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THE LARGE SCALE BEHAVIOR OF SUPER-BROWNIAN MOTION IN THREE DIMENSIONS WITH A SINGLE POINT SOURCE *

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ABSTRACT. In a recent work, Fleischmann and Mueller (2004) showed the existence of a super-Brownian motion in $\mathbb{R}^d$, $d = 2, 3$, with extra birth at the origin. Their construction made use of an analytical approach based on the fundamental solution of the heat equation with a one point potential worked out by Albeverio et al. (1995). The present note addresses two properties of this measure-valued process in the three-dimensional case, namely the scaling of the process and the large scale behavior of its mean.

1. Introduction and result

A super-Brownian motion in $\mathbb{R}$ with a single point source δ_0 was constructed in Engländer & Fleischmann [3]. It was shown that its expected mass grows exponentially in time, and is in the mass-rescaled limit distributed in space as $x \mapsto \mathrm{e}^{-|x|}$. In Engländer & Turaev [4] it is even proved that the random measures themselves grow in law exponentially as time increases, and are otherwise in the mass-rescaled limit spatially situated with the same shape except an overall random factor. The probabilistic effect behind the non-trivial existence of the model is the fact that a Brownian particle in $\mathbb{R}$ hits the origin with certainty and that it has there a non-degenerate local time, serving as an additional birth rate for the random creation of mass.

In higher dimensions, a Brownian particle fails to hit the origin, and a local time would degenerate. Nevertheless, Fleischmann & Mueller [6] succeeded in constructing a super-Brownian motion in $\mathbb{R}^d$, $d = 2, 3$, with a single point source. They heavily used well-known analytical facts from mathematical physics concerning Laplace operators with one-point-potentials. Heuristically, some additional rescaling enters the regularization of the delta function (serving as single point source). Properties of this new super-Brownian motion are not known so far. The *purpose* of the present note is to get some progress by studying its scaling and the large scale behavior of its expectation in the three-dimensional case.

1.1. The heat equation with one-point-potential. The Schrödinger equation with a one-point-potential is studied in quantum theory to describe singular

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electromagnetic effects on quantum particles, see e.g. the monograph Albeverio et al. [2, Part I]. By analytic continuation, solutions to the Schrödinger equation can be (at least formally) obtained via solutions of the heat equation.

Formally, the heat equation with a one-point-potential is given by

$$\partial_t u = \Delta u + \delta_0^{(\alpha)} u =: \Delta^{(\alpha)} u, \tag{1.1}$$

where ∂_t denotes the derivative with respect to time, Δ is the d–dimensional Laplacian, and $u : (0,\infty) \times \dot{\mathbb{R}}^d \to \mathbb{R}_+$ is a time-space field, where $\dot{\mathbb{R}}^d := \mathbb{R}^d \setminus \{0\}$ with the Euclidean metric is locally compact. If we denote by $B_\varepsilon(y)$ an open ball around $y \in \mathbb{R}^d$ of radius $\varepsilon > 0$, then having in mind that $\varepsilon^{-d}\mathbf{1}_{B_\varepsilon(0)} \approx \delta_0$, the operator $\Delta^{(\alpha)} := \Delta + \delta_0^{(\alpha)}$ is heuristically the limit as $\varepsilon \downarrow 0$ of the operator

$$\Delta_\varepsilon^{(\alpha)} := \Delta + h(d,\alpha,\varepsilon)\, \varepsilon^{-d}\mathbf{1}_{B_\varepsilon(0)}, \tag{1.2}$$

where $h(d,\alpha,\varepsilon)$ is some additional rescaling factor which depends on a parameter α at least. Restricting to $d = 3$, the function h can be chosen as

$$h(3,\alpha,\varepsilon) := \tfrac{\pi^2}{4}\,\varepsilon - 8\pi^2\alpha\varepsilon^2, \qquad \alpha \in \mathbb{R},\ \ \varepsilon > 0, \tag{1.3}$$

(cf. [2, (H.74)]).

Physically, α in the case $\alpha < 0$ is related to the *scattering length* $\mathrm{sl}_\alpha := -(4\pi\alpha)^{-1}$ of the free Laplace operator Δ with respect to the interaction Laplacian $\Delta^{(\alpha)}$. Roughly speaking, the scattering length describes the average distance a free particle manages to go before any interaction takes place. So, if $\alpha \downarrow -\infty$ the scattering length $\mathrm{sl}_\alpha \downarrow 0$ becomes smaller and we expect more interaction. For $\alpha \geq 0$ there is *no* proper physical interpretation of sl_α as the point spectrum of $\Delta^{(\alpha)}$ is empty (see [2, Theorem I.1.4]).

