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APPLICATION OF WHITE NOISE CALCULUS IN EVALUATING THE PATH INTEGRAL IN RELATIVISTIC QUANTUM MECHANICS

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Abstract. The Hida-Streit method of evaluating the Feynman path integral is applied to relativistic quantum mechanical systems. A charged particle in a uniform magnetic field is taken as an example where the Green function for the Dirac particle is obtained.

1. Introduction

Since Feynman introduced his path integral formulation of quantum mechanics [11], various approaches aimed at providing a mathematically rigorous meaning to the path integral have been given. Notable among these would be the prodistributions of DeWitt-Morette [10], the oscillatory integrals of Albeverio and Høegh-Krohn [1] which explores the Fresnel integrals, and the white noise analysis approach of Hida and Streit [27]. In this paper, we shall consider the Hida-Streit approach which, through the years, has solved various classes of potentials [2, 3, 5, 6, 7, 8, 13, 18, 19, 21, 27] in nonrelativistic quantum mechanics. In white noise analysis, the velocity of Brownian motion B, i.e., $\omega(\tau) = dB/d\tau$, $\tau \in \mathbb{R}$, which are independent random variables at each point in time, form the coordinates of an infinite dimensional space [14]. We note that white noise analysis as applied here is also referred to as Hida calculus [14, 16, 20, 23], and differs from the use of white noise in Parisi-Wu stochastic quantization [25]. The calculus allows the generalization of concepts in finite dimensions to the infinite dimensional case, including differential operators and integral transforms such as Fourier and Fourier-Mehler transforms [16]. This makes it particularly suitable for the evaluation of the Feynman path integral. Here we extend the applicability of the Hida-Streit white noise path integral to solve systems in relativistic quantum mechanics where the Green function for the Dirac equation is obtained.

In Section 2 we briefly discuss a general procedure for obtaining the Dirac Green function using the path integral method. We then take as an example in Section 3 the case of a relativistic charged particle in a uniform magnetic field. In Section 4, the paths of this particle are parametrized in terms of the white noise variable ω and the effective potential becomes similar in form to that of Lévy's stochastic area. As in the nonrelativistic case, the path integral can be evaluated with the

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help of the T-transform in white noise analysis from which we obtain the Dirac Green function. Concluding remarks are given in Section 5 where we cite other relativistic problems that may be solved using this approach.

2. Quantum Relativistic Systems

The Hida-Streit white noise path integral can be used to solve the Dirac equation for a spin $\frac{1}{2}$ particle of mass m given by

$$\left(m-\widehat{M}\right) G\left(\mathbf{r}'', \mathbf{r}'\right) = \delta\left(\mathbf{r}''-\mathbf{r}'\right), \tag{2.1}$$

where we have defined an operator $\widehat{M}$ in terms of the Dirac matrices α and β of the form,

$$\widehat{M} = -\beta\alpha\cdot(\mathbf{p}-e\mathbf{A})-\beta V+\beta E. \tag{2.2}$$

Here V and $\mathbf{A}$ are the scalar and vector potentials, respectively. By expressing the Green function $G\left(\mathbf{r}'', \mathbf{r}'\right)$ in Eq. (2.1) as,

$$G\left(\mathbf{r}'', \mathbf{r}'\right) = \left(m+\widehat{M}\right) g\left(\mathbf{r}'', \mathbf{r}'\right). \tag{2.3}$$

we obtain an iterated Dirac equation,

$$\left(m^2-\widehat{M}^2\right) g\left(\mathbf{r}'', \mathbf{r}'\right) = \delta\left(\mathbf{r}''-\mathbf{r}'\right). \tag{2.4}$$

The solution of Eq. (2.1) are also solutions of Eq. (2.4), but not conversely.

