# MATH 4064: Numerical Linear Algebra

## Homework 1: Matrices in Computation and Learning

Name: ______________________ Due date: September 3, 2026

Show the main steps in your work. Check dimensions before carrying out a matrix product. Unless a problem says otherwise, no software is needed.

**Problem 1 (10 points).** *A linear combination.* Let

$$\boldsymbol{v}_1 = \begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 8 \\ -1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 6 \end{bmatrix}.$$

Compute the linear combination $-\boldsymbol{v}_1 + 3\boldsymbol{v}_2 - 2\boldsymbol{v}_3$.

**Problem 2 (10 points).** *Dimensions first.* Let

$$A \in \mathbb{R}^{4\times 3}, \qquad B \in \mathbb{R}^{3\times 2}, \qquad \boldsymbol{x} \in \mathbb{R}^3, \qquad \boldsymbol{y} \in \mathbb{R}^4.$$

For each expression below, state whether it is defined. If it is defined, give the dimensions of the result. If it is not defined, explain which dimensions fail to match.

$$A\boldsymbol{x}, \qquad A^T\boldsymbol{y}, \qquad AB, \qquad BA, \qquad \boldsymbol{x}^T A^T \boldsymbol{y}.$$

**Problem 3 (10 points).** *Rows, columns, and a matrix-vector product.* Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

(a) Write $A$ as a list of its three columns and as a stack of its two rows.

(b) Compute $A\boldsymbol{x}$ using dot products with the rows of $A$.

(c) Compute $A\boldsymbol{x}$ as a linear combination of the columns of $A$.

(d) Briefly explain why the answers in parts (b) and (c) must agree.

**Problem 4 (10 points).** *Equations and matrix-vector notation.*

(a) Write the following system as one matrix-vector equation $A\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 7x_4 &= 0, \\ -2x_1 - x_2 + 4x_3 + 5x_4 &= 1, \\ 2x_1 + 3x_2 - 6x_3 + 4x_4 &= 3. \end{aligned}$$

State the dimensions of $A$, $\boldsymbol{x}$, and $\boldsymbol{b}$.

(b) Write the following matrix-vector equation as a system of two scalar equations:

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

**Problem 5 (10 points).** *Dot product and outer product.* Let

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

(a) Compute the dot product $\boldsymbol{v}^T\boldsymbol{w}$. Is the result a scalar, a row vector, or a column vector?

(b) Compute the outer product $\boldsymbol{v}\boldsymbol{w}^T$. State its dimensions.

(c) Describe the column space of $\boldsymbol{v}\boldsymbol{w}^T$ and state its rank.

**Problem 6 (10 points).** *Matrix multiplication and transpose.* Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & 2 \end{bmatrix}.$$

(a) Compute $AB$.

(b) Compute $(AB)^T$ and $B^TA^T$ separately.

(c) Verify from your calculations that $(AB)^T = B^TA^T$. Why would $A^TB^T$ not be defined in this example?

**Problem 7 (10 points).** *Column space and solvability.* Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

(a) Express the second column of $A$ in terms of the first column, and use this observation to describe $\mathrm{col}(A)$.

(b) Decide whether $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ belong to $\mathrm{col}(A)$.

(c) For each vector that belongs to $\mathrm{col}(A)$, find one $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}_i$.

**Problem 8 (10 points).** *Using low-rank factors.* Suppose

$$U \in \mathbb{R}^{1000\times 10}, \qquad V \in \mathbb{R}^{1000\times 10}, \qquad \boldsymbol{x} \in \mathbb{R}^{1000},$$

and a matrix $W$ is represented as $W = UV^T$.

(a) State the dimensions of $V^T\boldsymbol{x}$, $W$, and $W\boldsymbol{x}$.

(b) Explain how to compute $W\boldsymbol{x}$ without forming $W$.

(c) Compare the number of stored scalars for a dense $W$ with the number stored in the two factors $U$ and $V$.

(d) Using the leading-term estimates from Chapter 1, compare the cost of a dense product $W\boldsymbol{x}$ with the cost of $U(V^T\boldsymbol{x})$.

**Problem 9 (10 points).** *The heat matrix.* Divide the interval $[0, L]$ into $n = 5$ equal subintervals, so that the four interior temperatures are $u_1, u_2, u_3, u_4$. The boundary temperatures are $u_0 = u_5 = 0$, and the finite-difference equation is

$$\frac{-u_{k-1} + 2u_k - u_{k+1}}{h^2} = g(x_k), \qquad k = 1, 2, 3, 4,$$

where $h = L/5$.

(a) Write the four equations as $A\boldsymbol{u} = \boldsymbol{g}$, including the factor $1/h^2$ in $A$.

(b) How many entries of $A$ are nonzero?

(c) Name the matrix structure that makes $A$ cheaper to store than a general dense matrix.

**Problem 10 (10 points).** *One affine layer.* Let

$$W = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Define $\operatorname{ReLU}(t) = \max\{0, t\}$, applied componentwise to a vector.

(a) Compute $W\boldsymbol{x}$, then $W\boldsymbol{x} + \boldsymbol{b}$, and finally $\operatorname{ReLU}(W\boldsymbol{x} + \boldsymbol{b})$.

(b) Identify which step is linear, which expression is affine in $\boldsymbol{x}$, and which step is nonlinear.

(c) Augment $W$ and $\boldsymbol{x}$ to $\widetilde{W}$ and $\widetilde{\boldsymbol{x}}$, respectively, such that

$$W\boldsymbol{x} + \boldsymbol{b} = \widetilde{W}\widetilde{\boldsymbol{x}}.$$

Write $\widetilde{W}$ and $\widetilde{\boldsymbol{x}}$ explicitly for the given $W$, $\boldsymbol{x}$, and $\boldsymbol{b}$, and state their dimensions.

*End of Homework 1*