The *fundamental solution* p^α to the equation

$$\partial_t u = \Delta^{(\alpha)} u \quad \text{ on } (0,\infty) \times \dot{\mathbb{R}}^d, \quad d = 2,3, \tag{1.4}$$

which provides the basis for the analytical construction of the superprocess in [6], has been computed in Albeverio et al. [1]. In $d = 3$ (the two-dimensional case is more delicate, which is the reason we restrict to $d = 3$), the *one-point-interaction heat kernel* p^α for $\alpha \in \mathbb{R}$ is given by

$$\begin{aligned} &p_t^\alpha(x,y) \\ &= p_t(x,y) + \frac{2t}{|x||y|}\, p_t\big(|x| + |y|\big) - \frac{8\pi\alpha t}{|x||y|} \int_0^\infty \mathrm{d}u\ \mathrm{e}^{-4\pi\alpha u}\, p_t\big(u + |x| + |y|\big), \end{aligned} \tag{1.5}$$

$t > 0,\ x, y \in \dot{\mathbb{R}}^3$, where p is the usual *free* heat kernel defined by,

$$p_t(x,y) := (4\pi t)^{-d/2} \exp\big(-|y-x|^2/4t\big), \tag{1.6}$$

and with a slight abuse of notation,

$$p_t(r) := (4\pi t)^{-d/2} \exp(-r^2/4t), \qquad t > 0, \quad r \geq 0. \tag{1.7}$$

Also recall the scaling of the free heat kernel, i.e. for all $k, t > 0$ and $x, y \in \mathbb{R}^d$,

$$p_t(x,y) = k^{d/2} p_{kt}(k^{1/2}x, k^{1/2}y). \tag{1.8}$$

Note, that the last term in (1.5) is always finite and disappears for $\alpha = 0$. Moreover, $\alpha \mapsto p^\alpha$ is pointwise continuous and decreasing, and we have the (pointwise) convergences $p^\alpha \uparrow +\infty$ as $\alpha \downarrow -\infty$ (i.e. the fundamental solution explodes which can be interpreted as *immediate interaction*), whereas $p^\alpha \downarrow p$ as $\alpha \uparrow +\infty$ leads the free case (i.e. the interaction disappears).

Rigorously, the family $\{\Delta^{(\alpha)} : \alpha \in \mathbb{R}\}$ of operators are defined as *all* self-adjoint extensions on the Hilbert space $\mathcal{L}^2(\dot{\mathbb{R}}^d, \mathrm{d}x)$ of the Laplacian Δ acting on $\mathcal{C}^\infty_{\mathrm{com}}(\dot{\mathbb{R}}^d)$, the space of infinitely differentiable functions on $\dot{\mathbb{R}}^d = \mathbb{R}^d \setminus \{0\}$ with compact support (see e.g. [2, Chapters I.1 and I.5]). Hence, although the p^α from (1.5) differ from the free heat kernel p, they solve the heat equation

$$\partial_t p^\alpha_t(x,y) = \Delta p^\alpha_t(x,y) \quad \text{on } (0,\infty) \times \dot{\mathbb{R}}^3, \tag{1.9}$$

with the Laplacian Δ acting either on the variable x or y. In particular, $(t,x,y) \mapsto p^\alpha_t(x,y)$ is jointly continuous on $(0,\infty) \times \dot{\mathbb{R}}^3 \times \dot{\mathbb{R}}^3$. Let us denote by S^α the semigroup associated with the kernel p^α, i.e.

$$S^\alpha_t \varphi(x) := \int_{\dot{\mathbb{R}}^3} \mathrm{d}y\, \varphi(y)\, p^\alpha_t(x,y). \tag{1.10}$$

Note that S^α is *not* a contraction semigroup and so there is *no* stochastic process generated by this flow. The following lemma shows that in the present three-dimensional case the kernel p^α has a similar scaling behavior as the free heat kernel p.