The Green functions, $G\left(\mathbf{r}'', \mathbf{r}'\right)=\langle\mathbf{r}''|G|\mathbf{r}'\rangle$ and $g\left(\mathbf{r}'', \mathbf{r}'\right)=\langle\mathbf{r}''|g|\mathbf{r}'\rangle$, are matrix elements of the operators, $G=(m-\widehat{M})^{-1}$, and $g=(m^2-\widehat{M}^2)^{-1}$, respectively. We can also write, for example operator g, in integral form using the relation (see, e.g., [11], Eq. 5-17),

$$\lim_{\epsilon\rightarrow 0}\left[-\left(\theta+i\epsilon\right)^{-1}\right] = i\int_0^\infty \exp\left(i\theta\Lambda\right)\, d\Lambda. \tag{2.5}$$

Defining

$$H = \left(m^2-\widehat{M}^2\right)/2m, \tag{2.6}$$

we can express the operator $g=(1/2m)\,(1/H)$ as (see also [9, 26])

$$g=(i/2m)\int_0^\infty \exp\left(-iH\Lambda\right) d\Lambda\ . \tag{2.7}$$

Taking the matrix element of Eq. (2.7),

$$g\left(\mathbf{r}'', \mathbf{r}'\right)=(i/2m)\int_0^\infty \langle\mathbf{r}''|\exp\left(-iH\Lambda\right)|\mathbf{r}'\rangle\, d\Lambda, \tag{2.8}$$

we observe that the integrand in Eq. (2.8) is analogous in form to a quantum propagator evolving in $\Lambda-$time with an effective Hamiltonian H. Following Feynmans's prescription for handling quantum propagators [11] we can express the integrand $\langle\mathbf{r}''|\exp\left(-iH\Lambda\right)|\mathbf{r}'\rangle$ as a path integral. After evaluating the propagator $\langle\mathbf{r}''|\exp\left(-iH\Lambda\right)|\mathbf{r}'\rangle$, we can integrate out Λ as in Eq. (2.8) to get $g\left(\mathbf{r}'', \mathbf{r}'\right)$, and from Eq. (2.3), the Green function $G(\mathbf{r}'', \mathbf{r}')$ for the Dirac equation can then be subsequently calculated. This procedure, therefore, enables us to arrive at a solution for the Dirac equation via the path integral method. We evaluate as an example a charged particle in a uniform magnetic field where white noise analytical methods are used.

3. Particle in a Uniform Magnetic Field

Let us consider an electron of charge e ($e < 0$) and mass m subjected to a uniform magnetic field $\mathbb{B}$ along the z-axis and represented by the vector potential $\mathbf{A} = (1/2)(-\mathcal{B}y, \mathcal{B}x, 0)$. Here we represent the magnetic field by $\mathbb{B} = \mathcal{B}\widehat{k}$ in order to distinguish it from our notation for the Brownian motion, B. With $\widehat{M} = -\beta\alpha \cdot (\mathbf{p} - e\mathbf{A}) + \beta E$, the operator H in Eq. (2.6) has the form,

$$\begin{aligned} H &= (1/2m)\left[(p_x + e\mathcal{B}y/2)^2 + (p_y - e\mathcal{B}x/2)^2 + p_z^2\right] \\ &\quad - \left(k_0^2/2m\right) - (e\sigma \cdot \mathbb{B}/2m), \end{aligned} \tag{3.1}$$

where $k_0^2 = E^2 - m^2$. The spin part, with the magnetic field $\mathbb{B}$ along the z-axis, is just $-e\sigma \cdot \mathbb{B}/2m = -e\mathcal{B}\sigma_z/2m$. We can then use Eq. (3.1) as the effective Hamiltonian in $\langle \mathbf{r}''| \exp(-iH\Lambda) |\mathbf{r}'\rangle$ of Eq. (2.8). Defining the operator $S_z = (1/2)\sigma_z$, then S_z acting upon an eigenspinor has the eigenvalues, $s_i = \pm 1/2$. Noting that $|\mathbf{r}> = |xyz>|\chi>$, with $|\chi>$ a spinor, we can use $\sum |s_i><s_i| = 1$, to write the integrand in Eq. (2.8) as,

