot \mathbf{n}) \cdot \mathbf{u_\tau} - \int_\Omega \eta|\mathbf{S}(\mathbf{u})|^2$$

$$+ \frac{\lambda}{2}\int_\Omega \mathbf{u} \cdot \nabla(q^2) + \lambda\int_\Omega \nabla q \cdot \mathbf{J_D}. \tag{2.10}$$

From (2.10), we can identify the power of the system, i.e. the time derivative of the work $\mathfrak{W}$ of internal forces, upon collecting all terms having a scalar product with the velocity $\mathbf{u}$,

$$\dot{\mathfrak{W}} = \int_\Omega \mathbf{F} \cdot \mathbf{u} - \int_\Omega \mu\nabla\phi \cdot \mathbf{u} - \int_\Omega q\mathbf{E} \cdot \mathbf{u} + \frac{\lambda}{2}\nabla(q^2) \cdot \mathbf{u} - \frac{1}{2}\int_\Omega \rho'(\phi)\phi_t\mathbf{u} \cdot \mathbf{u}.$$

We assume that the system is closed, i.e. there are no external forces. This implies that $\dot{\mathfrak{W}} \equiv 0$ and we obtain an expression for the forces $\mathbf{F}$ acting on the fluid,

$$\mathbf{F} = \mu\nabla\phi + q\mathbf{E} + \frac{1}{2}\rho'(\phi)\phi_t\mathbf{u} - \nabla\left(\frac{\lambda}{2}q^2\right).$$

Using the first law of thermodynamics

$$\frac{d\mathfrak{E}}{dt} = \frac{d\mathfrak{W}}{dt} - \mathcal{T}\frac{d\mathfrak{S}}{dt},$$

where the absolute temperature is denoted by $\mathcal{T}$ and the entropy by $\mathfrak{S}$, we can conclude that

$$\mathcal{T}\dot{\mathfrak{S}} = \int_\Omega \eta |\mathbf{S}(\mathsf{u})|^2 - \int_\Omega \mathbf{E} \cdot \mathbf{J_D} + \int_\Omega \mathbf{J}_\phi \cdot \nabla\mu + \lambda \int_\Omega \nabla q \cdot \mathbf{J_D}$$

$$+ \int_\Gamma \eta (\mathbf{S}(\mathsf{u}) \cdot \mathbf{n}) \cdot \mathsf{u}_\tau + \int_\Gamma L(\dot{\phi} - \mathsf{u}_\tau \partial_\tau \phi).$$

To find an expression for the fluxes we introduce, in the spirit of Onsager,[53,58] a dissipation function Φ. Since this must be a positive-definite function on the fluxes, the simplest possible expression for a dissipation function is quadratic and diagonal in the fluxes, e.g.

$$\Phi = \frac{1}{2} \int_\Omega \frac{1}{M} |\mathbf{J}_\phi|^2 + \frac{\alpha}{2} \int_\Gamma \dot{\phi}^2 + \frac{1}{2} \int_\Omega \frac{1}{K} |\mathbf{J_D}|^2 + \frac{1}{2} \int_\Gamma \beta |\mathsf{u}_\tau|^2,$$

where all the proportionality constants, in principle, can depend on the phase ϕ. Here, M is known as the mobility, K the conductivity and β the slip coefficient. Using Onsager's relation

$$\langle D_\mathbf{J}(\dot{\mathcal{E}}(\mathbf{J}) + \Phi(\mathbf{J})), \bar{\mathbf{J}} \rangle = 0, \quad \forall \bar{\mathbf{J}}$$

and (2.10), we find that

$$\begin{cases} \mathbf{J}_\phi = -M\nabla\mu, \\ \mathbf{J}_\phi = -M\nabla\mu, \\ \mathbf{J_D} = K(\mathbf{E} - \lambda\nabla q), \\ \beta\mathsf{u}_\tau = -\eta\mathbf{S}(\mathsf{u})_{\mathbf{n}\tau} + L\partial_\tau\phi, \\ \alpha\dot{\phi} = -L, \end{cases} \tag{2.11}$$

where $\mathbf{S}(\mathsf{u})_{\mathbf{n}\tau} := (\mathbf{S}(\mathsf{u}) \cdot \mathbf{n})_\tau$.

Remark 2.5. (Constitutive relations) Definitions (2.11) can also be obtained by simply saying that the constitutive relations of the fluxes depend linearly on the gradients, which is implicitly postulated in the form of the dissipation function Φ.

Since, in practical settings, there is an externally applied voltage (which is going to act as the control mechanism) we introduce a potential V and then the electric field is given by $\mathbf{E} = -\nabla V$ with $V = V_0$ on $\partial^\star \Omega^\star$, where V_0 is the voltage applied.

To summarize, we obtain the following system of equations for the phase variable ϕ and the chemical potential μ,

$$\begin{cases} \phi_t + \mathsf{u} \cdot \nabla\phi = \nabla \cdot (M(\phi)\nabla\mu), & \text{in } \Omega, \\ \mu = \gamma\left(\frac{1}{\delta}\mathcal{W}'(\phi) - \delta\Delta\phi\right) - \frac{1}{2}\varepsilon'(\phi)|\nabla V|^2 + \frac{1}{2}\rho'(\phi)|\mathsf{u}|^2, & \text{in } \Omega, \quad (2.12) \\ \alpha(\phi_t + \mathsf{u}_\tau\partial_\tau\phi) + \gamma(\Theta'_{fs}(\phi) + \delta\partial_\mathbf{n}\phi) = 0, \quad M(\phi)\partial_n\mu = 0, & \text{on } \Gamma, \end{cases}$$

and the velocity u and pressure p,

$$
\begin{cases}
\dfrac{D(\rho(\phi)\mathsf{u})}{Dt} - \nabla \cdot (\eta(\phi)\mathbf{S}(\mathsf{u})) + \nabla \mathsf{p} = \mu\nabla\phi - q\nabla(V + \lambda q) \\
\qquad\qquad\qquad\qquad\qquad\quad + \dfrac{1}{2}\rho'(\phi)\phi_t\mathsf{u}, & \text{on } \Omega, \\
\nabla \cdot \mathsf{u} = 0, & \text{in } \Omega, \\
\mathsf{u} \cdot \mathbf{n} = 0, & \text{on } \Gamma, \\
\beta(\phi)\mathsf{u}_{\boldsymbol\tau} + \eta(\phi)\mathbf{S}(\mathsf{u})_{\mathbf{n}\boldsymbol\tau} = \gamma(\Theta'_{fs}(\phi) + \delta\partial_{\mathbf{n}}\phi)\partial_{\boldsymbol\tau}\phi, & \text{on } \Gamma,
\end{cases}
\tag{2.13}
$$

where we have set

$$
\frac{D(\rho(\phi)\mathsf{u})}{Dt} := \sigma(\phi)(\sigma(\phi)\mathsf{u})_t + \rho(\phi)\mathsf{u} \cdot \nabla\mathsf{u} + \frac{1}{2}\nabla \cdot (\rho(\phi)\mathsf{u})\mathsf{u}.
$$

In addition, we have the equation for the electric charges q,

$$
\begin{cases}
q_t + \nabla \cdot (q\mathsf{u}) = \nabla \cdot [K(\phi)\nabla(\lambda q + V)], & \text{in } \Omega, \\
K(\phi)\nabla(\lambda q + V) \cdot \mathbf{n} = 0, & \text{on } \Gamma,
\end{cases}
\tag{2.14}
$$

and voltage V,

$$
\begin{cases}
-\nabla \cdot (\varepsilon^{\star}(\phi)\nabla V) = q\chi_{\Omega}, & \text{in } \Omega^{\star}, \\
V = V_0, & \text{on } \partial^{\star}\Omega^{\star}, \\
\partial_{\mathbf{n}}V = 0, & \text{on } \partial\Omega^{\star} \cap \Gamma,
\end{cases}
\tag{2.15}
$$

where

$$
\varepsilon^{\star}(\phi) = \varepsilon(\phi)\chi_{\Omega} + \varepsilon_D\chi_{\Omega^{\star}\setminus\Omega},
$$

with ε_D being the value of the permittivity on the dielectric plates $\Omega^{\star}\setminus\Omega$, so ε_D is constant there.

Remark 2.6. (Generalized Navier boundary condition) In (2.13), the boundary condition for the tangential velocity is known as the generalized Navier boundary condition (GNBC), and it is aimed at resolving the so-called contact line paradox of the movement of a two-phase fluid on a solid wall. The reader is referred to, for instance, Refs. 57, 58 and 32 for a discussion of its derivation. Another remedy for the paradox is discussed in Ref. 41 by considering Navier–Stokes coupled to a Cahn–Hilliard–van der Waals phase-field model. Through an analytic solution and asymptotic analysis, they show that their model allows for a moving contact line even with no-slip conditions for the velocity.

Another type of GNBC has been developed in Refs. 70 and 71 which proposes an "interface formation" model (Shikhmurzaev model). Extra equations are introduced to model variable surface tension (with an equation of state) and is coupled to a slip boundary condition (see Ref. 50 for numerical simulations of this model in a microscopic region near the contact line). An asymptotic analysis of the

Shikhmurzaev model is given in Refs. 13 and 14, but raises an issue of a "missing boundary condition."

Despite the large controversy and discussion around the validity of this boundary condition, see for instance Refs. 19 and 71, we shall take the GNBC as given and will not discuss its applicability and/or consequences here.

Remark 2.7. (Galilean invariance) As discussed in Refs. 3 and 1 (Remark 2.2), the term $\frac{1}{2}\rho'(\phi)|u|^2$ is not an objective scalar, which makes our model not frame invariant. This basically amounts to choosing a frame of reference and, consequently, should not be seen as a serious limitation of our approach. In contrast, it is not completely clear how to carry out an analysis of, for instance, the frame indifferent model presented in Ref. 1.

2.2. *Nondimensionalization*

Here we present appropriate scalings so that we may write Eqs. (2.12)–(2.15) in nondimensional form. Table 1 shows some typical values for the material parameters appearing in the model (see also Ref. 46). Consider the following scalings:

$$\tilde{\rho} = \rho/\rho_s \quad (\text{choose } \rho_s), \qquad \tilde{\eta} = \eta/\eta_s \quad (\text{choose } \eta_s), \qquad \tilde{\beta} = \beta/\beta_s,$$

$$\beta_s = \eta_s/L_s, \qquad\qquad \tilde{p} = p/p_s, \qquad\qquad p_s = \rho_s U_s^2,$$

Table 1.   Physical parameters at standard temperature (25°C) and pressure (1 bar). A Farad (F) is C^2/J. For drinking water, K is 5×10^{-4} to 5×10^{-2}.

Parameter	Value
Surface tension γ	(air/water) 0.07199 J/m^2
Dynamic viscosity η_s	(water) 8.68×10^{-4}, (air) 1.84×10^{-5} kg/m · s
Density ρ_s	(water) 996.93, (air) 1.1839 kg/m^3
Length scale (Channel height) L_s	50×10^{-6} to 100×10^{-6} m
Velocity scale U_s	0.001 to 0.05 m/s
Voltage scale V_s	10 to 50 Volts
Permittivity of vacuum ε_{vac}	8.854×10^{-12} F/m
Permittivity ε_s	(water) $78.36 \cdot \varepsilon_{\text{vac}}$, (air) $1.0 \cdot \varepsilon_{\text{vac}}$
Charge (regularization) parameter λ	0.5 J · m^3/C^2
Mobility M_s	0.01 m^5/(J · s)
Phase field parameter α	0.001 J · s/m^2
Electrical conductivity K_s	(deionized water) 5.5×10^{-6}, (air) ≈ 0.0 C^2/(J · m · s) $\equiv$ Amp/(Volt · m)

$$\tilde{u} = u/U_s \quad (\text{choose } U_s), \qquad \tilde{x} = x/L_s \quad (\text{choose } L_s), \qquad \tilde{t} = t/t_s,$$

$$t_s = L_s/U_s, \qquad\qquad \tilde{\mu} = \mu/\mu_s, \qquad\qquad \mu_s = \gamma/L_s,$$

$$\tilde{q} = q/q_s, \qquad\qquad q_s = V_s/\lambda, \qquad\qquad \tilde{V} = V/V_s, \quad (\text{choose } V_s),$$

$$\tilde{\varepsilon} = \varepsilon/\varepsilon_s, \qquad\qquad \tilde{\delta} = \delta/L_s, \qquad\qquad \widetilde{M} = M/M_s,$$

$$\widetilde{K} = K/K_s, \qquad\qquad \text{Ca} = \frac{\eta_s U_s}{\gamma}, \qquad\qquad \text{Re} = \frac{\rho_s U_s L_s}{\eta_s},$$

$$\text{We} = \frac{\rho_s U_s^2 L_s}{\gamma}, \qquad\qquad \text{Bo} = \frac{\varepsilon_s V_s^2}{L_s \gamma}, \qquad\qquad I_E = \frac{\rho_s U_s^2}{q_s V_s},$$

$$S_P = \frac{\gamma}{\alpha/t_s}, \qquad\qquad M_O = \frac{\gamma M_s}{L_s^2 U_s}, \qquad\qquad K_O = \frac{V_s K_s}{L_s q_s U_s},$$

$$C_H = \frac{q_s L_s^2}{V_s \varepsilon_s},$$

where Ca is the capillary number, Re is the Reynolds number, We is the Weber number, Bo is the electrowetting Bond number, I_E is the ratio of fluid forces to electrical forces, S_P is the ratio of surface tension to "phase field forces," M_O is a (non-dimensional) mobility coefficient, K_O is a conductivity coefficient, and C_H is an electric charge coefficient.

Let us now make the change of variables. To simplify notation, we drop the tildes, and consider all variables and differential operators as non-dimensional. The fluid equations read:

$$\begin{cases} \dfrac{D(\rho u)}{Dt} - \dfrac{1}{\text{Re}} \nabla \cdot (\eta S(u)) + \nabla p = \dfrac{1}{\text{We}} \mu \nabla \phi - \dfrac{1}{I_E} q \nabla (V + q) + \dfrac{1}{2} \rho'(\phi) \phi_t u, & \text{in } \Omega, \\[2mm] \nabla \cdot u = 0, & \text{in } \Omega, \\[2mm] u \cdot \mathbf{n} = 0, & \text{on } \Gamma, \\[2mm] \beta u_{\boldsymbol{\tau}} + \eta S(u)_{\mathbf{n}\boldsymbol{\tau}} = \dfrac{1}{\text{Ca}} (\Theta'_{fs}(\phi) + \delta \partial_{\mathbf{n}} \phi) \partial_{\boldsymbol{\tau}} \phi, & \text{on } \Gamma. \end{cases}$$

The phase-field equations change to (again dropping the tilde)

$$\begin{cases} \phi_t + u \cdot \nabla \phi = M_O \nabla \cdot (M(\phi) \nabla \mu), & \text{in } \Omega, \\[2mm] \mu = \left(\dfrac{1}{\delta} \mathcal{W}'(\phi) - \delta \Delta \phi \right) - \text{Bo} \dfrac{1}{2} \varepsilon'(\phi) |\nabla V|^2 + \text{We} \dfrac{1}{2} \rho'(\phi) |u|^2, & \text{in } \Omega, \\[2mm] \phi_t + u_{\boldsymbol{\tau}} \partial_{\boldsymbol{\tau}} \phi + S_P(\Theta'_{fs}(\phi) + \delta \partial_{\mathbf{n}} \phi) = 0, \quad \partial_n \mu = 0, & \text{on } \Gamma. \end{cases}$$

Performing the change of variables on the charge transport equation gives

$$\begin{cases} q_t + \nabla \cdot (q u) = K_O \nabla \cdot (K(\phi) \nabla (q + V)), & \text{in } \Omega, \\[2mm] \mathbf{n} \cdot \nabla (q + V) = 0, & \text{on } \Gamma. \end{cases}$$

Lastly, for the electrostatic equation we obtain

$$\begin{cases} -\nabla \cdot (\varepsilon^\star(\phi)\nabla V) = C_{\mathrm{H}} q \chi_\Omega, & \text{in } \Omega^\star, \\ V = V_0/V_{\mathrm{s}}, & \text{on } \partial^\star \Omega^\star, \\ \partial_{\mathbf{n}} V = 0, & \text{on } \partial\Omega^\star \cap \Gamma, \end{cases}$$

where $\varepsilon^\star(\phi)$ has been normalized by ε_{s}.