Lemma 1.1 (**Scaling of the** p^α)**.** *We have, for all* $k,t > 0$, $x,y \in \dot{\mathbb{R}}^3$, *and* $\alpha \in \mathbb{R}$,

$$p^\alpha_t(x,y) = k^{3/2}\, p^{k^{-1/2}\alpha}_{kt}(k^{1/2}x, k^{1/2}y). \tag{1.11}$$

Proof. It follows immediately from the definition (1.5) of the p^α and the scaling (1.8) of the free heat kernel p. □

1.2. The flow associated with the one-point-interaction heat kernel. This section is devoted to introducing a space of functions Φ on which the flow S^α acts as a strongly continuous linear semigroup (see [6, Section 2] for details). Let ϕ denote the *weight and reference function*

$$\phi(x) := |x|^{-1}, \qquad x \in \dot{\mathbb{R}}^3 = \mathbb{R}^3 \setminus \{0\}. \tag{1.12}$$

For fixed $\varrho \in (1,2)$, let $\mathcal{H} = \mathcal{H}^\varrho$ denote the space of measurable functions φ on $\dot{\mathbb{R}}^3$ for which

$$\|\varphi\|_{\mathcal{H}} := \Big(\int_{\dot{\mathbb{R}}^3} \mathrm{d}x\, \phi(x)\, |\varphi(x)|^\varrho \Big)^{1/\varrho} < \infty. \tag{1.13}$$

Then $\big(\mathcal{H}, \|\cdot\|_{\mathcal{H}}\big)$ is a Banach space, where as usual we do not distinguish between equivalence classes and their representatives. Now, let $\Phi = \Phi^\varrho$ denote the set of all *continuous* functions $\varphi : \dot{\mathbb{R}}^3 \to \mathbb{R}$ such that $\varphi \in \mathcal{H}$ and

$$0 \le \varphi \le C\,\phi \quad \text{for some constant } C = C_\varphi > 0. \tag{1.14}$$

We endow Φ with the topology inherited from $\mathcal{H}$. Note that the set $\mathcal{C}^+_{\mathrm{com}} = \mathcal{C}^+_{\mathrm{com}}(\dot{\mathbb{R}}^3)$ of all non-negative, continuous functions on $\dot{\mathbb{R}}^3$ with compact support is contained in Φ. We remark that $\varphi \in \Phi$ might have a singularity at $x = 0$ of order $|x|^{-\xi}$ with $0 < \xi < 1$. The linear semigroup S^α introduced in (1.10) is strongly continuous on the cone $\Phi = \Phi^\varrho$, cf. Corollary 2.12 in [6].

1.3. Super-Brownian motion with a single point source. The superprocess X we want to study was constructed in [6] via the so-called *log-Laplace approach*. Roughly speaking, the Markov process X is uniquely determined by its log-Laplace transition functionals (s. (1.15) below), and they are described by a function v, the so-called log-Laplace function. The point is, that v solves uniquely a non-linear equation, the so-called log-Laplace equation (s. (1.16) below). The main work in [6] was, to verify that the Cauchy problem for this equation is well-posed. Here uniqueness followed from a contraction argument, existence was shown via a Picard iteration, and non-negativity of v followed using an approximating linearized equation. This then allows to construct X via a Trotter product approach to the related log-Laplace semigroup, more precisely, via an approximating log-Laplace equation related to separating critical continuous-state branching and mass flow according to S^α on alternate small time intervals.

Now we come to a precise description of X. Denote by $\mathcal{M} = \mathcal{M}(\dot{\mathbb{R}}^3)$ the set of all (Radon) measures μ on $\dot{\mathbb{R}}^3$ such that $\langle \mu, \varphi\rangle := \int_{\dot{\mathbb{R}}^3} \mu(\mathrm{d}x)\,\varphi(x) < \infty$ for all $\varphi \in \Phi$. Recalling that $\mathcal{C}^+_{\mathrm{com}} \subset \Phi$, endow $\mathcal{M}$ with the vague topology.

Fix a constant $\eta > 0$ (branching rate). Suppose $0 < \beta < 1$ (branching index; the finite variance branching case $\beta = 1$ has been excluded in [6] for $d = 3$ for technical reasons). Then for each $\alpha \in \mathbb{R}$, there is a non-degenerate $\mathcal{M}$–valued (time-homogeneous) Markov process $X = X^\alpha$ such that for (deterministic) starting measures $\mu \in \mathcal{M}$ and for $\varphi \in \Phi$,

$$-\log \mathbb{E}_\mu \exp\langle X^\alpha_t, -\varphi\rangle = \big\langle \mu, v(t,\cdot)\big\rangle, \qquad t > 0, \tag{1.15}$$

where $v = \big\{v(t,x) : t \ge 0,\ x \in \dot{\mathbb{R}}^3\big\}$ is the unique non-negative solution of the integral equation related to the Φ-valued evolution equation

$$\left\{\begin{array}{l} \partial_t v = \Delta^{(\alpha)} v - \eta\, v^{1+\beta} \quad \text{ on } (0,\infty),\\ \qquad\qquad v(0+,\,\cdot\,) = \varphi \end{array}\right. \tag{1.16}$$