$$\begin{aligned} \langle \mathbf{r}''| \exp(-iH\Lambda) |\mathbf{r}'\rangle &\equiv \sum_{s_i} \langle \chi|s_i\rangle \langle x''y''z''| \exp(-iH_s\Lambda) |x'y'z'\rangle \langle s_i|\chi\rangle \\ &= \sum_{s_i} \eta_{s_i}\eta_{s_i}^{+} \langle x''y''z''| \exp(-iH_s\Lambda) |x'y'z'\rangle \end{aligned} \tag{3.2}$$

where $\eta_{s_i} = \langle \chi|s_i\rangle$. The coordinate part, $\langle x''y''z''| \exp(-iH_s\Lambda) |x'y'z'\rangle$, has an effective Hamiltonian of the form (setting $\gamma = e\mathcal{B}/2$),

$$\begin{aligned} H_s &= (1/2m)\left[(p_x + e\mathcal{B}y/2)^2 + (p_y - e\mathcal{B}x/2)^2 + p_z^2\right] \\ &\quad - \left(k_0^2/2m\right) - (2\gamma s_i/m). \end{aligned} \tag{3.3}$$

We then evaluate the coordinate part as the path integral

$$\langle x''y''z''| \exp(-iH_s\Lambda) |x'y'z'\rangle = \int \exp(iS)\ D[xyz] \tag{3.4}$$

for a system evolving with a time-like parameter Λ, and an action corresponding to Eq. (3.3) given by

$$S = \int_0^\Lambda \left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \gamma\left(x\dot{y} - y\dot{x}\right) + \left(k_0^2/2m\right) + (2\gamma s_i/m)\right] d\lambda. \tag{3.5}$$

The path integral, Eq. (3.4), with an action given by Eq. (3.5) can be written as,

$$\langle x''y''z''| \exp(-iH_s\Lambda) |x'y'z'\rangle = e^{i\left[\left(k_0^2/2m\right) + (2\gamma s_i/m)\right]\Lambda} K(x'', y''; x', y')\, K(z'', z') \tag{3.6}$$

where the propagator along the z-coordinate is similar to the free particle case, i.e.,

$$\begin{aligned} K(z'', z') &= \int \exp\left[(im/2)\,\dot{z}^2\right] D[z] \\ &= (1/2\pi) \int \exp\left\{\left[ik_z(z''-z') - \left(ik_z^2/2m\right)\right]\Lambda\right\} dk_z. \end{aligned} \quad (3.7)$$

On the other hand, the propagator along the (x, y) axes is,

$$K(x''y''; x'y') = \int \exp\left\{ i\int_0^\Lambda \left[\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) + \gamma\left(x\dot{y} - y\dot{x}\right)\right] d\lambda \right\} D[xy]. \quad (3.8)$$

This equation is similar to the propagator for a nonrelativistic charged particle in a uniform magnetic field [6, 11] which can be evaluated using white noise calculus.

4. Evaluation in terms of White Noise

Let us now evaluate Eq. (3.8), $K(x'', y''; x', y')$, by parametrizing the paths in terms of Brownian motion variables as (setting, $m = \hbar = c = 1$),

$$\begin{aligned} x(\Lambda) &= x' + B_x(\Lambda) \\ &= x' + \int_0^\Lambda \omega_x(\lambda)\, d\lambda, \end{aligned} \quad (4.1)$$

where $\omega_x(\lambda) = dB_x(\lambda)/d\lambda$, is the corresponding white noise variable. Similarly,

$$\begin{aligned} y(\Lambda) &= y' + B_y(\Lambda) \\ &= y' + \int_0^\Lambda \omega_y(\lambda)\, d\lambda, \end{aligned} \quad (4.2)$$

with $\omega_y(\lambda) = dB_y(\lambda)/d\lambda$. Here x' and y' are the fixed initial points. With this, the velocity of the particle, for instance along the x-direction, becomes, $(dx/d\lambda) = \omega_x$, and the kinetic part acquires the form,

$$\exp\left[\frac{i}{2}\int_0^\Lambda \left(\dot{x}^2\right) d\lambda\right] = \exp\left[\frac{i}{2}\int_0^\Lambda \omega_x(\tau)^2\, d\lambda\right]. \quad (4.3)$$