To alleviate the notation, for the rest of our discussion we will set all the nondimensional groups (Ca, Re, We, Bo, $I_{\mathrm{E}}, S_{\mathrm{P}}, M_{\mathrm{O}}, K_{\mathrm{O}}$ and C_{H}) to one. If needed, the dependence of the constants on all these parameters can be traced by following our arguments. Moreover, we must note that if a simplification of this model is desired, then these scalings must serve as a guide to decide which effects are dominant.

2.3. *Tangential derivatives at the boundary*

As we can see from (2.12) and (2.13), our model incorporates tangential derivatives of the phase variable ϕ at the boundary Γ. Unfortunately, in the analysis, we are not capable of dealing with these terms. Therefore, we propose some simplifications.

The first possible simplification is simply to ignore the terms that contain this tangential derivative; see Ref. 27. However, it is our feeling that the presence of them is important, specially in dealing with the contact angle in the GNBC.

A second possibility would be to add an *ad hoc* term of the form $\Delta_\Gamma \phi$ on the boundary condition for the phase variable, where by Δ_Γ we denote the Laplace–Beltrami operator on Γ. A similar approach has been followed, in a somewhat different context, for instance, by Prüss *et al.*[56] and Cherfils *et al.*[21] However, this condition might lead to lack of conservation of ϕ, which is an important feature of phase field models based on the Cahn–Hilliard equation.

Finally, the approach that we propose is to recall that, in principle, the phase field variable must be constant in the bulk of each of the phases and so $\partial_\tau \phi \approx 0$ there. Moreover, in the sharp interface limit this tangential derivative must be a Dirac measure supported on the interface. Therefore we define a function

$$\psi(\phi) = \frac{1}{L_{\mathrm{s}}} \frac{1}{\delta} e^{-\frac{\phi^2}{2\delta}}, \quad \text{where } \delta \text{ is non-dimensional}, \tag{2.16}$$

and replace all the instances of $\partial_\tau \phi$ by $\psi(\phi)$. We are aware that this is a major assumption in our model. However, as we mentioned above, it is not possible to carry out any analysis if we leave such terms without modification. It is of no surprise then that, to the best of our knowledge, there is no existence results for systems with an unmodified GNBC.

2.4. *Contact line pinning*

Simply put, the contact line pinning (hysteresis) is a frictional effect that occurs at the three-phase contact line, and is rather controversial. We refer the reader to Refs. 74 and 73 for an explanation about its origins and possible dependencies. Let

us here only mention that, macroscopically, the pinning force has a threshold value and, thus, it should depend on the stress at the contact line. It is important to take into account contact line pinning since, as observed in Refs. **74** and **73**, it is crucial for capturing the true time scales of the problem.

We propose a phenomenological approach to deal with this effect. From the GNBC,

$$\beta\mathsf{u}_{\boldsymbol{\tau}} + \eta\mathbf{S}(\mathsf{u})_{\mathbf{n}\boldsymbol{\tau}} = \gamma(\Theta'_{fs}(\phi) + \delta\partial_{\mathbf{n}}\phi)\psi(\phi),$$

we can see that, to recover no-slip conditions, one must set the slip coefficient β sufficiently large. On the contrary, when β is small, one obtains an approximation of full slip conditions. A simple dimensional argument then shows that $\beta = \eta\ell$, where ℓ has the dimensions of inverse length. Therefore, we propose the slip coefficient to have the following form

$$\beta = \eta(\phi)\ell(\phi, \mathbf{S}),\tag{2.17}$$

with

$$\ell(\phi, \mathbf{S}) = \frac{C_\ell}{L_{\mathrm{s}}}\begin{cases}\dfrac{1}{\delta}, & |\phi| > \dfrac{1}{2}, \\[2mm] \dfrac{1}{\delta}, & |\phi| \le \dfrac{1}{2}, \text{ and } |\mathbf{S}(\mathsf{u})_{\mathbf{n}\boldsymbol{\tau}}| \ll T_p, \\[2mm] 1, & |\phi| \le \dfrac{1}{2}, \text{ and } |\mathbf{S}(\mathsf{u})_{\mathbf{n}\boldsymbol{\tau}}| \approx T_p,\end{cases}$$

where C_ℓ is a (phenomenological) constant and δ is the non-dimensional transition length. For the purposes of analysis, we face the same difficulties in this expression as in Sec. 2.3. Therefore, we will use this to model pinning in the numerical examples, but leave it out of the analysis.

Let us illustrate the effect of introducing such a term by means of a numerical example. We implemented the method of Ref. **62** and a variation of it for the slip coefficient β defined as in (2.17). We consider the evolution of a bubble on an inclined plane under the action of gravity. The material parameters are $\rho_1/\rho_2 = 50, \eta_1/\eta_2 = 10, M_1 = M_2 = 10^{-2}, \alpha = 10^{-3}, \gamma = 10$ and $\theta_s = 90°$. For the case without pinning we set $\beta_1 = \beta_2 = 1.5$ and $C_\ell = 1.5$ for the pinning case. The time step is set to $\Delta t = 5 \times 10^{-4}$. The mesh is adaptively refined near the interface and it is such that the local mesh size away from the interface is about 10^{-2} and near it 8×10^{-3}. The interface thickness is $\delta = 10^{-2}$. Figure 2 shows the evolution every 0.002 time units until $t = 2.2 \times 10^{-2}$. Notice that, although there is movement, the dynamics are indeed slowed down when compared to the case of no pinning. This, evidently, needs further investigation which we defer to a future work.

Remark 2.8. (Contact line pinning penalization) We essentially model pinning by a penalty approach. Thus, it does not *exactly* capture the pinning phenomena, i.e. a droplet getting stuck in a configuration with contact angles different from the equilibrium angles determined by the classic Young's equation. Instead, it only acts
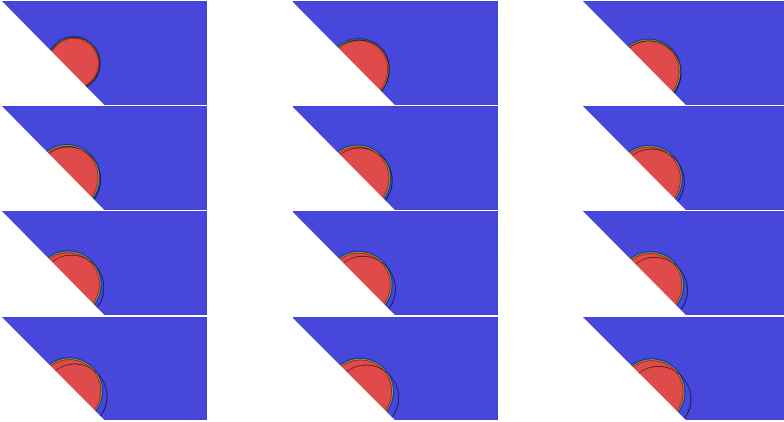
Fig. 2.   (Color online) Evolution of a droplet under the action of gravity with and without pinning. Colors represent the droplet with pinning as in (2.17) whereas the black solid line shows the position of the interface without pinning. The material parameters are $\rho_1/\rho_2 = 50, \eta_1/\eta_2 = 10, M_1 = M_2 = 10^{-2}, \alpha = 10^{-3}, \gamma = 10, \theta_s = 90°, \beta_1 = \beta_2 = C_\ell = 1.5$. The interface thickness is $\delta = 10^{-2}$. The configuration is shown for times $t = 0$ and then every 0.002 time units until $t = 2.2 \times 10^{-2}$.

as an additional retarding force to the contact line motion. However, the GNBC does allow for contact angle hysteresis when the droplet is in motion as was shown by a formal argument in Ref. 58 when using the GNBC.

3. Formal Weak Formulation and Formal Energy Estimate

In this section we obtain a weak formulation for problem (2.12)–(2.15) and show a formal energy estimate, which serves as an *a priori* estimate and the basic relation on which our existence theory is based. We mention that Ref. 15 considers a coupled Cahn–Hilliard and Navier–Stokes model with different densities (see Ref. 16 for simulations of their model and comparisons with experiments). However, they could only show local existence of a very unique regular solution; existence of weaker solutions was left open in Ref. 15.

The derivation of our formal weak formulation is on the basis of energy arguments. The main novelty in our approach is that we do not need to assume that the density is constant nor that the density contrast is small. This is possible thanks to the way we have written the convective term in (2.2a), since this allows us to use identity (2.3). See Remark 2.2 for more details.

3.1. *Formal weak formulation*

To obtain a weak formulation of the problem, we begin by multiplying the first equation of (2.12) by $\bar{\phi}$, the second by $\bar{\mu}$ and integrating in Ω. After integration by parts, taking into account the boundary conditions, we arrive at

$$\langle \phi_t, \bar{\phi} \rangle + \langle \mathsf{u} \cdot \nabla \phi, \bar{\phi} \rangle + \langle M(\phi)\nabla\mu, \nabla\bar{\phi} \rangle = 0 \tag{3.1a}$$

and

$$\langle\mu,\bar\mu\rangle = \frac{\gamma}{\delta}\langle\mathcal{W}'(\phi),\bar\mu\rangle + \gamma\delta\langle\nabla\phi,\nabla\bar\mu\rangle - \frac{1}{2}\langle\varepsilon'(\phi)|\nabla V|^2,\bar\mu\rangle + \frac{1}{2}\langle\rho'(\phi)|\mathsf{u}|^2,\bar\mu\rangle$$
$$+ \alpha[\phi_t + \mathsf{u}_{\boldsymbol\tau}\psi(\phi),\bar\mu] + \gamma[\Theta'_{fs}(\phi),\bar\mu]. \tag{3.1b}$$

Multiply the first equation of (2.13) by $\mathbf{w}$ such that $\mathbf{w}\cdot\mathbf{n}|_\Gamma = 0$, the second by $\bar p$ and integrate in Ω. Integration by parts on the first equation, in conjunction with the boundary conditions and (2.16), yields

$$-\langle\nabla\cdot(\eta(\phi)\mathbf{S}(\mathsf{u})),\mathbf{w}\rangle = \langle\eta(\phi)\mathbf{S}(\mathsf{u}),\mathbf{S}(\mathbf{w})\rangle - [\eta(\phi)\mathbf{S}(\mathsf{u})_\mathbf{n},\mathbf{w}_{\boldsymbol\tau}]$$
$$= \langle\eta(\phi)\mathbf{S}(\mathsf{u}),\mathbf{S}(\mathbf{w})\rangle + [\beta(\phi)\mathsf{u}_{\boldsymbol\tau},\mathbf{w}_{\boldsymbol\tau}]$$
$$- \gamma[\Theta'_{fs}(\phi) + \delta\partial_\mathbf{n}\phi,\mathbf{w}_{\boldsymbol\tau}\psi(\phi)]$$
$$= \langle\eta(\phi)\mathbf{S}(\mathsf{u}),\mathbf{S}(\mathbf{w})\rangle + [\beta(\phi)\mathsf{u}_{\boldsymbol\tau},\mathbf{w}_{\boldsymbol\tau}]$$
$$+ \alpha[\phi_t + \mathsf{u}_{\boldsymbol\tau}\psi(\phi),\mathbf{w}_{\boldsymbol\tau}\psi(\phi)],$$

where we used the third equation of (2.12). With these manipulations we obtain

$$\left\langle\frac{D(\rho(\phi)\mathsf{u})}{Dt},\mathbf{w}\right\rangle + \langle\eta(\phi)\mathbf{S}(\mathsf{u}),\mathbf{S}(\mathbf{w})\rangle - \langle\mathsf{p},\nabla\cdot\mathbf{w}\rangle + [\beta(\phi)\mathsf{u}_{\boldsymbol\tau},\mathbf{w}_{\boldsymbol\tau}]$$
$$+ \alpha[\mathsf{u}_{\boldsymbol\tau}\psi(\phi),\mathbf{w}_{\boldsymbol\tau}\psi(\phi)]$$
$$= \langle\mu\nabla\phi,\mathbf{w}\rangle - \langle q\nabla(\lambda q + V),\mathbf{w}\rangle + \frac{1}{2}\langle\rho'(\phi)\phi_t\mathsf{u},\mathbf{w}\rangle - \alpha[\phi_t\psi(\phi),\mathbf{w}_{\boldsymbol\tau}], \tag{3.2a}$$

for all $\mathbf{w}$, and

$$\langle\bar p,\nabla\cdot\mathsf{u}\rangle = 0, \tag{3.2b}$$

for all $\bar p$. Multiply (2.14) by r and integrate in Ω to get

$$\langle q_t,r\rangle - \langle q\mathsf{u},\nabla r\rangle + \langle K(\phi)\nabla(\lambda q + V),\nabla r\rangle = 0. \tag{3.3}$$

Let W be a function that equals zero on $\partial^\star\Omega^\star$. Multiply the equation for the electric potential (2.15) by W, integrate in $\Omega^\star$ to obtain

$$\langle\varepsilon^\star(\phi)\nabla V,\nabla W\rangle_{\Omega^\star} = \langle q,W\rangle. \tag{3.4}$$

Given the way the model has been derived, it is clear that an energy estimate must exist. Before we obtain it, let us show a comparison result *à la* Grönwall.

Lemma 3.1. (Grönwall) *Let $f,g,h,w : [0,T] \to \mathbb{R}$ be measurable and positive functions such that*

$$f(t)^2 + \int_0^t g(s)\,ds \le h(t) + \int_0^t f(s)w(s)\,ds, \quad \forall\,t\in[0,T]. \tag{3.5}$$

Then

$$\sup_{s\in[0,T]} f(s)^2 + \frac{1}{2}\int_0^T g(s)ds \leq 4\sup_{s\in[0,T]} h(s) + 4T\int_0^T w^2(s)ds, \quad \forall t\in[0,T].$$

Proof. Take, in (3.5), $t = t_0$, where

$$t_0 = \mathrm{argmax}\{f(s) : s\in[0,T]\},$$

then

$$f(t_0)^2 + \int_0^{t_0} g(s)\mathrm{d}s \leq \max_{s\in[0,T]} h(s) + f(t_0)\int_0^{t_0} w(s)\mathrm{d}s$$

$$\leq \max_{s\in[0,T]} h(s) + \frac{1}{2}f(t_0)^2 + \left(\int_0^T w(s)\mathrm{d}s\right)^2.$$

Canceling the common factors, applying the Cauchy–Schwarz inequality on the right and taking the supremum on the left-hand side we obtain the result. □

Remark 3.1. (Exponential in time estimates) The main advantage of using Lemma 3.1 to obtain *a priori* estimates, as opposed to a standard argument invoking Grönwall's inequality, is that we can avoid exponential dependence on the final time T.

The following result provides the formal energy estimate.