(see [6, Theorem 4.4]). That is,

$$v(t,x) \;=\; \int_{\dot{\mathbb{R}}^3} \mathrm{d}y\; p^\alpha_t(x,y)\,\varphi(y) - \eta \int_0^t \mathrm{d}s \int_{\dot{\mathbb{R}}^3} \mathrm{d}y\; p^\alpha_{t-s}(x,y)\, v^{1+\beta}(s,y), \tag{1.17}$$

$t > 0,\ x \in \dot{\mathbb{R}}^3$. Clearly, the first moments of X^α are determined by the S^α flow to be

$$\mathbb{E}_\mu \langle X^\alpha_t, \varphi\rangle = \langle \mu, S^\alpha_t \varphi\rangle, \tag{1.18}$$

for all starting measures $\mu \in \mathcal{M}$, $t \ge 0$, and $\varphi \in \Phi$.

1.4. Large scale behavior of the mean. Before we can state our result, we have to introduce some notation. The limiting measure will be expressed by means of the kernel

$$\vartheta^\alpha_t(x,y) := \frac{2t}{|x|\,|y|}\, p_t\big(|y|\big) - \frac{8\pi\alpha t}{|x|\,|y|} \int_0^\infty \mathrm{d}u\; \mathrm{e}^{-4\pi\alpha u}\, p_t\big(u + |y|\big), \tag{1.19}$$

for $\alpha \in \mathbb{R},\ t > 0$, and $x, y \in \dot{\mathbb{R}}^3$. Note that the integral is always finite, hence for $\alpha = 0$ the second term disappears. Moreover, the kernel ϑ^α is always non-negative.

This holds trivially whenever $\alpha < 0$, and to see this for $\alpha > 0$, use the estimate

$$p_t\big(u + |y|\big) \le p_t\big(|y|\big). \tag{1.20}$$

We extend the definition of ϑ^α by setting

$$\vartheta_t^\alpha(x,y) :\equiv \begin{cases} 0, & \text{if} \quad \alpha = +\infty, \\ +\infty, & \text{if} \quad \alpha = -\infty. \end{cases} \tag{1.21}$$

The so defined kernels ϑ^α turn out to be pointwise continuous in $\alpha \in [-\infty, +\infty]$ (which follows from the arguments of the proof of Theorem 1.2 in Section 2.2 below).

Theorem 1.2 (**Large scale behavior of the mean**). *For $t > 0$, $\alpha, \lambda_k \in \mathbb{R}$, and all starting measures $X_0^\alpha = \mu \in \mathcal{M}$ satisfying $\langle \mu, \phi \rangle < \infty$, we have the convergence in $\mathcal{M}$,*

$$\lim_{k\uparrow\infty} k^{-1/2}\, \mathbb{E}_\mu \big[X_{kt}^{\lambda_k \alpha}(k^{1/2}\,\mathrm{d}y)\big] \;=\; \big\langle \mu, \vartheta_t^{\alpha^*}(\,\cdot\,, y)\big\rangle\,\mathrm{d}y, \tag{1.22}$$

provided that $\alpha^* := \lim_{k\uparrow\infty} k^{1/2}\lambda_k\alpha \in [-\infty, +\infty]$.

Note that in the special case $\lambda_k = k^{-1/2}$ we have $\alpha^* = \alpha \in \mathbb{R}$, whereas the particular case $\alpha = 0$ implies $\alpha^* = 0$ giving the simplified ϑ^0 in the limit term (without the second term in (1.19)).

Remark 1.3 (*Large scale total mass).* Taking $\lambda_k \equiv 1$ and insert formally $\varphi = 1$ as test function into convergence statement (1.22) yields

$$\lim_{k\uparrow\infty} k^{-1/2}\, \mathbb{E}_\mu\, \langle X_{kt}^\alpha, 1\rangle \;=\; \begin{cases} 0 & \text{if} \;\; \alpha > 0, \\ 2t\,\langle \mu, \phi\rangle \displaystyle\int_{\dot{\mathbb{R}}^3} \mathrm{d}y\, \frac{1}{|y|}\, p_t(y) & \text{if} \;\; \alpha = 0, \\ \infty & \text{if} \;\; \alpha < 0. \end{cases} \tag{1.23}$$

