SOLUTION OF HW6

- Problem 6.1

(i) We have

$$\|\mathbf{v}\|_\infty = \max_{1\le i\le n} |v_i| \ge 0.$$

(ii) From $\|\mathbf{v}\|_\infty = 0$, we have

$$\max_{1\le i\le n} |v_i| = 0,$$

i.e.

$$0 \le |v_i| \le \max_{1\le i\le n} |v_i| = 0, \quad \forall i,$$

which yields

$$v_i = 0, \quad \forall i.$$

(iii) By definition, we have

$$\|\alpha\mathbf{v}\|_\infty = \max_{1\le i\le n} |\alpha v_i| = |\alpha| \max_{1\le i\le n} |v_i| = |\alpha|\|\mathbf{v}\|_\infty.$$

(iv) By definition, we have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|_\infty &= \max_{1\le i\le n} |u_i + v_i| \\ &\le \max_{1\le i\le n} (|u_i| + |v_i|) \\ &\le \max_{1\le i\le n} |u_i| + \max_{1\le i\le n} |v_i| \\ &= \|\mathbf{u}\|_\infty + \|\mathbf{v}\|_\infty.\end{aligned}$$

- Problem 6.4

We have

$$\begin{aligned}\tau_i &= \frac{Y_i - Y_{i-2}}{2h} - f(x_{i-1}, Y_{i-1}) \\ &= \frac{(Y_{i-1} + hY'_{i-1} + h^2Y''_{i-1}/2 + O(h^3)) - (Y_{i-1} - hY'_{i-1} + h^2Y''_{i-1}/2 + O(h^3))}{2h} - Y'_{i-1} \\ &= (Y'_{i-1} + O(h^2)) - Y'_{i-1} = O(h^2).\end{aligned}$$

- Problem 6.6

We have

$$\begin{aligned}\tau_i =& \frac{Y_i - Y_{i-1}}{h} - \frac{3}{2}f(x_{i-1}, Y_{i-1}) + \frac{1}{2}f(x_{i-2}, Y_{i-2}) \\ =& \frac{(Y_{i-1} + hY'_{i-1} + h^2Y''_{i-1}/2 + O(h^3)) - Y_{i-1}}{h} - \frac{3}{2}Y'_{i-1} + \frac{1}{2}Y'_{i-2} \\ =& \frac{(Y_{i-1} + hY'_{i-1} + h^2Y''_{i-1}/2 + O(h^3)) - Y_{i-1}}{h} - \frac{3}{2}Y'_{i-1} + \frac{1}{2}(Y'_{i-1} - Y''_{i-1}h + O(h^2)) \\ =& (Y'_{i-1} + \frac{1}{2}hY''_{i-1} + O(h^2)) - \frac{3}{2}Y'_{i-1} + \frac{1}{2}(Y'_{i-1} - Y''_{i-1}h + O(h^2)) \\ =& O(h^2)\end{aligned}$$

We can use a second-order one step method to obtain y_1 by equation (6.35).

- Problem 6.7

The scheme is A-B-III:

$$\frac{y_i - y_{i-1}}{h} - \left[\frac{23}{12}f(x_{i-1}, y_{i-1}) - \frac{16}{12}f(x_{i-2}, y_{i-2}) + \frac{5}{12}f(x_{i-3}, y_{i-3})\right] = 0.$$

We have

$$\begin{aligned}\tau_i =& \frac{Y_i - Y_{i-1}}{h} - \left[\frac{23}{12}Y'_{i-1} - \frac{16}{12}Y'_{i-2} + \frac{5}{12}Y'_{i-3}\right] \\ =& Y'_{i-1} + \frac{h}{2}Y''_{i-1} + \frac{h^2}{6}Y'''_{i-1} + O(h^3) \\ & - \frac{23}{12}Y'_{i-1} \\ & + \frac{16}{12}(Y'_{i-1} - Y''_{i-1}h + \frac{1}{2}Y'''_{i-1}h^2 + O(h^3)) \\ & - \frac{5}{12}(Y'_{i-1} - 2Y''_{i-1}h + 2Y'''_{i-1}h^2 + O(h^3)) \\ =& O(h^3).\end{aligned}$$

We can use a third-order one step method to obtain y_1 and y_2 by equation (6.35).

- Problem 6.8

The scheme is A-M-II:

$$\frac{y_i - y_{i-1}}{h} - \left[\frac{5}{12}f(x_i, y_i) + \frac{8}{12}f(x_{i-1}, y_{i-1}) - \frac{1}{12}f(x_{i-2}, y_{i-2})\right] = 0.$$

We have

$$\begin{aligned}
\tau_i =& \frac{Y_i - Y_{i-1}}{h} - \left[\frac{5}{12}Y_i' + \frac{8}{12}Y_{i-1}' - \frac{1}{12}Y_{i-2}'\right] \\
=& Y_{i-1}' + \frac{h}{2}Y_{i-1}'' + \frac{h^2}{6}Y_{i-1}''' + O(h^3) \\
&- \frac{5}{12}(Y_{i-1}' + Y_{i-1}''h + \frac{1}{2}Y_{i-1}'''h^2 + O(h^3)) \\
&- \frac{8}{12}Y_{i-1}' \\
&+ \frac{1}{12}(Y_{i-1}' - Y_{i-1}''h + \frac{1}{2}Y_{i-1}'''h^2 + O(h^3)) \\
=& O(h^3).
\end{aligned}$$

We can use a third-order one step method to obtain y_1 by equation (6.35).

- Problem 6.9

We have

$$\begin{aligned}
\tau_i =& \frac{1}{h}(\frac{3}{2}Y_i - 2Y_{i-1} + \frac{1}{2}Y_{i-2}) - f(x_i, Y_i) \\
=& \frac{3}{2h}(Y_{i-1} + Y_{i-1}'h + \frac{1}{2}h^2Y_{i-1}'' + O(h^3)) \\
&- \frac{2}{h}Y_{i-1} \\
&+ \frac{1}{2h}(Y_{i-1} - Y_{i-1}'h + \frac{1}{2}h^2Y_{i-1}'' + O(h^3)) \\
&- (Y_{i-1}' + Y_{i-1}''h + O(h^2)) \\
=& (O)(h^2)
\end{aligned}$$

The characteristic equation is

$$\frac{3}{2}\lambda^2 - 2\lambda + \frac{1}{2} = 0$$

and the roots are 1 and $\frac{1}{3}$. The root conditions are satisfied.