Theorem 3.1. (Stability) *If there is a solution to* (2.12)–(2.15), *then it must satisfy the following estimate*:

$$\sup_{s\in(0,T]} \left\{ \int_\Omega \left[\frac{1}{2}\rho(\phi)|\mathsf{u}|^2 + \frac{\lambda}{4}q^2 + \gamma\left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi) \right) \right] + \int_{\Omega^\star} \frac{1}{4}\varepsilon^\star(\phi)|\nabla V|^2 \right.$$

$$+ \gamma\int_\Gamma \Theta_{fs}(\phi) \Big\} + \int_0^T \left\{ \int_\Omega [\eta(\phi)|\mathsf{S}(\mathsf{u})|^2 + M(\phi)|\nabla\mu|^2 \right.$$

$$+ K(\phi)|\nabla(\lambda q + V)|^2] + \int_\Gamma [\beta(\phi)|\mathsf{u}_\tau|^2 + \alpha|\phi_t + \mathsf{u}_\tau\psi(\phi)|^2] \Big\}$$

$$\leq \left\{ \int_\Omega \left[\frac{1}{2}\rho(\phi)|\mathsf{u}|^2 + q^2 + \gamma\left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi) \right) + \frac{1}{2}|\bar{V}_0|^2 \right] \right.$$

$$+ \int_{\Omega^\star} \left(\varepsilon^\star(\phi)|\nabla V|^2 + \varepsilon_M|\nabla\bar{V}_0|^2 \right) + \gamma\int_\Gamma \Theta_{fs}(\phi) \Big\} \Big|_{t=0}$$

$$+ \sup_{s\in[0,T]} \left\{ \int_{\Omega^\star} \varepsilon_M|\nabla\bar{V}_0|^2 + \int_\Omega \frac{1}{\lambda}|\bar{V}_0|^2(t) \right\}$$

$$+ cT\int_0^T \left[\int_{\Omega^\star} \varepsilon_M|\nabla\bar{V}_{0,t}|^2 + \frac{4}{\lambda}\int_\Omega |\bar{V}_{0,t}|^2 \right], \tag{3.6}$$

where c does not depend on T.

Proof. We first deal with the Navier–Stokes and Cahn–Hilliard equations in a way very similar to Theorem 3.1 of Ref. 62. Set $\mathbf{w} = \mathbf{u}$ in (3.2a) and notice that

$$\left\langle \frac{D(\rho\mathbf{u})}{Dt}, \mathbf{u} \right\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|\mathbf{u}|^2,$$

because

$$\frac{D(\rho\mathbf{u})}{Dt} = \sigma(\sigma\mathbf{u})_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}\nabla\cdot(\rho\mathbf{u})\mathbf{u}.$$

We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2 + \int_\Omega \eta|\mathbf{S}(\mathbf{u})|^2 + \int_\Gamma \beta(\phi)|\mathbf{u}_\tau|^2 + \alpha\int_\Gamma |\mathbf{u}_\tau\psi(\phi)|^2$$

$$= \langle\mu\nabla\phi, \mathbf{u}\rangle - \langle q\nabla(\lambda q + V), \mathbf{u}\rangle + \frac{1}{2}\langle\rho'(\phi)\phi_t, |\mathbf{u}|^2\rangle - \alpha[\phi_t\mathbf{u}_\tau, \psi(\phi)]. \qquad (3.7)$$

Set $\bar{\phi} = \mu$ in (3.1a) to get

$$\langle\mu, \phi_t\rangle + \langle\mu\nabla\phi, \mathbf{u}\rangle + \int_\Omega M(\phi)|\nabla\mu|^2 = 0. \qquad (3.8)$$

Set $\bar{\mu} = -\phi_t$ in (3.1b) to write

$$-\langle\phi_t, \mu\rangle = -\gamma\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega\left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi)\right) + \int_\Gamma \Theta_{fs}(\phi)\right] + \frac{1}{2}\langle\varepsilon'(\phi)\phi_t, |\nabla V|^2\rangle$$

$$- \frac{1}{2}\langle\rho'(\phi)\phi_t, |\mathbf{u}|^2\rangle - \alpha\int_\Gamma (\phi_t)^2 - \alpha[\phi_t, \mathbf{u}_\tau\psi(\phi)]. \qquad (3.9)$$

Add (3.7), (3.8) and (3.9) to arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega\left(\frac{1}{2}\rho(\phi)|\mathbf{u}|^2 + \gamma\left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi)\right)\right) + \gamma\int_\Gamma \Theta_{fs}(\phi)\right] + \int_\Omega \eta(\phi)|\mathbf{S}(\mathbf{u})|^2$$

$$+ \int_\Gamma \beta(\phi)|\mathbf{u}_\tau|^2 + \int_\Omega M(\phi)|\nabla\mu|^2 + \alpha\int_\Gamma (\phi_t + \mathbf{u}_\tau\psi(\phi))^2$$

$$= -\langle q\nabla(\lambda q + V), \mathbf{u}\rangle + \frac{1}{2}\langle\varepsilon'(\phi)\phi_t, |\nabla V|^2\rangle. \qquad (3.10)$$

We next deal with the electrostatic equations. Set $r = \lambda q + V$ in (3.3) to get

$$\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega q^2 + \langle V, q_t\rangle - \langle q\nabla(\lambda q + V), \mathbf{u}\rangle + \int_\Omega K(\phi)|\nabla(\lambda q + V)|^2 = 0. \qquad (3.11)$$

Take the time derivative of (3.4) and set $W = V - \bar{V}_0$, where by $\bar{V}_0$ we mean an extension of V_0 to $\Omega^\star$. We obtain

$$\int_{\Omega^\star} \partial_t(\varepsilon^\star(\phi))|\nabla V|^2 + \frac{1}{2}\int_{\Omega^\star} \varepsilon^\star(\phi)\partial_t(|\nabla V|^2)$$

$$= \langle q_t, V\rangle - \langle q_t, \bar{V}_0\rangle + \langle\partial_t(\varepsilon^\star(\phi)\nabla V), \nabla\bar{V}_0\rangle_{\Omega^\star}. \qquad (3.12)$$

Add (3.10), (3.11) and (3.12) and recall that $\varepsilon^\star(\phi)$ is constant on $\Omega^\star\backslash\Omega$. We thus obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega\left[\frac{1}{2}\rho(\phi)|\mathsf{u}|^2 + \frac{\lambda}{2}q^2 + \gamma\left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi)\right)\right] + \int_{\Omega^\star}\frac{1}{2}\varepsilon^\star(\phi)|\nabla V|^2\right.
$$

$$
\left. + \gamma\int_\Gamma\Theta_{fs}(\phi)\right\} + \int_\Omega[\eta(\phi)|\mathsf{S}(\mathsf{u})|^2 + M(\phi)|\nabla\mu|^2 + K(\phi)|\nabla(\lambda q + V)|^2]
$$

$$
+ \int_\Gamma[\beta(\phi)|\mathsf{u}_\tau|^2 + \alpha|\phi_t + \mathsf{u}_\tau\psi(\phi)|^2] = \langle\partial_t(\varepsilon^\star(\phi)\nabla V),\nabla\bar{V}_0\rangle_{\Omega^\star} - \langle q_t,\bar{V}_0\rangle.
$$

Integrate in time over $[0,t]$, with $0 < t < T$ and integrate by parts the right-hand side. Repeated applications of the Cauchy–Schwarz inequality give us

$$
\left\{\int_\Omega\left[\frac{1}{2}\rho(\phi)|\mathsf{u}|^2 + \frac{\lambda}{4}q^2 + \gamma\left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi)\right)\right] + \int_{\Omega^\star}\frac{1}{4}\varepsilon^\star(\phi)|\nabla V|^2\right.
$$

$$
\left. + \gamma\int_\Gamma\Theta_{fs}(\phi)\right\}\bigg|_t + \int_0^t\int_\Omega[\eta(\phi)|\mathsf{S}(\mathsf{u})|^2 + M(\phi)|\nabla\mu|^2
$$

$$
+ K(\phi)|\nabla(\lambda q + V)|^2] + \int_0^t\int_\Gamma[\beta(\phi)|\mathsf{u}_\tau|^2 + \alpha|\phi_t + \mathsf{u}_\tau\psi(\phi)|^2]
$$

$$
\leq\left\{\int_\Omega\left[\frac{1}{2}\rho(\phi)|\mathsf{u}|^2 + q^2 + \gamma\left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi)\right)\right] + \int_{\Omega^\star}\varepsilon^\star(\phi)|\nabla V|^2\right.
$$

$$
\left. + \gamma\int_\Gamma\Theta_{fs}(\phi)\right\}\bigg|_{t=0} + \int_{\Omega^\star}\varepsilon_M(|\nabla\bar{V}_0|^2(t) + |\nabla\bar{V}_0|^2(0))
$$

$$
+ \int_\Omega\left[\frac{1}{\lambda}|\bar{V}_0|^2(t) + \frac{1}{2}|\bar{V}_0|^2(0)\right] + c\int_0^t\left\{\int_{\Omega^\star}\varepsilon_M|\nabla\bar{V}_{0,t}|^2 + \frac{4}{\lambda}\int_\Omega|\bar{V}_{0,t}|^2\right\}^{1/2}
$$

$$
\times\left[\frac{\lambda}{4}\int_\Omega q^2 + \int_{\Omega^\star}\frac{1}{4}\varepsilon^\star(\phi)|\nabla V|^2\right]^{1/2},
$$

where ε_M is the maximal value of the function $\varepsilon^\star(\phi)$.

Finally, if we set

$$
f(t) = \left\{\int_\Omega\left[\frac{1}{2}\rho(\phi)|\mathsf{u}|^2 + \frac{\lambda}{4}q^2 + \gamma\left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi)\right)\right]\right.
$$

$$
\left. + \int_{\Omega^\star}\frac{1}{4}\varepsilon^\star(\phi)|\nabla V|^2 + \gamma\int_\Gamma\Theta_{fs}(\phi)\right\}(t),
$$

$$
g(t) = \left\{\int_\Omega[\eta(\phi)|\mathsf{S}(\mathsf{u})|^2 + M(\phi)|\nabla\mu|^2 + K(\phi)|\nabla(\lambda q + V)|^2]\right.
$$

$$
\left. + \int_\Gamma[\beta(\phi)|\mathsf{u}_\tau|^2 + \alpha|\phi_t + \mathsf{u}_\tau\psi(\phi)|^2]\right\}(t),
$$

$$h(t) = \left\{ \int_{\Omega} \left[\frac{1}{2}\rho(\phi)|\mathbf{u}|^2 + q^2 + \gamma \left(\frac{\delta}{2}|\nabla\phi|^2 + \frac{1}{\delta}\mathcal{W}(\phi) \right) \right] \right.$$

$$+ \int_{\Omega^\star} \varepsilon^\star(\phi)|\nabla V|^2 + \gamma \int_{\Gamma} \Theta_{fs}(\phi) \Big\} \Big|_0$$

$$+ \int_{\Omega^\star} \varepsilon_M (|\nabla\bar{V}_0|^2(t) + |\nabla\bar{V}_0|^2(0)) + \int_{\Omega} \left[\frac{1}{\lambda}|\bar{V}_0|^2(t) + \frac{1}{2}|\bar{V}_0|^2(0) \right],$$

$$w(t) = \left\{ \int_{\Omega^\star} \varepsilon_M |\nabla\bar{V}_{0,t}|^2 + \frac{4}{\lambda} \int_{\Omega} |\bar{V}_{0,t}|^2 \right\}^{1/2},$$

then an application of Lemma 3.1 gives the desired estimate. $\qquad\square$

4. The Fully Discrete Problem and Its Analysis

In this section we introduce a spacetime discrete problem that is used to approximate the electrowetting problem (3.1)–(3.4). Using this discrete problem, and the result of Theorem 3.1, we will prove that a time-discrete version of our problem always has a solution. Moreover, in Sec. 5, we will base our numerical experiments on a variant of the problem defined here.

The fully discrete problem that we present below uses a backward Euler technique to handle time discretization and finite-element-like techniques for space. In this respect our approach differs from, for instance Refs. 27, 15 and 30, where there is no time discretization and the space discretization is usually handled via a special basis consisting of eigenfunctions of the underlying (linearized) operators. Our approach, although makes the analysis more complicated, is motivated by the fact that, in principle, it is possible to implement our fully discrete problem. Therefore, the results of this section provide convergence of a fully practical numerical scheme.

The particular structure of the fully discrete problem is such that, again using energy arguments, an analogue of Theorem 3.1 can be obtained. Being that we are now in finite dimension, this *a priori* estimate guarantees, via a fixed point argument, the existence of solutions.

4.1. *Definition of the fully discrete problem*

To discretize in time, as discussed in Sec. 1.1, we divide the time interval $[0, T]$ into subintervals of length $\Delta t > 0$. Recall that the time increment operator $\eth$ was introduced in (1.4) and the time average operator $\overline{(\cdot)}$ in (1.5).

To discretize in space, we introduce a parameter $h > 0$ and let $\mathbb{W}_h \subset H^1_\star(\Omega^\star), \mathbb{Q}_h \subset H^1(\Omega), \mathbb{X}_h \subset \mathbf{V}$ and $\mathbb{M}_h \subset L^2_{\bar{j}=0}(\Omega)$ be finite dimensional subspaces. We require the following compatibility condition between the spaces $\mathbb{W}_h$ and $\mathbb{Q}_h$:

$$W_h|_\Omega \in \mathbb{Q}_h, \quad \forall W_h \in \mathbb{W}_h. \tag{4.1}$$

Moreover, we require that the pair of spaces $(\mathbb{X}_h, \mathbb{M}_h)$ satisfies the so-called LBB condition (see Refs. 33, 18 and 28), that is, there exists a constant c independent of h such that

$$c\|\bar{p}_h\|_{L^2} \le \sup_{\mathbf{v}_h \in \mathbb{X}_h} \frac{\int_\Omega \bar{p}_h \nabla \cdot \mathbf{v}_h}{\|\mathbf{v}_h\|_{\mathbf{H}^1}}, \quad \forall \bar{p}_h \in \mathbb{M}_h. \tag{4.2}$$

Finally, we assume that if $\mathbb{Y}$ is any of the continuous spaces and $\mathbb{Y}_h$ the corresponding subspace, then $h_1 < h_2$ implies $\mathbb{Y}_{h_2} \subset \mathbb{Y}_{h_1}$. Moreover, the family of spaces $\{\mathbb{Y}_h\}_{h>0}$, is "dense in the limit." In other words, for every $h > 0$ there is a continuous operator $\mathcal{I}_h : \mathbb{Y} \to \mathbb{Y}_h$ such that when $h \to 0$

$$\|y - \mathcal{I}_h y\|_{\mathbb{Y}} \to 0, \quad \forall y \in \mathbb{Y}.$$

The space $\mathbb{W}_h$ will be used to approximate the voltage; $\mathbb{Q}_h$ the charge, phase field and chemical potential; and $\mathbb{X}_h, \mathbb{M}_h$ the velocity and pressure, respectively. Finally, to account for the boundary conditions on the voltage, we denote

$$\mathbb{W}_h(\bar{V}_0^{k+1}) = \mathbb{W}_h + \bar{V}_0^{k+1}.$$

Remark 4.1. (Finite elements) The introduced spaces can be easily constructed using, for instance, finite elements, see Refs. 33, 18, 28 and 25 for details. The compatibility condition (4.1) can be easily attained. For instance, one can require that the mesh is constructed in such a way that for all cells $\mathcal{K}$ in the triangulation $\mathcal{T}_h$,

$$\mathcal{K} \cap \bar{\Omega} \ne \emptyset \Leftrightarrow \mathcal{K} \cap (\Omega^\star \backslash \bar{\Omega}) = \emptyset,$$

and the polynomial degree of the space $\mathbb{Q}_h$ is no less than that of $\mathbb{W}_h$. Finally, we remark that the nestedness assumption is done merely for convenience.

The fully discrete problem searches for

$$\{V_{h\Delta t} - \bar{V}_{0,\Delta t}, q_{h\Delta t}, \phi_{h\Delta t}, \mu_{h\Delta t}, \mathbf{u}_{h\Delta t}, p_{h\Delta t}\} \subset \mathbb{W}_h \times \mathbb{Q}_h^3 \times \mathbb{X}_h \times \mathbb{M}_h,$$

that solve:

Initialization: For $n = 0$, let q_h^0, ϕ_h^0 and $\mathbf{u}_h^0$ be suitable approximations of the initial charge, phase field and velocity, respectively.