A rigorous argument can be given along the same lines as the proof of Theorem 1.2 below. ◊

1.5. Discussion and open problems. Let us comment on the *three cases* $\alpha^* = +\infty$, $\alpha^* \in \mathbb{R}$, and $\alpha^* = -\infty$ in Theorem 1.2. In the first case, the limiting mass disappears, more precisely, the scaled expression $\mathbb{E}_\mu\big[X_{kt}^{\lambda_k\alpha}(k^{1/2}\,\mathrm{d}y)\big]$ is of order $o(k^{1/2})$. Roughly speaking, if $\alpha^* = +\infty$, then there are no interactions in the scaling limit (free case). In the second case, $\alpha^* \in \mathbb{R}$, the former expectation is about $k^{1/2}\,\big\langle \mu, \vartheta_t^{\alpha^*}(\,\cdot\,,y)\big\rangle\,\mathrm{d}y$. Note that these measures are decreasing in α^*. Finally, if $\alpha^* = -\infty$, we have immediate interaction in the large scale limit leading to the explosion of the expected mass.

Clearly, to describe only the large scale behavior of the *expected processes* is unsatisfactory. It is desirable to get insight into the processes themselves. Recall that in the one-dimensional case the large time behavior in law of the process itself is known from [4] (for a sharpening of some results from [4], see Engländer & Winter [5]). However, we stress the fact, that the process in three dimensions is expected to have quite different features. For instance, if $\alpha = 0$, then according to Remark 1.3 the total mass grows with a power order, whereas in one dimension the growth is exponential. Moreover, in the three-dimensional case one needs additionally to contract the normalized measures to get a limit. For the measures themselves, scaled as in Theorem 1.2, *there might be extinction in law* despite convergence of their expectations as in (1.22).

Another *open problem* is the large scale behavior of $\mathbb{E}X^{\alpha}$ in the two-dimensional case, in which the fundamental solutions p^{α} from [1] are analytically more delicate, see e.g. [6, formula (2.30)]. In particular, a scaling property as in Lemma 1.1 is not available.

Remark 1.4 (*Discrete version).* In physics literature, Redner & Kang [7], the following somehow related model occurs: Discrete time random walkers produce a fixed number of additional particles if they hit the origin in $\mathbb{Z}^d$. Dimension effects occur. ◊

2. Proofs

2.1. A scaling property. Recall that we are dealing with the three-dimensional case.

Proposition 2.1 (A scaling property). *Let* $t,k>0,\ \mu\in\mathcal{M},$ *and* $\alpha,\lambda_k\in\mathbb{R}.$ *Then*

$$\begin{aligned}\Big\{k^{-1/\beta}X_{kt}^{\lambda_k\alpha}(k^{1/2}\,\cdot\,)\,\Big|\,X_0^{\lambda_k\alpha}=k^{1/\beta}\mu(k^{-1/2}\,\cdot\,)\Big\}&\\ \stackrel{\mathcal{L}}{=}\ \Big\{X_t^{k^{1/2}\lambda_k\alpha}\,\Big|\,X_0^{k^{1/2}\lambda_k\alpha}=\mu\Big\}.&\end{aligned}\tag{2.1}$$

Note that besides $\lambda_k\equiv 1,$ the cases $\lambda_k=k^{-1/2}$ or even $\alpha=0$ are particularly nice, since here the right hand side in (2.1) is independent of k (a kind of self-similarity).

Proof. For $\varphi\in\Phi$ fixed,

$$\big\langle k^{-1/\beta}X_{kt}^{\lambda_k\alpha}(k^{1/2}\,\mathrm{d}y),\,\varphi\big\rangle\ =\ \big\langle X_{kt}^{\lambda_k\alpha},\,k^{-1/\beta}\varphi(k^{-1/2}\,\cdot\,)\big\rangle,\tag{2.2}$$

hence, by (1.15) and (1.17),

$$\begin{aligned}&-\log\mathbb{E}_{k^{1/\beta}\mu(k^{-1/2}\,\cdot\,)}\exp\big\langle k^{-1/\beta}X_{kt}^{\lambda_k\alpha}(k^{1/2}\,\mathrm{d}y),-\varphi\big\rangle\\ &=\ -\log\mathbb{E}_{k^{1/\beta}\mu(k^{-1/2}\,\cdot\,)}\exp\big\langle X_{kt}^{\lambda_k\alpha},\,-k^{-1/\beta}\varphi(k^{-1/2}\,\cdot\,)\big\rangle\\ &=\ \big\langle k^{1/\beta}\mu(k^{-1/2}\,\cdot\,),v(kt,\,\cdot\,)\big\rangle\ =\ \big\langle\mu,k^{1/\beta}v(kt,k^{1/2}\,\cdot\,)\big\rangle,\end{aligned}\tag{2.3}$$