The integration over all paths ($\lim_{N\to\infty}\prod d[x_j]$ or $d^\infty x$), on the other hand, leads to an integration over the Gaussian white noise measure of the form, $d\mu(\omega_x) = N_\omega \exp\left(-\frac{1}{2}\int \omega_x(\tau)^2\, d\tau\right) d^\infty\omega_x$. The exponential factor $\exp\left(-\frac{1}{2}\int \omega_x(\tau)^2\, d\tau\right)$ in $d\mu(\omega_x)$, which is responsible for the Gaussian fall-off, spoils the direct correspondence between $d^\infty x$ of the Feynman path integral and $d^\infty\omega_x$ of the white noise measure. A better comparison can, therefore, be attained by multiplying $d\mu(\omega_x)$ by a factor $\exp\left(\frac{1}{2}\int \omega_x(\tau)^2\, d\tau\right)$ to obtain the correspondence, $d^\infty x \to d^\infty\omega$. Together with Eq. (4.3), the translation of the integrand for the Feynman integral into the language of white noise analysis leads us to consider the Gauss kernel (for the x-degree of freedom),

$$I_0 = \mathcal{N}\ \exp\left[\left(\frac{i+1}{2}\right)\int\limits_0^\Lambda \omega_x(\lambda)^2\ d\lambda\right], \tag{4.4}$$

where $\mathcal{N}$ is a normalization.

The kinetic part of Eq. (3.8) leads to a white noise functional which is a two-dimensional version of Eq. (4.4). We have,

$$I_{xy} = N\exp\left\{\left(\frac{i+1}{2}\right)\int\limits_0^\Lambda \left[\omega_x(\lambda)^2 + \omega_y(\lambda)^2\right]\ d\lambda\right\}. \tag{4.5}$$

On the other hand, the interaction part in Eq. (3.8) can be written as,

$$\begin{aligned}\exp\left[i\gamma\int\limits_0^\Lambda \left(x\dot{y} - y\dot{x}\right)d\lambda\right] &= \exp\left\{i\gamma\int\limits_0^\Lambda \left[x' + B_x(\lambda)\right]\omega_y(\lambda)\,d\lambda\right.\\ &\qquad \left.-i\gamma\int\limits_0^\Lambda \left[y' + B_y(\lambda)\right]\omega_x(\lambda)\,d\lambda\right\}\\ &= \exp\left\{i\gamma\int\limits_0^\Lambda \left[x'\omega_y(\lambda) - y'\omega_x(\lambda)\right]d\lambda\right\}\\ &\qquad \times\exp\left(2i\gamma S_T\right),\end{aligned} \tag{4.6}$$

where S_T is Lévy's stochastic area given by

$$S_T = \frac{1}{2}\int\limits_0^\Lambda \left[B_x(\lambda)\,dB_y(\lambda) - B_y(\lambda)\,dB_x(\lambda)\right]. \tag{4.7}$$

This may be handled by noting that the two-dimensional Brownian motion, $B_x(\lambda)$ and $B_y(\lambda)$, can be realized on the probability space of a one-dimensional white noise [6, 15, 22], where

$$B_x(\lambda) = \int\limits_0^\lambda \omega(\tau)\ d\tau; \qquad dB_x(\lambda) = \omega(\lambda)\ d\lambda. \tag{4.8}$$

$$B_y(\lambda) = \int\limits_{-\lambda}^0 \omega(\tau)\ d\tau; \qquad dB_y(\lambda) = \omega(-\lambda)\ d\lambda. \tag{4.9}$$