Time Marching: For $0 \le n \le N - 1$ we compute

$$(V_h^{n+1}, q_h^{n+1}, \phi_h^{n+1}, \mu_h^{n+1}, \mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbb{W}_h(\bar{V}_0^{n+1}) \times \mathbb{Q}_h^3 \times \mathbb{X}_h \times \mathbb{M}_h,$$

that solve:

$$\langle \varepsilon^\star(\phi_h^{n+1}) \nabla V_h^{n+1}, \nabla W_h \rangle_{\Omega^\star} = \langle q_h^{n+1}, W_h \rangle, \quad \forall W_h \in \mathbb{W}_h, \tag{4.3}$$

$$\left\langle \frac{\partial q_h^{n+1}}{\Delta t}, r_h \right\rangle - \langle q_h^n \mathbf{u}_h^{n+1}, \nabla r_h \rangle$$

$$+ \langle K(\phi_h^n) \nabla (\lambda q_h^{n+1} + V_h^{n+1}), \nabla r_h \rangle = 0, \quad \forall r_h \in \mathbb{Q}_h, \tag{4.4}$$

$$\left\langle \frac{\eth\phi_h^{n+1}}{\Delta t}, \bar{\phi}_h \right\rangle + \langle \mathbf{u}_h^{n+1} \cdot \nabla\phi_h^n, \bar{\phi}_h \rangle$$

$$+ \langle M(\phi_h^n)\nabla\mu_h^{n+1}, \nabla\bar{\phi}_h \rangle = 0, \quad \forall\bar{\phi}_h \in \mathbb{Q}_h \tag{4.5}$$

$$\langle \mu_h^{n+1}, \bar{\mu}_h \rangle = \frac{\gamma}{\delta}\langle \mathcal{W}'(\phi_h^n) + \mathcal{A}\eth\phi_h^{n+1}, \bar{\mu}_h \rangle + \gamma\delta\langle \nabla\phi_h^{n+1}, \nabla\bar{\mu}_h \rangle$$

$$- \frac{1}{2}\langle \mathcal{E}(\phi_h^{n+1}, \phi_h^n)|\nabla V_h^{n+1}|^2, \bar{\mu}_h \rangle + \frac{1}{2}\langle \rho'(\phi_h^n)\mathbf{u}_h^n \cdot \mathbf{u}_h^{n+1}, \bar{\mu}_h \rangle$$

$$+ \alpha\left[\frac{\eth\phi_h^{n+1}}{\Delta t} + \mathbf{u}_{h\boldsymbol{\tau}}^{n+1}\psi(\phi_h^n), \bar{\mu}_h\right] + \gamma[\Theta'_{fs}(\phi_h^n) + \mathcal{B}\eth\phi_h^{n+1}, \bar{\mu}_h]$$

$$\forall\bar{\mu}_h \in \mathbb{Q}_h, \tag{4.6}$$

where we introduced

$$\mathcal{E}(\varphi_1, \varphi_2) = \int_0^1 \varepsilon'(s\varphi_1 + (1-s)\varphi_2)\mathrm{d}s, \tag{4.7}$$

$$\left\langle \overline{\frac{\rho(\phi_h^{n+1})\mathbf{u}_h^{n+1} - \rho(\phi_h^n)\mathbf{u}_h^n}{\Delta t}}, \mathbf{w}_h \right\rangle + \langle \rho(\phi_h^n)\mathbf{u}_h^n \cdot \nabla\mathbf{u}_h^{n+1}, \mathbf{w}_h \rangle$$

$$+ \frac{1}{2}\langle \nabla \cdot (\rho(\phi_h^n)\mathbf{u}_h^n)\mathbf{u}_h^{n+1}, \mathbf{w}_h \rangle + \langle \eta(\phi_h^n)\mathbf{S}(\mathbf{u}_h^{n+1}), \mathbf{S}(\mathbf{w}_h) \rangle - \langle p_h^{n+1}, \nabla \cdot \mathbf{w}_h \rangle$$

$$+ [\beta(\phi_h^n)\mathbf{u}_{h\boldsymbol{\tau}}^{n+1}, \mathbf{w}_{h\boldsymbol{\tau}}] + \alpha[\mathbf{u}_{h\boldsymbol{\tau}}^{n+1}\psi(\phi_h^n), \mathbf{w}_{h\boldsymbol{\tau}}\psi(\phi_h^n)]$$

$$= \langle \mu_h^{n+1}\nabla\phi_h^n, \mathbf{w}_h \rangle - \langle q_h^n\nabla(\lambda q_h^{n+1} + V_h^{n+1}), \mathbf{w}_h \rangle$$

$$+ \frac{1}{2}\left\langle \rho'(\phi_h^n)\frac{\eth\phi_h^{n+1}}{\Delta t}\mathbf{u}_h^n, \mathbf{w}_h \right\rangle - \alpha\left[\frac{\eth\phi_h^{n+1}}{\Delta t}, \mathbf{w}_{h\boldsymbol{\tau}}\psi(\phi_h^n)\right] \quad \forall\mathbf{w}_h \in \mathbb{X}_h, \tag{4.8a}$$

$$\langle \bar{p}_h, \nabla \cdot \mathbf{u}_h^{n+1} \rangle = 0, \quad \forall\bar{p}_h \in \mathbb{M}_h. \tag{4.8b}$$

Remark 4.2. (Stabilization parameters) Notice that, in (4.6), we have introduced two stabilization parameters, namely $\mathcal{A}$ and $\mathcal{B}$. Their purpose is twofold. First, they will allow us to treat the nonlinear terms explicitly while still being able to maintain stability of the scheme, see Proposition 4.1 below. Second, when studying convergence of this problem, the presence of these terms will allow us to obtain further *a priori* estimates on discrete solutions which, in turn, will help in passing to the limit, see Theorem 6.1. We must mention that, this way of writing nonlinearities is related to the splitting of the energy into a convex and concave part proposed in Ref. 75. See also Refs. 68 and 67.

Remark 4.3. (Derivative of the permittivity) Notice that (4.7), i.e. the definition of the term $\mathcal{E}$, is a highly nonlinear function of its arguments (unless ε is of a very specific type). As the reader has seen in the derivation of the energy law (Theorem 3.1), the treatment of the term involving the derivative of the permittivity

is subtle. In the fully discrete setting this is additionally complicated by the fact that we need to deal with quantities at different time layers. The reason to write the derivative of the permittivity in this form is that

$$
\mathcal{E}(\varphi_1, \varphi_2) = \begin{cases} \dfrac{\varepsilon(\varphi_1) - \varepsilon(\varphi_2)}{\varphi_1 - \varphi_2}, & \varphi_1 \neq \varphi_2, \\ \varepsilon'(\varphi_1), & \varphi_1 = \varphi_2, \end{cases}
$$

which will allow us to obtain the desired cancellations.

The following subsections will be devoted to the analysis of problem (4.3)–(4.8). For convenience, we define

$$
\dot{\phi}_h^n := \frac{\partial \phi_h^n}{\Delta t} + \mathbf{u}_{h\tau}^n \psi(\phi_h^{n-1}).
$$

4.2. A priori *estimates and existence*

Let us show that, if problem (4.3)–(4.8) has a solution, it satisfies a discrete energy inequality similar to the one stated in Theorem 3.1. To do this, we first require the following formula, whose proof is straightforward.

Lemma 4.1. (Summation by parts) *Let* $\{f^n\}_{n=0}^{m-1}$ *and* $\{g^n\}_{n=0}^{m-1}$ *be sequences and assume* $f^{-1} = g^{-1} = 0$. *Then we have*

$$
\sum_{n=0}^{m-1} (\partial g^n) f^n = f^{m-1} g^{m-1} - \sum_{n=0}^{m-2} g^n (\partial f^{n+1}). \tag{4.9}
$$

Proposition 4.1. (Discrete stability) *Assume that the stabilization parameters* $\mathcal{A}$ *and* $\mathcal{B}$ *are chosen so that*

$$
\mathcal{A} \geq \frac{1}{2} \sup_{\xi \in \mathbb{R}} \mathcal{W}''(\xi), \quad \mathcal{B} \geq \frac{1}{2} \sup_{\xi \in \mathbb{R}} \Theta_{fs}''(\xi). \tag{4.10}
$$

The solution to (4.3)–(4.8), *if it exists, satisfies the following a priori estimate*

$$
\|\mathbf{u}_{h\Delta t}\|_{\ell^\infty(\mathbf{L}^2)} + \|\partial \mathbf{u}_{h\Delta t}\|_{\mathfrak{h}^{1/2}(\mathbf{L}^2)} + \|\mathbf{u}_{h\Delta t}\|_{\ell^2(\mathbf{V})} + \|q_{h\Delta t}\|_{\ell^\infty(L^2)}
$$

$$
+ \|\partial q_{h\Delta t}\|_{\mathfrak{h}^{1/2}(L^2)} + \|\nabla \phi_{h\Delta t}\|_{\ell^\infty(\mathbf{L}^2)} + \|\nabla \partial \phi_{h\Delta t}\|_{\mathfrak{h}^{1/2}(\mathbf{L}^2)}
$$

$$
+ \|\mathcal{W}(\phi_{h\Delta t})\|_{\ell^\infty(L^1)} + \|\nabla V_{h\Delta t}\|_{\ell^\infty(\mathbf{L}^2(\Omega^\star))} + \|\nabla \partial V_{h\Delta t}\|_{\mathfrak{h}^{1/2}(\mathbf{L}^2(\Omega^\star))}
$$

$$
+ \|\dot{\phi}_{h\Delta t}\|_{\ell^2(L^2(\Gamma))} + \|\Theta_{fs}(\phi_{h\Delta t})\|_{\ell^\infty(L^1(\Gamma))} + \|\nabla \mu_{h\Delta t}\|_{\ell^2(\mathbf{L}^2)}
$$

$$
+ \|\nabla (\lambda q + V)_{h\Delta t}\|_{\ell^2(\mathbf{L}^2)} \leq c, \tag{4.11}
$$

where we have set $\mu_h^0 \equiv 0, V_h^0 \equiv 0$ *for convenience in writing* (4.11). *The constant* c *depends on the constants* γ, δ, α, *the data of the problem* $\mathbf{u}_h^0, \phi_h^0, q_h^0, \bar{V}_{0,\Delta t}$ *and* T, *but it does not depend on the discretization parameters* h *or* Δt, *nor the solution of the problem.*

Proof. We repeat the steps used to prove Theorem 3.1, i.e. set $\mathbf{w}_h = 2\Delta t\mathbf{u}_h^{n+1}$ in (4.8a), $\bar{p}_h = p_h^{n+1}$ in (4.8b), $\bar{\phi}_h = 2\Delta t\mu_h^{n+1}$ in (4.5), $\bar{\mu}_h = -2\eth\phi_h^{n+1}$ in (4.6) and $r_h = 2\Delta t(\lambda q_h^{n+1} + V_h^{n+1})$ in (4.4). To treat the time-derivative terms in the discrete momentum equation, we use the identity

$$2\mathbf{u}_h^{n+1} \cdot (\overline{\rho(\phi_h^{n+1})}\mathbf{u}_h^{n+1} - \rho(\phi_h^n)\mathbf{u}_h^n)$$

$$= \rho(\phi_h^{n+1})|\mathbf{u}_h^{n+1}|^2 - \rho(\phi_h^n)|\mathbf{u}_h^n|^2 + \rho(\phi_h^n)|\eth\mathbf{u}_h^{n+1}|^2;$$

see Refs. 38 and 39. To obtain control on the explicit terms involving the derivatives of the Ginzburg–Landau potential $\mathcal{W}$ and the surface energy density Θ_{fs}, notice that, for instance,

$$\mathcal{W}(\phi_h^{n+1}) - \mathcal{W}(\phi_h^n) = \mathcal{W}'(\phi_h^n)\eth\phi_h^{n+1} + \frac{1}{2}\mathcal{W}''(\xi)(\eth\phi_h^{n+1})^2,$$

for some ξ. Choosing the stabilization constant according to (4.10) (cf. Refs. 68, 67, 69 and 62), we deduce that

$$\int_\Omega (\mathcal{W}'(\phi_h^n) + A\eth\phi_h^{n+1})\eth\phi_h^{n+1} \geq \int_\Omega \eth\mathcal{W}(\phi_h^{n+1}).$$

Adding (4.4)–(4.8) yields,

$$\eth\|\sigma(\phi_h^{n+1})\mathbf{u}_h^{n+1}\|_{\mathbf{L}^2}^2 + \|\sigma(\phi_h^n)\eth\mathbf{u}_h^{n+1}\|_{\mathbf{L}^2}^2 + \lambda(\eth\|q_h^{n+1}\|_{L^2}^2 + \|\eth q_h^{n+1}\|_{L^2}^2)$$

$$+ \gamma\delta(\eth\|\nabla\phi_h^{n+1}\|_{\mathbf{L}^2}^2 + \|\nabla\eth\phi_h^{n+1}\|_{\mathbf{L}^2}^2) + \frac{2\gamma}{\delta}\int_\Omega \eth\mathcal{W}(\phi_h^{n+1})$$

$$+ 2\gamma\int_\Gamma \eth\Theta_{fs}(\phi_h^{n+1}) + 2\Delta t\left[\left\|\sqrt{\eta(\phi_h^n)}\mathbf{S}(\mathbf{u}_h^{n+1})\right\|_{\mathbf{L}^2}^2\right.$$

$$+ \left\|\sqrt{\beta(\phi_h^n)}\mathbf{u}_{h\tau}^{n+1}\right\|_{\mathbf{L}^2(\Gamma)}^2 + \left\|\sqrt{M(\phi_h^n)}\nabla\mu_h^{n+1}\right\|_{\mathbf{L}^2}^2$$

$$+ \left\|\sqrt{K(\phi_h^n)}\nabla(\lambda q_h^{n+1} + V_h^{n+1})\right\|_{\mathbf{L}^2}^2 + \alpha\left\|\frac{\eth\phi_h^{n+1}}{\Delta t} + \mathbf{u}_{h\tau}^{n+1}\psi(\phi_h^n)\right\|_{L^2(\Gamma)}^2\right]$$

$$+ 2\langle\eth q_h^{n+1}, V_h^{n+1}\rangle \leq \langle\mathcal{E}(\phi_h^{n+1}, \phi_h^n)|\nabla V_h^{n+1}|^2, \eth\phi_h^{n+1}\rangle. \tag{4.12}$$

Take the difference of (4.3) at time-indices $n+1$ and n to obtain

$$\langle\eth(\varepsilon^\star(\phi_h^{n+1})\nabla V_h^{n+1}), \nabla W_h\rangle_{\Omega^\star} = \langle\eth q_h^{n+1}, W_h\rangle,$$

and set $W_h = 2(V_h^{n+1} - \bar{V}_0^{n+1})$. In view of (1.6) we have

$$2\eth(\varepsilon^\star(\phi_h^{n+1})\nabla V_h^{n+1}) \cdot \nabla V_h^{n+1}$$

$$= \eth(\varepsilon^\star(\phi_h^{n+1})|\nabla V_h^{n+1}|^2) + \varepsilon^\star(\phi_h^n)|\nabla\eth V_h^{n+1}|^2 + \eth(\varepsilon^\star(\phi_h^{n+1}))|\nabla V_h^{n+1}|^2,$$

whence

$$\eth\left\|\sqrt{\varepsilon^{\star}(\phi_h^{n+1})}\nabla V_h^{n+1}\right\|_{\mathbf{L}^2(\Omega^{\star})}^2 + \left\|\sqrt{\varepsilon^{\star}(\phi_h^n)}\nabla\eth V_h^{n+1}\right\|_{\mathbf{L}^2(\Omega^{\star})}^2$$

$$+ \int_{\Omega^{\star}} \eth\varepsilon^{\star}(\phi_h^{n+1})|\nabla V_h^{n+1}|^2$$

$$= 2\langle\eth q_h^{n+1}, V_h^{n+1}\rangle - 2\langle\eth q_h^{n+1}, \bar{V}_0^{n+1}\rangle$$

$$+ 2\langle\eth(\varepsilon^{\star}(\phi_h^{n+1})\nabla V_h^{n+1}), \nabla\bar{V}_0^{n+1}\rangle_{\Omega^{\star}}. \tag{4.13}$$