where $\big\{v(t',x')\,:\,t'\geq 0,\,x'\in\dot{\mathbb{R}}^3\big\}$ is the non-negative solution of the integral equation related to the function-valued evolution equation

$$\begin{cases}\partial_t v=\Delta^{(\lambda_k\alpha)}v-\eta\,v^{1+\beta}\qquad\text{on }(0,\infty),\\ \quad v(0+,\,\cdot\,)=k^{-1/\beta}\varphi(k^{-1/2}\,\cdot\,).\end{cases}\tag{2.4}$$

More precisely,

$$\begin{aligned}k^{1/\beta}v(kt,k^{1/2}x)\ &=k^{1/\beta}\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\ p_{kt}^{\lambda_k\alpha}(k^{1/2}x,y)\,k^{-1/\beta}\varphi(k^{-1/2}y)\\ &\qquad-\ k^{1/\beta}\eta\int_0^{kt}\mathrm{d}s\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\ p_{kt-s}^{\lambda_k\alpha}(k^{1/2}x,y)\,v^{1+\beta}(s,y).\end{aligned}\tag{2.5}$$

By a change of variable,

$$k^{1/\beta}v(kt,k^{1/2}x) \;=\; \int_{\dot{\mathbb{R}}^3}\mathrm{d}y\; k^{3/2}p_{kt}^{\lambda_k\alpha}(k^{1/2}x,k^{1/2}y)\,\varphi(y) \qquad (2.6)$$

$$-\;k^{1/\beta}\eta\int_0^t \mathrm{d}s\,k\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\; k^{3/2}p_{kt-ks}^{\lambda_k\alpha}(k^{1/2}x,k^{1/2}y)\,v^{1+\beta}(ks,k^{1/2}y).$$

Hence, by Lemma 1.1,

$$k^{1/\beta}v(kt,k^{1/2}x) \qquad (2.7)$$

$$=\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\; p_t^{k^{1/2}\lambda_k\alpha}(x,y)\,\varphi(y)\;-\;k^{1/\beta}\eta\int_0^t\mathrm{d}s\,k\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\; p_{t-s}^{k^{1/2}\lambda_k\alpha}(x,y)\,v^{1+\beta}(ks,k^{1/2}y).$$

Since $1/\beta+1-(1/\beta)(1+\beta)=0$ we see that $k^{1/\beta}v(kt,k^{1/2}x)=:w_k(t,x)$ satisfies the equation

$$w_k(t',x') \;=\; \int_{\dot{\mathbb{R}}^3}\mathrm{d}y\; p_{t'}^{k^{1/2}\lambda_k\alpha}(x',y)\,\varphi(y)-\eta\int_0^{t'}\mathrm{d}s\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\; p_{t'-s}^{k^{1/2}\lambda_k\alpha}(x',y)\,w_k^{1+\beta}(s,y),$$

$t'>0,\ x'\in\dot{\mathbb{R}}^3$. By uniqueness of solutions of the log-Laplace equation (1.17) and by (1.15), claim (2.1) follows. □

2.2. Proof of Theorem 1.2. Fix $\varphi\in\mathcal{C}^+_{\mathrm{com}}(\dot{\mathbb{R}}^3)$. Using formula (1.18) for the first moment of X^α and substitution, we obtain

$$k^{-1/2}\,\mathbb{E}_\mu\big\langle X_{kt}^{\lambda_k\alpha},\varphi(k^{-1/2}\,\cdot\,)\big\rangle = k^{-1/2}\int_{\dot{\mathbb{R}}^3}\mu(\mathrm{d}x)\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\,p_{kt}^{\lambda_k\alpha}(x,y)\,\varphi(k^{-1/2}y)$$

$$= k^{-1/2}\int_{\dot{\mathbb{R}}^3}\mu(\mathrm{d}x)\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\,k^{3/2}\,p_{kt}^{\lambda_k\alpha}(x,k^{1/2}y)\,\varphi(y). \qquad (2.8)$$

By Lemma 1.1 this is equal to

$$k^{-1/2}\int_{\dot{\mathbb{R}}^3}\mu(\mathrm{d}x)\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\,p_t^{k^{1/2}\lambda_k\alpha}(k^{-1/2}x,y)\,\varphi(y). \qquad (2.9)$$

Using formula (1.5) for the expression of p^α, we get three terms, we will deal with separately.