Eqs. (4.8) and (4.9) enable us to write Eq. (4.7) as

$$\begin{aligned}S_T &= \int\limits_{\mathbf{R}^2} \omega(\tau_1)\ F_S(\tau_1,\tau_2)\ \omega(\tau_2)\ d\tau_1 d\tau_2,\\ &= \langle \omega, F_S(\tau_1,\tau_2)\ \omega\rangle\end{aligned} \tag{4.10}$$

where

$$F_S(\tau_1,\tau_2) = \frac{1}{4}\left[\chi_{[-\Lambda,0]}(\tau_1)\ \chi_{[0,-\tau_1]}(\tau_2) + \chi_{[-\Lambda,0]}(\tau_2)\ \chi_{[0,-\tau_2]}(\tau_1)\right] - \frac{1}{4}\left[\chi_{[0,\Lambda]}(\tau_1)\ \chi_{[-\tau_1,0]}(\tau_2) + \chi_{[0,\Lambda]}(\tau_2)\ \chi_{[-\tau_2,0]}(\tau_1)\right] . \tag{4.11}$$

The $\chi_{[\alpha,\beta]}(\tau_j)$, $j=1,2$, with $[\alpha,\beta]$ as the limits of integration, denotes the integration over τ_j in Eq. (4.10).

On the other hand, the terms in the exponent of the kinetic part, Eq. (4.5), can be written as,

$$\left(\frac{i+1}{2}\right)\int_0^\Lambda \left[\omega_x(\tau)^2 + \omega_y(\tau)^2\right] d\tau = \left(\frac{i+1}{2}\right)\int_0^\Lambda \left[\omega(\tau)^2 + \omega(-\tau)^2\right] d\tau = -\frac{1}{2}\int_{-\Lambda}^{\Lambda} \omega(\tau)\ K\ \omega(\tau)\ d\tau, \tag{4.12}$$

where $K = -(i+1)$. Furthermore, to fix the endpoints x'' and y'', we use the Donsker delta function $\delta(x(\Lambda) - x'')$ and $\delta(y(\Lambda) - y'')$. where $x(\Lambda)$ and $y(\Lambda)$ are given by Eqs. (4.1) and (4.2). With Eqs. (4.6) to (4.12), we can now write Eq. (3.8) as,

$$\begin{aligned} K(x'',y'';x',y') = &\int I_{xy} \exp\left[i\gamma\left(x'\int_{-\Lambda}^{0}\omega(\lambda)\,d\lambda - y'\int_0^\Lambda \omega(\lambda)\,d\lambda\right)\right] \\ &\times \exp\left(-\frac{1}{2}\langle\omega, L\ \omega\rangle\right) \\ &\times\delta\left(x' - x'' + \int_0^\Lambda \omega(\lambda)\ d\lambda\right) \\ &\times\delta\left(y' - y'' + \int_{-\Lambda}^0 \omega(\lambda)\ d\lambda\right) d\mu(\omega), \end{aligned} \tag{4.13}$$

where $L = -4i\gamma F_S(\tau_1,\tau_2)$ and $F_S(\tau_1,\tau_2)$ is given by Eq. (4.11). To evaluate the integration over the white noise measure, we write the delta functions in terms of their Fourier representation so that Eq. (4.13) becomes,

$$\begin{aligned} K(x'',y'';x',y') = &\frac{1}{(2\pi)^2}\int_{\mathbf{R}^2} d^2\mathbf{p}\exp\left[i\mathbf{p}\cdot(\mathbf{x}' - \mathbf{x}'')\right] \\ &\times\int N \exp\left[i\left(p_x - \gamma y'\right)\int_0^\Lambda \omega(\lambda)\ d\lambda\right] \end{aligned}$$