Add (4.12) and (4.13). Notice that, since the permittivity is assumed constant on $\Omega^{\star}\backslash\bar{\Omega}$, on the left-hand side of the resulting inequality we have the following term:

$$\int_{\Omega} (\eth\varepsilon(\phi_h^{n+1}) - \mathcal{E}(\phi_h^{n+1}, \phi_h^n)\eth\phi_h^{n+1})|\nabla V_h^{n+1}|^2 = 0,$$

where we used the definition of $\mathcal{E}$, see (4.7) and Remark 4.3. Therefore, we obtain

$$\eth\|\sigma(\phi_h^{n+1})\mathbf{u}_h^{n+1}\|_{\mathbf{L}^2}^2 + \|\sigma(\phi_h^n)\eth\mathbf{u}_h^{n+1}\|_{\mathbf{L}^2}^2 + \lambda\left(\eth\|q_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\eth q_h^{n+1}\|_{L^2}^2\right)$$

$$+ \gamma\delta(\eth\|\nabla\phi_h^{n+1}\|_{\mathbf{L}^2}^2 + \|\nabla\eth\phi_h^{n+1}\|_{\mathbf{L}^2}^2) + \frac{2\gamma}{\delta}\int_{\Omega}\eth\mathcal{W}(\phi_h^{n+1})$$

$$+ \eth\left\|\sqrt{\varepsilon^{\star}(\phi_h^{n+1})}\nabla V_h^{n+1}\right\|_{\mathbf{L}^2(\Omega^{\star})}^2 + \left\|\sqrt{\varepsilon^{\star}(\phi_h^n)}\nabla\eth V_h^{n+1}\right\|_{\mathbf{L}^2(\Omega^{\star})}^2$$

$$+ 2\gamma\int_{\Gamma}\eth\Theta_{fs}(\phi_h^{n+1}) + 2\Delta t\left[\left\|\sqrt{\eta(\phi_h^n)}\mathbf{S}(\mathbf{u}_h^{n+1})\right\|_{\mathbf{L}^2}^2\right.$$

$$+ \left\|\sqrt{\beta(\phi_h^n)}\mathbf{u}_{h\boldsymbol{\tau}}^{n+1}\right\|_{\mathbf{L}^2(\Gamma)}^2 + \left\|\sqrt{M(\phi_h^n)}\nabla\mu_h^{n+1}\right\|_{\mathbf{L}^2}^2$$

$$+ \left\|\sqrt{K(\phi_h^n)}\nabla(\lambda q_h^{n+1} + V_h^{n+1})\right\|_{\mathbf{L}^2}^2 + \alpha\left\|\frac{\eth\phi_h^{n+1}}{\Delta t} + \mathbf{u}_{h\boldsymbol{\tau}}^{n+1}\psi(\phi_h^n)\right\|_{L^2(\Gamma)}^2\right]$$

$$\leq -2\langle\eth q_h^{n+1}, \bar{V}_0^{n+1}\rangle + 2\langle\eth(\varepsilon^{\star}(\phi_h^{n+1})\nabla V_h^{n+1}), \nabla\bar{V}_0^{n+1}\rangle_{\Omega^{\star}}. \tag{4.14}$$

Summing (4.14) for $n = 0, \ldots, m-1$, using summation by parts (4.9) (set $\mu_h^0 \equiv 0, V_h^0 \equiv 0$), applying the Cauchy–Schwarz and weighted Young's inequality, we obtain the result. □

Remark 4.4. (Compatibility) Notice that condition (4.1) is needed to obtain the stability estimate, otherwise $2\Delta t(\lambda q_h^{n+1} + V_h^{n+1})$ would not be an admissible test function for (4.4).

The *a priori* estimate (4.11) allows us to conclude that, for all $h > 0$ and $\Delta t > 0$, problem (4.3)–(4.8) has a solution.

Theorem 4.1. (Existence) *Assume that the discrete spaces satisfy assumptions (4.1) and (4.2), the stabilization parameters $\mathcal{A}, \mathcal{B}$ are chosen as in Proposition 4.1. Then, for all $h > 0$ and $\Delta t > 0$, problem (4.3)–(4.8) has a solution. Moreover, any solution satisfies estimate (4.11).*

Proof. The idea of the proof is to use the "*method of a priori estimates*" at each time step. In other words, for each time step we define a map $\mathcal{L}^{n+1}$ in such a way that a fixed point of $\mathcal{L}^{n+1}$, if it exists, is a solution of our problem. Then, with the aid of the previously shown *a priori* estimates we show that $\mathcal{L}^{n+1}$ does indeed have a fixed point.

We proceed by induction in the discrete time and assume that we have shown that the problem has a solution up to n. For each $n = 0, \ldots, N - 1$, we define

$$\mathcal{L}^{n+1} : \mathbb{W}_h(\bar{V}_0^{n+1}) \times \mathbb{Q}_h^3 \times \mathbb{X}_h \times \mathbb{M}_h \quad \rightarrow \quad \mathbb{W}_h(\bar{V}_0^{n+1}) \times \mathbb{Q}_h^3 \times \mathbb{X}_h \times \mathbb{M}_h,$$

$$(V_h, q_h, \phi_h, \mu_h, \mathbf{u}_h, p_h) \stackrel{\mathcal{L}^{n+1}}{\longmapsto} (\hat{V}_h, \hat{q}_h, \hat{\phi}_h, \hat{\mu}_h, \hat{\mathbf{u}}_h, \hat{p}_h),$$

where the quantities with hats solve

$$\langle \varepsilon^\star(\phi_h) \nabla \hat{V}_h, \nabla W_h \rangle_{\Omega^\star} = \langle \hat{q}_h, W_h \rangle, \quad \forall W_h \in \mathbb{W}_h, \tag{4.15}$$

$$\left\langle \frac{\hat{q}_h - q_h^n}{\Delta t}, r_h \right\rangle - \langle q_h \mathbf{u}_h, \nabla r_h \rangle$$

$$+ \langle K(\phi_h^n) \nabla(\lambda \hat{q}_h + \hat{V}_h), \nabla r_h \rangle = 0, \quad \forall r_h \in \mathbb{Q}_h, \tag{4.16}$$

$$\left\langle \frac{\hat{\phi}_h - \phi_h^n}{\Delta t}, \bar{\phi}_h \right\rangle + \langle \mathbf{u}_h \cdot \nabla \phi_h^n, \bar{\phi}_h \rangle + \langle M(\phi_h^n) \nabla \hat{\mu}_h, \nabla \bar{\phi}_h \rangle = 0, \quad \forall \bar{\phi}_h \in \mathbb{Q}_h, \tag{4.17}$$

$$\langle \hat{\mu}_h, \bar{\mu}_h \rangle = \frac{\gamma}{\delta} \langle \mathcal{W}'(\phi_h^n) + \mathcal{A}(\phi_h - \phi_h^n), \bar{\mu}_h \rangle + \gamma \delta \langle \nabla \hat{\phi}_h, \nabla \bar{\mu}_h \rangle$$

$$+ \frac{1}{2} \langle \rho'(\phi_h^n) \mathbf{u}_h^n \cdot \mathbf{u}_h, \bar{\mu}_h \rangle - \frac{1}{2} \langle \mathcal{E}(\phi_h, \phi_h^n) \nabla V_h \cdot \nabla \hat{V}_h, \bar{\mu}_h \rangle$$

$$+ \alpha \left[\frac{\hat{\phi}_h - \phi_h^n}{\Delta t} + \mathbf{u}_{h\tau} \psi(\phi_h^n), \bar{\mu}_h \right] + \gamma [\Theta'_{fs}(\phi_h^n) + \mathcal{B}(\phi_h - \phi_h^n), \bar{\mu}_h]$$

$$\forall \bar{\mu}_h \in \mathbb{Q}_h, \tag{4.18}$$

$$\left\langle \frac{\frac{1}{2}(\rho(\phi_h) + \rho(\phi_h^n)) \hat{\mathbf{u}}_h - \rho(\phi_h^n) \mathbf{u}_h^n}{\Delta t}, \mathbf{w}_h \right\rangle + \langle \rho(\phi_h^n) \mathbf{u}_h^n \cdot \nabla \hat{\mathbf{u}}_h, \mathbf{w}_h \rangle$$

$$+ \frac{1}{2} \langle \nabla \cdot (\rho(\phi_h^n) \mathbf{u}_h^n) \hat{\mathbf{u}}_h, \mathbf{w}_h \rangle + \langle \eta(\phi_h^n) \mathbf{S}(\hat{\mathbf{u}}_h), \mathbf{S}(\mathbf{w}_h) \rangle - \langle \hat{p}_h, \nabla \cdot \mathbf{w}_h \rangle$$

$$+ [\beta(\phi_h^n) \hat{\mathbf{u}}_{h\tau}, \mathbf{w}_{h\tau}] + \alpha [\mathbf{u}_{h\tau} \psi(\phi_h^n), \mathbf{w}_{h\tau} \psi(\phi_h^n)]$$

$$= \langle \mu_h \nabla \phi_h^n, \mathbf{w}_h \rangle - \langle q_h \nabla (\lambda q_h + V_h), \mathbf{w}_h \rangle + \frac{1}{2} \left\langle \rho'(\phi_h^n) \frac{\phi_h - \phi_h^n}{\Delta t} \mathbf{u}_h^n, \mathbf{w}_h \right\rangle$$

$$- \alpha \left[\frac{\phi_h - \phi_h^n}{\Delta t}, \mathbf{w}_{h\tau} \psi(\phi_h^n) \right] \quad \forall \mathbf{w}_h \in \mathbb{X}_h, \tag{4.19}$$

$$\langle \bar{p}_h, \nabla \cdot \hat{\mathbf{u}}_h \rangle = 0, \quad \forall \bar{p}_h \in \mathbb{M}_h. \tag{4.20}$$

Notice that a fixed point of $\mathcal{L}^{n+1}$ is precisely a solution of the discrete problem (4.3)–(4.8).

To show the existence of a fixed point we must prove that:

- The operator $\mathcal{L}^{n+1}$ is well defined.
- If there is a $\mathcal{X} = (V_h, q_h, \phi_h, \mu_h, \mathbf{u}_h, p_h)$ for which $\mathcal{X} = \omega \mathcal{L}^{n+1} \mathcal{X}$, for some $\omega \in [0, 1]$, then

$$\|\mathcal{X}\| \leq M, \tag{4.21}$$

where $M > 0$ does not depend on $\mathcal{X}$ or ω.

Then, an application of the Leray–Schauder theorem[29,76] will allow us to conclude. Moreover, since a fixed point of $\mathcal{L}^{n+1}$ is precisely a solution of our problem, Proposition 4.1 gives us the desired stability estimate for this solution.

Let us then proceed to show these two points:

The operator $\mathcal{L}^{n+1}$ is well defined: Clearly, for any given ϕ_h, and q_h, the system (4.15)–(4.16) is positive definite and, thus, there are unique $\hat{V}_h$ and $\hat{q}_h$. Having computed $\hat{V}_h$ and $\hat{q}_h$ we then notice that (4.19) and (4.20) are nothing but a discrete version of a generalized Stokes problem. Assumption (4.2) then shows that there is a unique pair $(\hat{\mathbf{u}}_h, \hat{p}_h)$. To conclude, use $(\hat{V}_h, \hat{q}_h, \hat{\mathbf{u}}_h, \hat{p}_h)$ as data in (4.17) and (4.18). The fact that this linear system has a unique solution can then be seen, for instance, by noticing that the system matrix is positive definite.

Bounds on the operator: Notice, first of all, that one of the assumptions of the Leray–Schauder theorem is the compactness of the operator for which we are looking for a fixed point. However, this is trivial since the spaces we are working on are finite dimensional. Let us now show the bounds noticing that, at this stage, we do not need to obtain bounds that are independent of $h, \Delta t$ or the solution at the previous step. This will be a consequence of Proposition 4.1. Let us then assume that for some $\mathcal{X} = (V_h, q_h, \phi_h, \mu_h, \mathbf{u}_h, p_h)$ we have $\mathcal{X} = \omega \mathcal{L}^{n+1} \mathcal{X}$. Notice, first of all, that if $\omega = 0$ then $\mathcal{X} = 0$ and the bound is trivial. If $\omega \in (0, 1]$, the existence of such element can be identified with replacing, in (4.15)–(4.20), $(\hat{V}_h, \hat{q}_h, \hat{\phi}_h, \hat{\mu}_h, \hat{\mathbf{u}}_h, \hat{p}_h)$ by $\omega^{-1}(V_h, q_h, \phi_h, \mu_h, \mathbf{u}_h, p_h)$. Having done that, set $\mathbf{w}_h = 2\Delta t \mathbf{u}_h$ in (4.19), $r_h = 2\Delta t(\lambda q_h + V_h)$ in (4.16), $\bar{\phi}_h = 2\Delta t \mu_h$ in (4.17) and $\bar{\mu}_h = 2(\phi_h - \phi_h^n)$ in (4.18). Next we observe that, by induction, the equation has a solution at the previous time step, therefore there are functions that satisfy (4.3) for time n. Multiply this identity by

ω and subtract it from (4.15). Arguing as in the proof of Proposition 4.1 we see that condition (4.10) implies that to obtain the desired bound we must prove estimates for the terms

$$\langle \rho'(\phi_h^n)\mathbf{u}_h^n\mathbf{u}_h, \phi_h^n \rangle, \quad \langle \mu_h, \phi_h^n \rangle, \quad \langle q_h^n, V_h \rangle,$$

which are, in a sense, the price we are paying for not being fully implicit. All these terms are linear $\mathcal{X}$ and, thus, can be easily bounded by taking into account that we are in finite dimensions and that the estimates need not be uniform in h and Δt.

$\square$

5. Numerical Experiments

In this section we present a series of numerical examples aimed at showing the capabilities of the model we have proposed and analyzed. The implementation of all the numerical experiments has been carried out with the help of the `deal.II` library[9,8] and the details will be presented in Ref. 63. In all the numerical experiments the, classical and necessary, scaling $h = \mathcal{O}(\delta)$ holds. This is standard for phase field models. In addition, as the experiments conducted in Ref. 62 show the fluid component of our solution procedure possesses good mass conservation properties.