1° (*First term*). The first term equals,

$$k^{-1/2}\int_{\dot{\mathbb{R}}^3}\mu(\mathrm{d}x)\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\,p_t(k^{-1/2}x,y)\,\varphi(y). \qquad (2.10)$$

This double integral is finite and vanishes as $k\uparrow\infty$. To see this, let us restrict the outer integral first to $|x|>K$ where we specify $K\geq 1$ later. We call this restricted integral I_K. We use $\varphi\leq C\phi$ (since $\varphi\in\Phi$) and, with S denoting the free heat flow,

$$S_t\phi\leq C\phi,\qquad t\geq 0, \qquad (2.11)$$

with changed constant C (see [6, Lemma 2.1]) to arrive at

$$I_K\;\leq\; Ck^{-1/2}\int_{|x|>K}\mu(\mathrm{d}x)\,\phi(k^{-1/2}x)\;=\;C\int_{|x|>K}\mu(\mathrm{d}x)\,\phi(x), \qquad (2.12)$$

the last step by the particular form of ϕ. The latter integral can be made arbitrarily small (uniformly in k) by choosing K sufficiently large (by our assumption on μ).

It remains to deal with the case $|x| \leq K$ for fixed K. We split the internal integral in (2.10) as follows. First, if $|k^{-1/2}x - y| \geq |y|/2$, then

$$p_t(k^{-1/2}x, y) \leq p_t\big(|y|/2\big), \tag{2.13}$$

which leads to the bound

$$k^{-1/2} \int_{|x|\leq K} \mu(\mathrm{d}x) \int_{\dot{\mathbb{R}}^3} \mathrm{d}y\, p_t\big(|y|/2\big)\, \varphi(y) \longrightarrow 0 \quad \text{as } k \uparrow \infty \tag{2.14}$$

(the $\mu(\mathrm{d}x)$-integral is finite as $\langle \mu, \phi \rangle < \infty$). On the other hand, if $|k^{-1/2}x - y| < |y|/2$, then $-k^{-1/2}|x| + |y| < |y|/2$ which implies $|y| < 2k^{-1/2}|x| \leq 2K$. Hence as $p_t(k^{-1/2}x, y) \leq Ct^{-3/2}$ and $\varphi \leq C\phi$, we get the upper estimate

$$C_t\, k^{-1/2} \int_{|x|\leq K} \mu(\mathrm{d}x) \int_{|y|<\, 2K} \mathrm{d}y\, \phi(y) \longrightarrow 0 \quad \text{as } k \uparrow \infty \tag{2.15}$$

(the dy-integral is finite, since we are in dimension three).

2° (*Second term*). The second term reads

$$\int_{\dot{\mathbb{R}}^3} \mu(\mathrm{d}x) \int_{\dot{\mathbb{R}}^3} \mathrm{d}y\ \frac{2t}{|x|\,|y|}\ p_t\big(k^{-1/2}|x| + |y|\big)\, \varphi(y)\ =:\ \mathit{II}_k\,. \tag{2.16}$$

Observe that,

$$p_t\big(k^{-1/2}|x| + |y|\big) \uparrow p_t\big(|y|\big) \quad \text{as } \ k \uparrow \infty. \tag{2.17}$$

We can apply the monotone convergence theorem to obtain the limit,

$$\lim_{k\uparrow\infty}\ \mathit{II}_k\ =\ \int_{\dot{\mathbb{R}}^3} \mu(\mathrm{d}x)\, \frac{2t}{|x|} \int_{\dot{\mathbb{R}}^3} \mathrm{d}y\ \frac{1}{|y|}\, p_t(y)\, \varphi(y), \tag{2.18}$$

where finiteness follows from $\langle \mu, \phi \rangle < \infty$ and $\varphi \leq C\phi$. Hence, this summand gives the first part of the kernel ϑ^{α^*}.