Let us briefly describe the discretization technique. Its starting point is problem (4.3)–(4.8) which, being a nonlinear problem, we linearize with time-lagging of the variables. Moreover, for the Cahn–Hilliard Navier–Stokes part we employ the fractional time-stepping technique developed in Ref. 62. In other words, at each time step we know

$$(V_h^n, q_h^n, \phi_h^n, \mu_h^n, \mathbf{u}_h^n, p_h^n, \xi_h^n) \in \mathbb{W}_h(\bar{V}_0^n) \times \mathbb{Q}_h^3 \times \mathbb{X}_h \times \mathbb{M}_h^2,$$

with $\xi_h^0 := 0$ and, to advance in time, solve the following sequence of discrete and linear problems:

Step 1 (Potential). Find $V_h^{n+1} \in \mathbb{W}_h(\bar{V}_0^{n+1})$ that solves:

$$\langle \varepsilon^\star(\phi_h^n)\nabla V_h^{n+1}, \nabla W_h \rangle_{\Omega^\star} = \langle q_h^n, W_h \rangle, \quad \forall W_h \in \mathbb{W}_h.$$

Step 2 (Charge). Find $q_h^{n+1} \in \mathbb{Q}_h$ that solves:

$$\left\langle \frac{\eth q_h^{n+1}}{\Delta t}, r_h \right\rangle - \langle q_h^n \mathbf{u}_h^n, \nabla r_h \rangle + \langle K(\phi_h^n)\nabla(\lambda q_h^{n+1} + V_h^{n+1}), \nabla r_h \rangle = 0, \quad \forall r_h \in \mathbb{Q}_h.$$

Step 3 (Phase Field and Chemical Potential). Find $\phi_h^{n+1}, \mu_h^{n+1} \in \mathbb{Q}_h$ that solve:

$$\left\langle \frac{\eth \phi_h^{n+1}}{\Delta t}, \bar{\phi}_h \right\rangle + \langle \mathbf{u}_h^n \cdot \nabla \phi_h^n, \bar{\phi}_h \rangle + \langle M(\phi_h^n)\nabla \mu_h^{n+1}, \nabla \bar{\phi}_h \rangle = 0, \quad \forall \bar{\phi}_h \in \mathbb{Q}_h,$$

$$\langle \mu_h^{n+1}, \bar{\mu}_h \rangle = \frac{\gamma}{\delta} \langle \mathcal{W}'(\phi_h^n) + \mathcal{A}\eth\phi_h^{n+1}, \bar{\mu}_h \rangle + \gamma\delta\langle \nabla\phi_h^{n+1}, \nabla\bar{\mu}_h \rangle$$

$$- \frac{1}{2}\langle \varepsilon'(\phi_h^n)|\nabla V_h^{n+1}|^2, \bar{\mu}_h \rangle + \frac{1}{2}\langle \rho'(\phi_h^n)|\mathbf{u}_h^n|^2, \bar{\mu}_h \rangle$$

$$+ \alpha\left[\frac{\eth\phi_h^{n+1}}{\Delta t} + \mathbf{u}_{h\boldsymbol{\tau}}^n\psi(\phi_h^n), \bar{\mu}_h\right] + \gamma[\Theta'_{fs}(\phi_h^n) + \mathcal{B}\eth\phi_h^{n+1}, \bar{\mu}_h],$$

$$\forall\,\bar{\mu}_h \in \mathbb{Q}_h.$$

Step 4 (Velocity). Define $p_h^\sharp = p_h^n + \xi_h^n$, then find $\mathbf{u}_h^{n+1} \in \mathbb{X}_h$ such that

$$\left\langle \overline{\frac{\rho(\phi_h^{n+1})\mathbf{u}_h^{n+1} - \rho(\phi^n)\mathbf{u}_h^n}{\Delta t}}, \mathbf{w}_h \right\rangle + \langle \rho(\phi_h^n)\mathbf{u}_h^n \cdot \nabla\mathbf{u}_h^{n+1}, \mathbf{w}_h \rangle$$

$$+ \frac{1}{2}\langle \nabla \cdot (\rho(\phi_h^n)\mathbf{u}_h^n)\mathbf{u}_h^{n+1}, \mathbf{w}_h \rangle + \langle \eta(\phi_h^n)\mathbf{S}(\mathbf{u}_h^{n+1}), \mathbf{S}(\mathbf{w}_h) \rangle - \langle p_h^\sharp, \nabla \cdot \mathbf{w}_h \rangle$$

$$+ [\beta(\phi_h^n)\mathbf{u}_{h\boldsymbol{\tau}}^{n+1}, \mathbf{w}_{h\boldsymbol{\tau}}] + \alpha[\mathbf{u}_{h\boldsymbol{\tau}}^{n+1}\psi(\phi_h^n), \mathbf{w}_{h\boldsymbol{\tau}}\psi(\phi_h^n)]$$

$$= \langle \mu_h^{n+1}\nabla\phi_h^n, \mathbf{w}_h \rangle - \langle q_h^n\nabla(\lambda q_h^{n+1} + V_h^{n+1}), \mathbf{w}_h \rangle$$

$$+ \frac{1}{2}\left\langle \rho'(\phi_h^n)\frac{\eth\phi_h^{n+1}}{\Delta t}\mathbf{u}_h^n, \mathbf{w}_h \right\rangle - \alpha\left[\frac{\eth\phi_h^{n+1}}{\Delta t}, \mathbf{w}_{h\boldsymbol{\tau}}\psi(\phi_h^n)\right] \quad \forall\,\mathbf{w}_h \in \mathbb{X}_h.$$

Step 5 (Penalization and Pressure). Finally, ξ_h^{n+1} and p_h^{n+1} are computed via

$$\langle \nabla\xi_h^{n+1}, \nabla\bar{p}_h \rangle = -\frac{\varrho}{\Delta t}\langle \nabla \cdot \mathbf{u}_h^{n+1}, \bar{p}_h \rangle, \quad \forall\,\bar{p}_h \in \mathbb{M}_h,$$

where $\varrho := \min\{\rho_1, \rho_2\}$ and

$$p_h^{n+1} = p_h^n + \xi_h^{n+1}.$$

Remark 5.1. (CFL) A variant of the subscheme used to solve for the Cahn–Hilliard Navier–Stokes part of our problem was proposed in Ref. 62 and shown to be unconditionally stable. In that reference, however, the equations for the phase field and velocity are coupled via terms of the form $\langle \mathbf{u}_h^{n+1} \cdot \nabla\phi_h^n, \bar{\phi}_h \rangle$. If we adopt this approach, coupling Steps 2 and 3, and assume that the permittivity does not depend on the phase, it seems possible to show that this variant of the scheme described above is stable under

$$\Delta t \leq c\delta h.$$

On the other hand, if we work with full time-lagging of the variables, then it is possible to show that the scheme is stable under the, quite restrictive, assumption that

$$\Delta t \leq c\delta^2 h^2.$$

To assess how extreme these conditions are one must remember that, in practice, it is necessary to set $h = \mathcal{O}(\delta)$. Nevertheless, computations show that these conditions

are suboptimal and just a standard CFL condition is necessary to guarantee stability of the scheme.

5.1. *Movement of a droplet*

The first example aims at showing that, indeed, electric actuation can be used to manipulate a two-fluid system. The fluid occupies the domain $\Omega = (-5, 5) \times (0, 1)$ and above and below there are dielectric plates of thickness $1/2$, so that $\Omega^\star = (-5, 5) \times (-1/2, 3/2)$. A droplet of a heavier fluid shaped like half a circle of radius $1/2$ is centered at the origin and initially at rest. To the right half of lower plate we apply a voltage, so that

$$V_0 = V_{00} \chi_D, \quad D = \left\{ (x, y) \in \mathbb{R}^2 : x \geq 0, \ y = -\frac{1}{2} \right\}.$$

The density ratio between the two fluids is $\rho_1/\rho_2 = 100$, the viscosity ratio $\eta_1/\eta_2 = 10$ and the surface tension coefficient is $\gamma = 50$. The conductivity ratio is $K_1/K_2 = 10$ and the permittivity ratio $\varepsilon_1/\varepsilon_2 = 5$ and $\varepsilon_D/\varepsilon_2 = 100$. We have set the mobility parameter to be constant $M = 10^{-2}$ and $\alpha = 10^{-3}$. The slip coefficient is taken constant $\beta = 10$, and the equilibrium contact angle between the two fluids is $\theta_s = 120°$. The interface thickness is $\delta = 5 \times 10^{-2}$ and the regularization parameter $\lambda = 0.5$. The applied voltage is $V_{00} = 20$.

The time-step is set constant and $\Delta t = 10^{-3}$. The initial mesh consists of 5364 cells with two different levels of refinement. Away from the two-fluid interface the local mesh size is about 0.125 and, near the interface, the local mesh size is about 0.03125. As required in `deal.II`, the degree of nonconformity of the mesh is restricted to 1, i.e. there is only one hanging node per face. Every 10 time-steps the mesh is coarsened and refined using as, heuristic, refinement indicator the $\mathbf{L}^2$-norm of the gradient of the phase field variable ϕ. The number of coarsened and refined cells is such that we try to keep the number of cells constant.

The discrete spaces are constructed with finite elements with equal polynomial degree in each coordinate direction and

$$\deg \mathbb{W}_h = 1, \quad \deg \mathbb{Q}_h = 2, \quad \deg \mathbb{X}_h = 2, \quad \deg \mathbb{M}_h = 1,$$

that is the lowest order quadrilateral Taylor–Hood element. No stabilization is added to the momentum conservation equation, nor the convection diffusion equation used to define the charge density.

Figure 3 shows the evolution of the interface. Notice that, other than adapting the mesh so as to resolve the interfacial layer, no other special techniques are applied to obtain these results. As expected, the applied voltage creates a local variation in the value of the surface tension between the two fluids, which in turn generates a forcing term that drives the droplet. The observed results are robust with respect to time and space discretization, in the sense that further mesh refinement or reduction of the time step lead to no significant variation of the result.
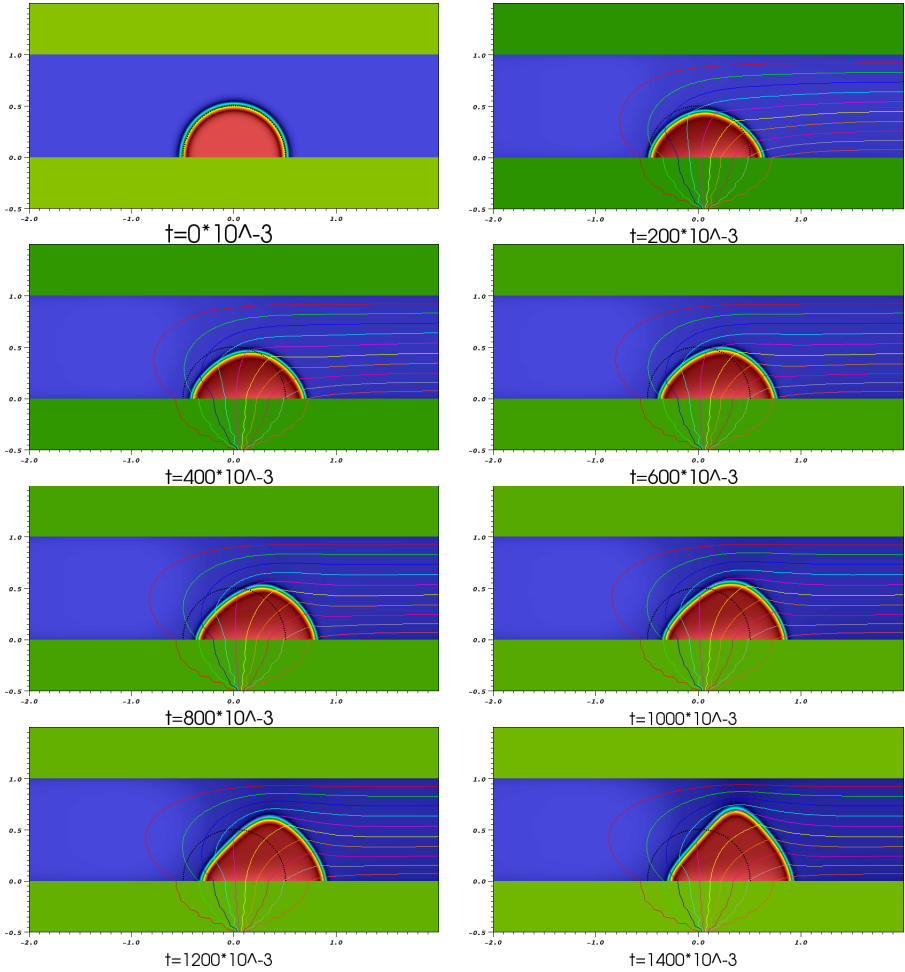
Fig. 3. (Color online) Movement of a droplet under the action of an external voltage. The material parameters are $\rho_1/\rho_2 = 100, \eta_1/\eta_2 = 10, \gamma = 50, K_1/K_2 = 10, \varepsilon_1/\varepsilon_2 = 5, \varepsilon_D/\varepsilon_2 = 100, M = 10^{-2}, \alpha = 10^{-3}, \beta = 10, \theta_s = 120°, \delta = 5 \times 10^{-2}, \lambda = 0.5$ and $V_{00} = 20$. The interface is shown at times $0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2$ and 1.4. Colored lines are used to represent the iso-values of the voltage. The black dotted line is the position of the interface at the beginning of the computations.

5.2. *Splitting of a droplet*

One of the main arguments in favor of diffuse interface models is their ability to handle topological changes automatically. The purpose of this numerical simulation is to illustrate this by showing that, using electrowetting, one can split a droplet and, thus, control fluids. Initially a drop of heavier material occupies

$$S_{\rho_2} = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{2.5^2} + \frac{y^2}{0.5^2} = 1 \right\}.$$
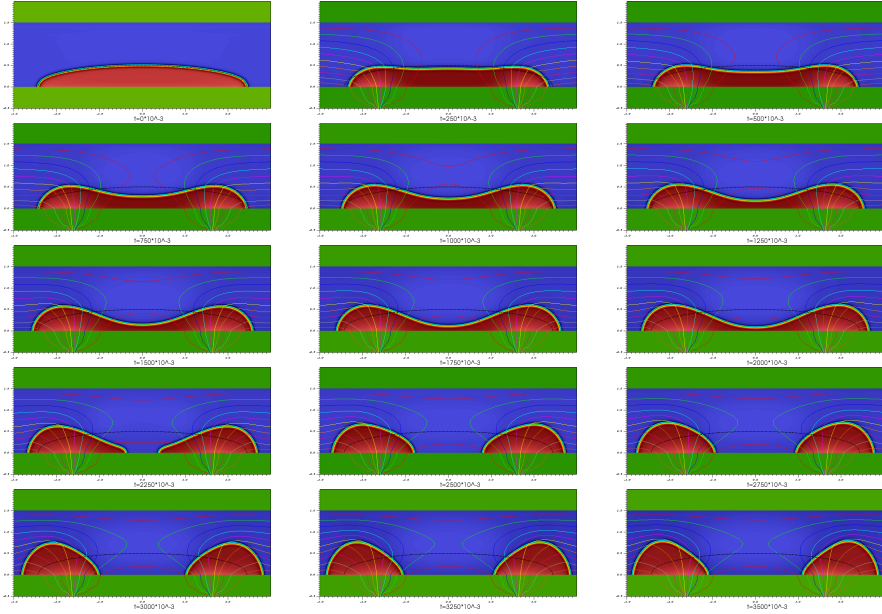
Fig. 4. (Color online) Splitting of a droplet under the action of an external voltage. The material parameters are $\rho_1/\rho_2 = 100, \eta_1/\eta_2 = 10, \gamma = 50, K_1/K_2 = 10, \varepsilon_1/\varepsilon_2 = 5, \varepsilon_D/\varepsilon_2 = 100, M = 10^{-2}, \alpha = 10^{-3}, \beta = 10, \theta_s = 120°, \delta = 5 \times 10^{-2}, \lambda = 0.5$ and $V_{00} = 20$. The interface is shown at times $0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5, 2.75, 3.0, 3.25$ and 3.5. Colored lines are used to represent the iso-values of the voltage. The black dotted line is the position of the interface at the beginning of the computations.

The material parameters are the same as in Sec. 5.1. To be able to split the droplet, the externally applied voltage is such that

$$D = \left\{ (x, y) \in \mathbb{R}^2 : |x| \geq \frac{3}{2}, \ y = -\frac{1}{2} \right\}.$$

Figure 4 shows the evolution of the system. Notice that, other than adapting the mesh so as to resolve the interfacial layer, nothing else is done and the topological change is handled without the necessity to detect it or to adapt the time-step. Again, the observed results are robust with respect to time and space discretization. In addition, we have plotted in Fig. 5 the velocity field during the evolution. It is interesting to notice that the velocity is rather small near the zone where the break-up occurs, as the evolution is being driven by the local change of surface tension caused by the applied voltage.

5.3. *Merging of two droplets*

To finalize let us show an example illustrating the merging of two droplets of the same material via electric actuation. The geometrical configuration is the same as in Sec. 5.2. In this case, however, there are initially two droplets of heavier material,
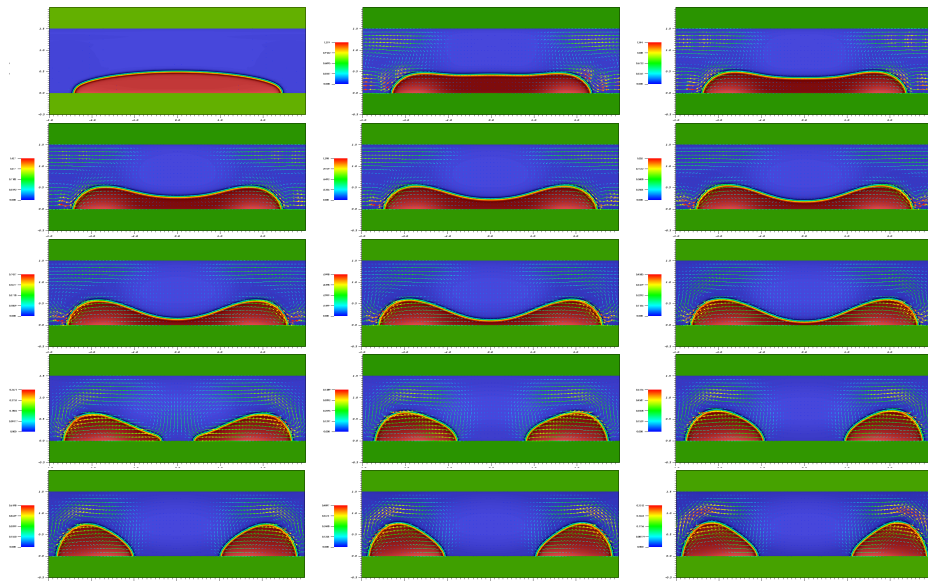
Fig. 5.   (Color online) Splitting of a droplet under the action of an external voltage. Refer to Fig. 4 for the material parameters and plotting times. Arrows represent the velocity field.

each of radius 0.5 and centered at $(-0.7, 0)$ and $(0.7, 0)$, respectively. The material parameters are the same as in Sec. 5.2, except the interfacial thickness, which is set to $\delta = 10^{-2}$. We apply an external voltage so that

$$D = \left\{ (x, y) \in \mathbb{R}^2 : |x| \leq \frac{1}{2}, \ y = -\frac{1}{2} \right\}.$$

To be able to capture the fine interface dynamics that merging possesses, we set the initial level of refinement to 4, with 2 extra refinements near the interface, so that the number of cells is 39,792 with a local mesh size away of the interface of about 0.045 and near the interface of about 6×10^{-3}. This amounts to a total of 105,237 degrees of freedom. The time-step, again, is set to $\Delta t = 10^{-3}$.

Figure 6 shows the evolution of the two droplets under the action of the voltage. Again, other than properly resolving the interfacial layer, we did not need to do anything special to handle the topological change. As in previous cases, the results are robust with respect to the discretization parameters. Figure 7 shows the velocity profile at the instant when merging occurs. Notice, again, that the motion is generated by the local change of surface tension effect created by the application of a voltage.

6. The Semi-Discrete Problem

In Sec. 4.2 we showed that the fully discrete problem always has a solution and that, moreover, this solution satisfies certain *a priori* estimates. Our purpose here is to
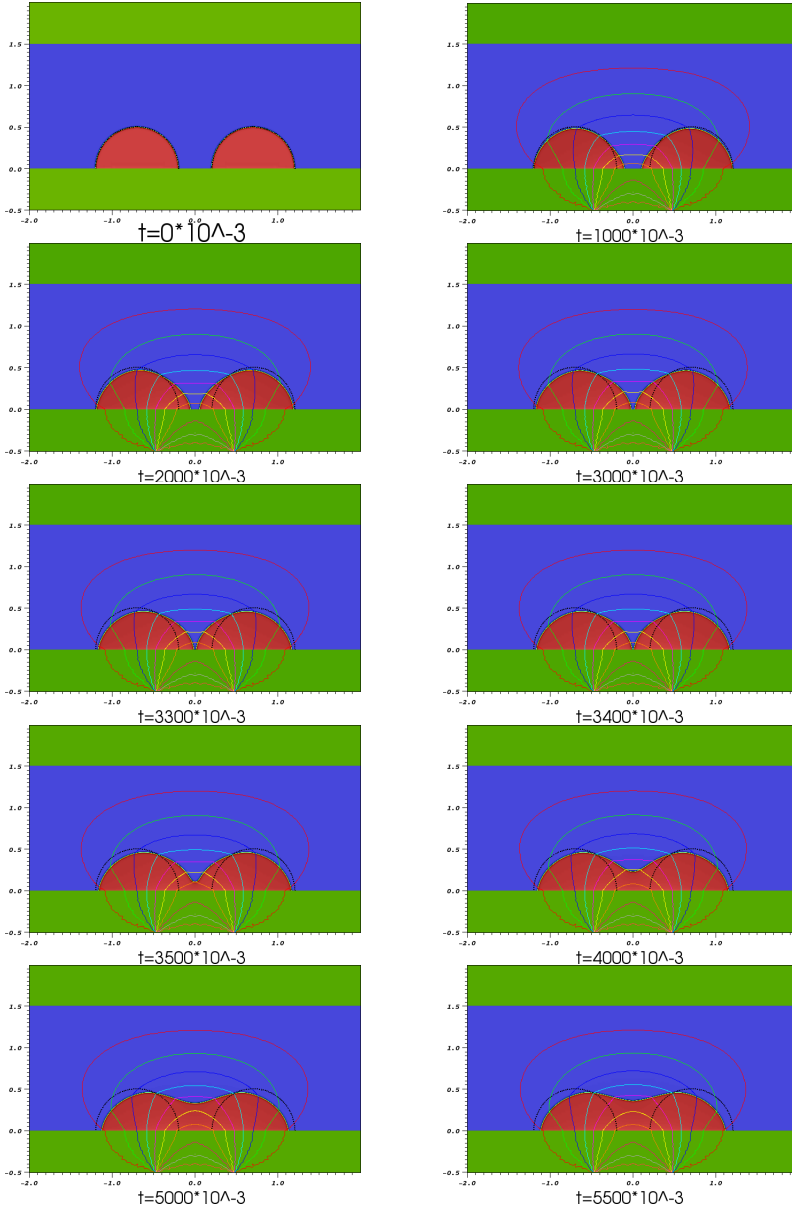
Fig. 6. (Color online) The material parameters are $\rho_1/\rho_2 = 100, \eta_1/\eta_2 = 10, \gamma = 50, K_1/K_2 = 10, \varepsilon_1/\varepsilon_2 = 5, \varepsilon_D/\varepsilon_2 = 100, M = 10^{-2}, \alpha = 10^{-3}, \beta = 10, \theta_s = 120°, \delta = 10^{-2}, \lambda = 0.5$ and $V_{00} = 20$. The interface is shown at times $0, 1, 2, 3, 3.3, 3.4, 3.5, 4, 5$ and 5.5. Colored lines are used to represent the iso-values of the voltage. The black dotted line is the position of the interface at the beginning of the computations.
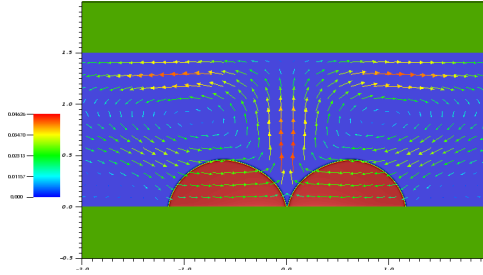
Fig. 7. (Color online) Merging of two droplets under the action of an externally applied voltage. Refer to Fig. 6 for material parameters. The interface is shown at the instant when merging occurs. Arrows represent the velocity field.

pass to the limit for $h \to 0$ so as to show that a semi-discrete (that is continuous in space and discrete in time) version of our electrowetting model always has a solution.

Let us begin by defining the semi-discrete problem. Given initial data and an external voltage, we find:

$$\{V_{\Delta t} - \bar{V}_{0,\Delta t}, q_{\Delta t}, \phi_{\Delta t}, \mu_{\Delta t}, \mathbf{u}_{\Delta t}, p_{\Delta t}\} \subset H^1_\star(\Omega^\star) \times H^1(\Omega)^3 \times \mathbf{V} \times L^2_{\int=0}(\Omega)$$

that solve:

Initialization: For $n = 0$, let q^0, ϕ^0 and $\mathbf{u}^0$ equal the initial charge, phase field and velocity, respectively.

Time Marching: For $0 \le n \le N - 1$ we compute

$$(V^{n+1}, q^{n+1}, \phi^{n+1}, \mu^{n+1}, \mathbf{u}^{n+1}, p^{n+1}) \in H^1_\star(\Omega^\star) + \bar{V}^{n+1}_0 \times H^1(\Omega)^3 \times \mathbf{V} \times L^2_{\int=0}(\Omega),$$

that solve:

$$\langle \varepsilon^\star(\phi^{n+1})\nabla V^{n+1}, \nabla W\rangle_{\Omega^\star} = \langle q^{n+1}, W\rangle, \quad \forall W \in H^1_0(\Omega^\star), \tag{6.1}$$

$$\left\langle \frac{\eth q^{n+1}}{\Delta t}, r\right\rangle - \langle q^n \mathbf{u}^{n+1}, \nabla r\rangle$$

$$+ \langle K(\phi^n)\nabla(\lambda q^{n+1} + V^{n+1}), \nabla r\rangle = 0, \quad \forall r \in H^1(\Omega) \tag{6.2}$$

$$\left\langle \frac{\eth \phi^{n+1}}{\Delta t}, \bar{\phi}\right\rangle + \langle \mathbf{u}^{n+1} \cdot \nabla \phi^n, \bar{\phi}\rangle + \langle M(\phi^n)\nabla \mu^{n+1}, \nabla\bar{\phi}\rangle = 0, \quad \forall \bar{\phi} \in H^1(\Omega) \tag{6.3}$$

$$\langle \mu^{n+1}, \bar{\mu}\rangle = \frac{\gamma}{\delta}\langle \mathcal{W}'(\phi^n) + \mathcal{A}\eth\phi^{n+1}, \bar{\mu}\rangle + \gamma\delta\langle\nabla\phi^{n+1}, \nabla\bar{\mu}\rangle + \frac{1}{2}\langle \rho'(\phi^n)\mathbf{u}^n \cdot \mathbf{u}^{n+1}, \bar{\mu}\rangle$$

$$- \frac{1}{2}\langle \mathcal{E}(\phi^{n+1}, \phi^n)|\nabla V^{n+1}|^2, \bar{\mu}\rangle + \alpha\left[\frac{\eth\phi^{n+1}}{\Delta t} + \mathbf{u}^{n+1}_\tau \psi(\phi^n), \bar{\mu}\right]$$

$$+ \gamma[\Theta'_{fs}(\phi^n) + \mathcal{B}\eth\phi^{n+1}, \bar{\mu}] \quad \forall \bar{\mu} \in H^1(\Omega) \cap L^\infty(\Omega), \tag{6.4}$$

$$\left\langle \frac{\overline{\rho(\phi^{n+1})}\mathbf{u}^{n+1} - \rho(\phi^n)\mathbf{u}^n}{\Delta t}, \mathbf{w} \right\rangle + \left\langle \rho(\phi^n)\mathbf{u}^n \cdot \nabla\mathbf{u}^{n+1} + \frac{1}{2}\nabla \cdot (\rho(\phi^n)\mathbf{u}^n)\mathbf{u}^{n+1}, \mathbf{w} \right\rangle$$

$$+ \langle \eta(\phi^n)\mathbf{S}(\mathbf{u}^{n+1}), \mathbf{S}(\mathbf{w}) \rangle - \langle p^{n+1}, \nabla \cdot \mathbf{w} \rangle + [\beta(\phi^n)\mathbf{u}_\tau^{n+1}, \mathbf{w}_\tau]$$

$$+ \alpha[\mathbf{u}_\tau^{n+1}\psi(\phi^n), \mathbf{w}_\tau\psi(\phi^n)]$$

$$= \langle \mu^{n+1}\nabla\phi^n, \mathbf{w} \rangle - \langle q^n\nabla(\lambda q^{n+1} + V^{n+1}), \mathbf{w} \rangle + \frac{1}{2}\left\langle \rho'(\phi^n)\frac{\partial\phi^{n+1}}{\Delta t}\mathbf{u}^n, \mathbf{w} \right\rangle$$

$$- \alpha\left[\frac{\partial\phi^{n+1}}{\Delta t}, \mathbf{w}_\tau\psi(\phi^n)\right], \quad \forall \mathbf{w} \in \mathbf{V}, \tag{6.5a}$$

$$\langle \bar{p}, \nabla \cdot \mathbf{u}^{n+1} \rangle = 0, \quad \forall \bar{p} \in L^2_{\int=0}(\Omega). \tag{6.5b}$$

Remark 6.1. (Permittivity) Notice that, in our definition of solution, the test function for Eq. (6.4) needs to be bounded. This is necessary to make sense of the term

$$\langle \mathcal{E}(\phi^{n+1}, \phi^n)|\nabla V^{n+1}|^2, \bar{\mu} \rangle,$$

since $\mathcal{E}$ is bounded by construction and $V^{n+1} \in H^1(\Omega)$. The authors of Ref. 27 used a similar choice of test functions and showed, using different techniques, existence of a solution for their model of electrowetting in the case when the permittivity is phase-dependent.

Since the solution to the fully discrete problem (4.3)–(4.8) exists for all values of $h > 0$ and satisfies uniform bounds, one expects the sequence of discrete solutions to converge, in some topology, and that the limit is a solution of problem (6.1)–(6.5). The following result shows that this is indeed the case.

Theorem 6.1. (Existence and stability) *For all $\Delta t > 0$, problem (6.1)–(6.5) has a solution. Moreover, this solution satisfies an energy estimate, analogous to (4.11), where the constant c might depend on Δt and the data of the problem, but not on the solution.*

Proof. Theorem 4.1 shows the existence, for every $h > 0$, of a solution to the fully discrete problem (4.3)–(4.8) which, moreover, satisfies estimate (4.11). This estimate implies that, for every n, as $h \to 0$:

- $\mathcal{W}(\phi_h^n)$ remains bounded in $L^1(\Omega)$. Since the modified Ginzburg–Landau potential is a quadratic function of its argument, this implies that there is a subsequence, again labeled by ϕ_h^n, that converges weakly in $L^2(\Omega)$.
- $\nabla\phi_h^n$ remains bounded in $\mathbf{L}^2$. This, together with the previous observation, gives us a subsequence that converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$.
- The strong L^2-convergence of ϕ_h^n implies that the convergence is almost everywhere and, since all the material functions are assumed continuous, the coefficients also converge almost everywhere.