3° (*Third term*). It remains to insert the scaled third term from representation (1.5) into (2.9) which reads as

$$\begin{aligned} k^{-1/2} \int_{\dot{\mathbb{R}}^3} \mu(\mathrm{d}x) \int_{\dot{\mathbb{R}}^3} \mathrm{d}y\ &\frac{-8\pi t k^{1/2} \lambda_k \alpha}{k^{-1/2}|x|\,|y|} \\ &\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-4\pi k^{1/2}\lambda_k \alpha u}\, p_t\big(u + k^{-1/2}|x| + |y|\big)\, \varphi(y). \end{aligned} \tag{2.19}$$

We distinguish several cases: If $\lambda_k \alpha = 0$ for all sufficiently large k, then the third term disappears and we are done. From now on assume $\lambda_k \alpha \neq 0$ for all k. Substituting $u \mapsto 4\pi k^{1/2} |\lambda_k \alpha| u$ into (2.19) yields,

$$\begin{aligned} \int_{\dot{\mathbb{R}}^3} \mu(\mathrm{d}x) \int_{\dot{\mathbb{R}}^3} \mathrm{d}y\, &\frac{-2t\, \mathrm{sign}(\lambda_k \alpha)}{|x|\,|y|} \\ &\int_0^\infty \mathrm{d}u\ \mathrm{e}^{-\mathrm{sign}(\rho_k \alpha)\, u}\, p_t\Big(\frac{u}{4\pi k^{1/2}|\lambda_k \alpha|} + k^{-1/2}|x| + |y|\Big)\, \varphi(y). \end{aligned} \tag{2.20}$$

Now let $k^{1/2}|\lambda_k \alpha| \to \infty$. We may consider a monotone subsequence of $k^{1/2}\lambda_k \alpha$. Clearly,

$$p_t\Big(\frac{u}{4\pi k^{1/2}|\lambda_k \alpha|} + k^{-1/2}|x| + |y|\Big) \uparrow p_t\big(|y|\big) \quad \text{as } k \uparrow \infty, \tag{2.21}$$

and by monotone convergence, (2.20) converges along the subsequence to

$$-\int_{\dot{\mathbb{R}}^3}\mu(\mathrm{d}x)\,\frac{2t\,\mathrm{sign}(\alpha^*)}{|x|}\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-\mathrm{sign}(\alpha^*)u}\int_{\dot{\mathbb{R}}^3}\mathrm{d}y\ \frac{1}{|y|}\,p_t(y)\,\varphi(y), \tag{2.22}$$

which is independent of the choice of the subsequence. Note that

$$\mathrm{sign}(\alpha^*)\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-\mathrm{sign}(\alpha^*)\,u} = \begin{cases} 1 & \text{if } \mathrm{sign}(\alpha^*) = 1, \\ +\infty & \text{if } \mathrm{sign}(\alpha^*) = -1. \end{cases} \tag{2.23}$$

In the first case the second and the third limiting terms cancel.

Next, let $k^{1/2}\lambda_k\alpha \to 0$. Note, that

$$p_t\Big(\frac{u}{4\pi k^{1/2}|\lambda_k\alpha|} + k^{-1/2}|x| + |y|\Big) \le p_t\Big(\frac{u}{4\pi k^{1/2}|\lambda_k\alpha|}\Big)\,\mathrm{e}^{-|y|^2/4t}\,. \tag{2.24}$$

In this case the double integral (2.20) is in absolute value bounded by

$$\int_{\dot{\mathbb{R}}^3}\mu(\mathrm{d}x)\ \int_{\dot{\mathbb{R}}^3}\mathrm{d}y\,\frac{2t}{|x|\,|y|}\,\mathrm{e}^{-|y|^2/4t}\,\varphi(y)\int_0^\infty \mathrm{d}u\ \mathrm{e}^u\,p_t\Big(\frac{u}{4\pi\,k^{1/2}|\lambda_k\alpha|}\Big), \tag{2.25}$$

which tends to 0 as $k \uparrow \infty$ as the $\mu(\mathrm{d}x)$ and $\mathrm{d}y$-integrals are finite and in the $\mathrm{d}u$-integral the p_t-term compensates the e^u.

It remains to deal with the case $k^{1/2}\lambda_k\alpha \to \alpha^* \in \dot{\mathbb{R}}^1$. Note, that we only have to justify to change the limit and integration in (2.20), as substituting $u \mapsto \big(4\pi|\alpha^*|\big)^{-1}u$ leads to the desired expression. To justify the interchange, we estimate as in (2.24). The resulting $\mu(\mathrm{d}x)$ and $\mathrm{d}y$-integrals are independent of k and finite, whereas to dominate in the second integral we use $k^{1/2}|\lambda_k\alpha| \le |\alpha^*|+1$ for all sufficiently large k. □

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