- There is a subsequence of $\mathbf{u}_h^{n+1}$ that converges weakly in $\mathbf{V}$ and strongly in $\mathbf{L}^2(\Omega)$.
- A subsequence of $V_h^n - \bar{V}_0^n$ converges weakly in $H_\star^1(\Omega^\star)$ and hence strongly in $L^2(\Omega^\star)$.
- There is a subsequence of q_h^{n+1} that converges weakly in $L^2(\Omega)$. Moreover, we know that $K(\phi_h^n)\nabla(\lambda q_h^{n+1} + V_h^{n+1})$ converges weakly. By the a.e. convergence of the coefficients and the L^2-weak convergence of ∇V_h^{n+1} we conclude that ∇q_h^{n+1} must converge weakly and, thus, the convergence is weak in $H^1(\Omega)$ and strong in $L^2(\Omega)$.
- The quantity $\dot{\phi}_h^{n+1} = \frac{\partial \phi_h^{n+1}}{\Delta t} + \mathbf{u}_{h\boldsymbol{\tau}}^{n+1}\psi(\phi_h^n)$ remains bounded in $L^2(\Gamma)$, which implies that there is a subsequence of $\dot{\phi}_h^{n+1}$ that converges weakly in $L^2(\Gamma)$.
- $\nabla\mu_h^n$ remains bounded in $\mathbf{L}^2(\Omega)$. Moreover, setting $\bar{\mu}_h = 1$ in (4.6) and the observations given above, imply

$$|\langle \mu_h^{n+1}, 1 \rangle| \leq \left| \frac{\gamma}{\delta} \langle \mathcal{W}'(\phi_h^k) + \mathcal{A}\partial\phi_h^{k+1}, 1 \rangle + \frac{1}{2}\langle \rho'(\phi_h^n)\mathbf{u}_h^n, \mathbf{u}_h^{n+1} \rangle \right.$$

$$\left. + \alpha[\dot{\phi}_h^{k+1}, 1] + \gamma[\Theta'_{fs}(\phi_h^n) + \mathcal{B}\partial\phi_h^{n+1}, 1] \right| \leq c,$$

which shows that $\int_\Omega \mu_h^{n+1}$ remains bounded and, thus, μ_h^n remains bounded in $H^1(\Omega)$ and so there is a subsequence that converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$.
- Finally, we use the compatibility condition (4.2) and the discrete momentum Eq. (4.8a) to obtain an estimate on the pressure p_h^{n+1},

$$c\|p_h^{n+1}\|_{L^2} \leq \frac{1}{\Delta t}\|\rho(\phi_h^n)\|_{L^\infty}\|\partial\mathbf{u}_h^{n+1}\|_{\mathbf{L}^2} + \frac{1}{\Delta t}\|\partial\rho(\phi_h^{n+1})\|_{L^\infty}\|\mathbf{u}_h^{n+1}\|_{\mathbf{L}^2}$$

$$+ \|\rho(\phi_h^n)\|_{L^\infty}\|\mathbf{u}_h^n\|_{\mathbf{H}^1}\|\mathbf{u}_h^{n+1}\|_{\mathbf{H}^1} + \|\rho'(\phi_h^n)\|_{L^\infty}\|\nabla\phi_h^n\|_{\mathbf{L}^2}$$

$$\times \|\mathbf{u}_h^n\|_{\mathbf{H}^1}\|\mathbf{u}_h^{n+1}\|_{\mathbf{H}^1} + \|\eta(\phi_h^n)\|_{L^\infty}\|\mathbf{S}(\mathbf{u}_h^{n+1})\|_{\mathbf{L}^2}$$

$$+ \|\beta(\phi_h^n)\|_{L^\infty}\|\mathbf{u}_h^{n+1}\|_{\mathbf{V}} + \alpha\|\psi(\phi_h^n)\|_{L^\infty(\Gamma)}\|\dot{\phi}_h^{n+1}\|_{L^2(\Gamma)}$$

$$+ \|\mu_h^{n+1}\|_{H^1}\|\nabla\phi_h^n\|_{\mathbf{L}^2} + \|q_h^{n+1}\|_{H^1}\|\nabla(\lambda q_h^{n+1} + V_h^{n+1})\|_{\mathbf{L}^2}$$

$$+ \frac{1}{\Delta t}\|\rho'(\phi_h^n)\|_{L^\infty}\|\partial\phi_h^{n+1}\|_{L^2}\|\mathbf{u}_h^n\|_{\mathbf{V}} \leq \frac{c}{\Delta t},$$

which, for a fixed and positive Δt, implies the existence of a L^2-weakly convergent subsequence.

Let us denote the limit by

$$\{V_{\Delta t} - \bar{V}_{0,\Delta t}, q_{\Delta t}, \phi_{\Delta t}, \mu_{\Delta t}, \mathbf{u}_{\Delta t}, p_{\Delta t}\} \subset H_\star^1(\Omega^\star) \times H^1(\Omega)^3 \times \mathbf{V} \times L_{f=0}^2(\Omega).$$

It remains to show that this limit is a solution of (6.1)–(6.5):

Equation (6.1). Notice that if we show that, as $h \to 0$, the sequence V_h^{n+1} converges to V^{n+1} strongly in $H_\star^1(\Omega^\star)$, then the a.e. convergence of the coefficients

implies

$$\langle \varepsilon^\star(\phi_h^{n+1})\nabla V_h^{n+1}, \nabla W\rangle_{\Omega^\star} \to \langle \varepsilon^\star(\phi^{n+1})\nabla V^{n+1}, \nabla W\rangle_{\Omega^\star}. \tag{6.6}$$

Let us then show the strong convergence by an argument similar to that of [Ref. 27, p. 2778]. For any function $V \in H_\star^1(\Omega^\star)$, we introduce the elliptic projection $\mathcal{P}_h V \in \mathbb{W}_h(V)$ as the solution to

$$\langle \nabla \mathcal{P}_h V, \nabla W_h\rangle_{\Omega^\star} = \langle \nabla V, \nabla W_h\rangle_{\Omega^\star}, \quad \forall\, W_h \in \mathbb{W}_h(0).$$

It is well known that $\mathcal{P}_h V \to V$ strongly in $H_\star^1(\Omega^\star)$. Given that ε is uniformly bounded,

$$
\begin{aligned}
c\|\nabla(V_h^{n+1} - V^{n+1})\|_{\mathbf{L}^2}^2 &\leq \langle \varepsilon^\star(\phi_h^{n+1})\nabla(V_h^{n+1} - V^{n+1}), \nabla(V_h^{n+1} - V^{n+1})\rangle_{\Omega^\star} \\
&= \langle \varepsilon^\star(\phi_h^{n+1})\nabla V_h^{n+1}, \nabla(\mathcal{P}_h V^{n+1} - V^{n+1})\rangle_{\Omega^\star} \\
&\quad + \langle \varepsilon^\star(\phi_h^{n+1})\nabla V_h^{n+1}, \nabla(V_h^{n+1} - \mathcal{P}_h V^{n+1})\rangle_{\Omega^\star} \\
&\quad + \langle \varepsilon^\star(\phi_h^{n+1})\nabla V^{n+1}, \nabla(V^{n+1} - V_h^{n+1})\rangle_{\Omega^\star} \\
&= I + II + III.
\end{aligned}
$$

Let us estimate each of the terms separately. Since the coefficients are bounded and the sequence ∇V_h^{n+1} is uniformly bounded in $\mathbf{L}^2(\Omega)$, the strong convergence of $\mathcal{P}_h V^{n+1}$ shows that $I \to 0$. For II we use the equation, namely

$$II = \langle \varepsilon^\star(\phi_h^{n+1})\nabla V_h^{n+1}, \nabla(V_h^{n+1} - \mathcal{P}_h V^{n+1})\rangle_{\Omega^\star} = \langle q_h^{n+1}, V_h^{n+1} - \mathcal{P}_h V^{n+1}\rangle \to 0$$

since q_h^{n+1} converges strongly in $L^2(\Omega)$. Finally, notice that the last term can be rewritten as

$$
\begin{aligned}
III &= \langle (\varepsilon^\star(\phi_h^{n+1})] - \varepsilon^\star(\phi^{n+1})\nabla V^{n+1}, \nabla(V^{n+1} - V_h^{n+1})\rangle_{\Omega^\star} \\
&\quad + \langle \varepsilon^\star(\phi^{n+1})\nabla V^{n+1}, \nabla(V^{n+1} - V_h^{n+1})\rangle_{\Omega^\star}.
\end{aligned}
$$

The uniform boundedness of ∇V_h^{n+1} in $\mathbf{L}^2(\Omega)$ implies that, for the first term, it suffices to show that $(\varepsilon^\star(\phi_h^{n+1}) - \varepsilon^\star(\phi^{n+1}))\nabla V^{n+1} \to 0$ in $\mathbf{L}^2(\Omega)$, which follows from the Lebesgue dominated convergence theorem. For the second term, use the weak convergence of ∇V_h^{n+1}. This, together with the strong L^2-convergence of q_h^{n+1} implies that the limit solves (6.1).

Equation (6.2). The strong L^2-convergence of q_h^{n+1} implies that $\frac{1}{\Delta t}\eth q_h^{n+1} \to \frac{1}{\Delta t}\eth q^{n+1}$ strongly in $L^2(\Omega)$. Using the compact embeddings $H^1(\Omega) \Subset L^4(\Omega)$ and $\mathbf{V} \Subset \mathbf{L}^4(\Omega)$, we see that

$$\langle q_h^n \mathbf{u}_h^{n+1}, \nabla r\rangle \to \langle q^n \mathbf{u}^{n+1}, \nabla r\rangle, \quad \forall\, r \in H^1(\Omega),$$

as $h \to 0$. The term $K(\phi_h^n)\nabla(\lambda q_h^{n+1} + V_h^{n+1})$ can be treated as in (6.6). These observations imply that the limit solves (6.2).

Equation (6.3). The strong $\mathbf{L}^2$-convergence of $\mathbf{u}_h^{n+1}$, the weak H^1-convergence of ϕ_h^{n+1} and an argument similar to (6.6) imply that the limit solves (6.3).

Equation (6.4). The smoothness of $\mathcal{W}$ and the fact that its growth is quadratic imply

$$|\langle \mathcal{W}'(\phi_h^n) - \mathcal{W}'(\phi^n), \bar{\mu} \rangle| \leq \max_\varphi |\mathcal{W}''(\varphi)| \|\phi_h^n - \phi^n\|_{L^2} \|\bar{\mu}\|_{L^2} \to 0.$$

A similar argument and the embedding $H^1(\Omega) \Subset L^2(\Gamma)$ can be used to show convergence of $\Theta'_{fs}(\phi_h^n)$. Since ρ is a bounded smooth function,

$$\langle \rho'(\phi_h^n) \mathbf{u}_h^n \cdot \mathbf{u}_h^{n+1}, \bar{\mu} \rangle \to \langle \rho'(\phi^n) \mathbf{u}^n \cdot \mathbf{u}^{n+1}, \bar{\mu} \rangle.$$

The strong $\mathbf{L}^2$-convergence of ∇V_h^{n+1} implies that

$$\langle \mathcal{E}(\phi_h^{n+1}, \phi_h^n) |\nabla V_h^{n+1}|^2, \bar{\mu} \rangle \to \langle \mathcal{E}(\phi^{n+1}, \phi^n) |\nabla V^{n+1}|^2, \bar{\mu} \rangle,$$

where it is necessary to have $\bar{\mu} \in L^\infty(\Omega)$. To conclude that (6.4) is satisfied by the limit, it is left to show that $\dot{\phi}_h^{n+1}$ converges strongly in $L^2(\Gamma)$. We know that $\dot{\phi}_h^{n+1}$ converges weakly in $L^2(\Gamma)$. On the other hand, $\frac{1}{\Delta t} \eth \phi_h^{n+1}$ converges strongly in $L^2(\Gamma)$, $\mathbf{u}_{h\tau}^{n+1}$ converges strongly in $\mathbf{L}^2(\Gamma)$ and $\psi(\phi_h^n)$ converges a.e. in Γ.

Equations (6.5). Clearly, (6.5b) is satisfied. To show that (6.5a) holds, notice that

$$\overline{\langle \rho(\phi_h^{n+1}) \mathbf{u}_h^{n+1}} - \overline{\rho(\phi^{n+1}) \mathbf{u}^{n+1}}, \mathbf{w} \rangle$$

$$= \langle \overline{\rho(\phi_h^{n+1})} (\mathbf{u}_h^{n+1} - \mathbf{u}^{n+1}), \mathbf{w} \rangle + \langle (\overline{\rho(\phi_h^{n+1})} - \overline{\rho(\phi^{n+1})}) \mathbf{u}^{n+1}, \mathbf{w} \rangle \to 0.$$

Since we assume that ψ is smooth and the slip coefficient β is smooth and depends only on the phase field, but not on the stress (as opposed to Sec. 2.4), we can get convergence of the terms $[\beta(\phi_h^n) \mathbf{u}_{h\tau}^{n+1}, \mathbf{w}_\tau]$ and $[\mathbf{u}_{h\tau}^{n+1} \psi(\phi_h^n), \mathbf{w}_\tau \psi(\phi_h^n)]$, respectively. The advection term can be treated using standard arguments and thus we will not give details here. The terms

$$\langle \mu_h^{n+1} \nabla \phi_h^n, \mathbf{w} \rangle, \quad \langle q_h^n \nabla (\lambda q_h^{n+1} + V_h^{n+1}), \nabla \mathbf{w} \rangle,$$

can be treated using arguments similar to the ones given before. The term

$$\left\langle \rho'(\phi_h^n) \frac{\eth \phi_h^{n+1}}{\Delta t} \mathbf{u}_h^n, \mathbf{w} \right\rangle,$$

can be easily shown to converge since all terms converge strongly. The convergence of the term

$$\left[\frac{\eth \phi_h^{n+1}}{\Delta t}, \mathbf{w}_\tau \psi(\phi_h^n) \right],$$

follows again from the compact embedding $H^1(\Omega) \Subset L^2(\Gamma)$. Finally, the convergence of the viscous stress term follows the lines of the proof of (6.6).

To conclude, we notice that we do not need to reprove estimates similar to (4.11). These are uniformly valid, in h, for all terms in the sequence and, therefore, valid

for the limit. Moreover, if one wanted to obtain an energy estimate by repeating the arguments used to obtain Proposition 4.1 it would be necessary first to obtain uniform L^∞ bounds on the sequence $\eth\phi_{h,\Delta t}$, since one of the steps in the proof requires setting $\bar\mu_h = 2\eth\phi_h^{n+1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.2. (Limit $\Delta t \to 0$) We are unable to pass to the limit when $\Delta t \to 0$ for several reasons. First, the estimates on the pressure depend on the time-step and getting around this would require finer estimates on the time derivative of the velocity, this is standard for Navier–Stokes. In addition, the terms

$$\left\langle \rho'(\phi^n)\frac{\eth\phi^{n+1}}{\Delta t}\mathbf{u}^n, \mathbf{w} \right\rangle, \quad \left\langle \frac{\eth\rho(\phi^{n+1})}{\Delta t}\mathbf{u}^{n+1}, \mathbf{w} \right\rangle,$$

would require finer estimates on the time derivative of the phase field which we have not been able to show. It might be possible, however, to circumvent these two restrictions by defining the weak solution to the continuous problem with an unconstrained formulation for the momentum equation (i.e. solution and test functions in $\mathbf{V}$) and modifying the Cahn–Hilliard equations to their "viscous version", in other words suitably adding a term of the form ϕ_t. We will not pursue this direction.

7. Conclusions and Perspectives

Some possible directions for future work would be to extend the analysis by passing to the limit as $\Delta t \to 0$, or investigate the phenomenological pinning model more thoroughly. It would also be interesting to look at the use of open boundary conditions on $\partial^\star\Omega^\star$, which is more physically correct for some electrowetting devices. As far as we know, this is an open area of research in the context of phase-field methods. Other extensions of the model could include the transport of surfactant at the liquid–gas interface, though this would make the model more complicated. We want to emphasize that our model gives physically reasonable results when modeling actual electrowetting systems, and so could be used within an optimization framework for improving device design.

Concerning numerics, an important issue that has not been addressed is how to actually solve the discretized systems. Even in the fully uncoupled case, the presence of the dynamic boundary condition in the Cahn–Hilliard system (Step 2 in the scheme of Sec. 5) makes this problem extremely ill-conditioned and standard preconditioning techniques (for instance the one in Ref. 10) inapplicable.

Acknowledgments

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