$$
\begin{aligned}
&\times \exp\left[i\left(p_y+\gamma x'\right)\int_{-\Lambda}^{0}\omega\left(\lambda\right)d\lambda\right] \\
&\times \exp\left(-\frac{1}{2}\left\langle\omega,K\omega\right\rangle\right)\exp\left(-\frac{1}{2}\left\langle\omega,L\ \omega\right\rangle\right)\ d\mu\left(\omega\right) \\
=\ &\frac{1}{\left(2\pi\right)^2}\int_{\mathbf{R}^2}d^2\mathbf{p}\exp\left[i\mathbf{p}\cdot\left(\mathbf{x}'-\mathbf{x}''\right)\right]\int N\exp\left(i\left\langle\omega,\xi\right\rangle\right) \\
&\times \exp\left(-\frac{1}{2}\left\langle\omega,K\omega\right\rangle\right)\exp\left(-\frac{1}{2}\left\langle\omega,L\ \omega\right\rangle\right)\ d\mu\left(\omega\right),
\end{aligned}
\tag{4.14}
$$

where $\mathbf{x}=(x,y)$, $\mathbf{p}=(p_x,p_y)$, and, $\xi=(p_x-\gamma y')\,\chi_{[0,\Lambda]}+(p_y+\gamma x')\,\chi_{[-\Lambda,0]}$. We then note that the integration over $d\mu(\omega)$ is just the T-transform of the white noise functional, $\Phi=N\exp\left(-\frac{1}{2}\langle\omega,K\omega\rangle\right)\exp\left(-\frac{1}{2}\langle\omega,L\ \omega\rangle\right)$, i.e.,

$$
\begin{aligned}
(T\Phi)(\xi) &= \int N\exp\left(i\left\langle\omega,\xi\right\rangle\right)\exp\left(-\frac{1}{2}\left\langle\omega,K\omega\right\rangle\right)\exp\left(-\frac{1}{2}\left\langle\omega,L\ \omega\right\rangle\right)\ d\mu\left(\omega\right) \\
&= \left[\det\left(1+L\left(1+K\right)^{-1}\right)\right]^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\left\langle\xi,\left(1+K+L\right)^{-1}\xi\right\rangle\right],
\end{aligned}
\tag{4.15}
$$

where we used Eq. (33) of ref. [6]. With $K=-(i+1)$, we also have,

$$
\begin{aligned}
\left[\det\left(1+L\left(1+K\right)^{-1}\right)\right]^{-\frac{1}{2}} &= \left[\det\left(1+iL\right)\right]^{-\frac{1}{2}} \\
&= \left[\cos\left(\gamma\Lambda\right)\right]^{-1},
\end{aligned}
\tag{4.16}
$$

and [6],

$$
\begin{aligned}
\left\langle\xi,\left(1+K+L\right)^{-1}\xi\right\rangle &= \left(i/\gamma\right)\tan\left(\gamma\Lambda\right)\left[\left(p_x^2+p_y^2\right)\right. \\
&\quad \left.+\gamma^2\left(x'^2+y'^2\right)+2\gamma\left(p_y x'-p_x y'\right)\right].
\end{aligned}
\tag{4.17}
$$

With Eqs. (4.15), (4.16) and (4.17), Eq. (4.14) becomes,

$$
\begin{aligned}
K\left(x'',y'';x',y'\right) &= \frac{\cos\left(\gamma\Lambda\right)^{-1}}{\left(2\pi\right)^2}\exp\left[-\frac{i}{2}\gamma\tan\left(\gamma\Lambda\right)\left(x'^2+y'^2\right)\right] \\
&\quad\times\int_{-\infty}^{+\infty}\exp\left\{-i\tan\left(\gamma\Lambda\right)\left(2\gamma\right)^{-1}p_x^2\right\} \\
&\quad\times\exp\left\{i\left[\left(x'-x''\right)+\tan\left(\gamma\Lambda\right)y'\right]p_x\right\}dp_x \\
&\quad\times\int_{-\infty}^{+\infty}\exp\left\{-i\tan\left(\gamma\Lambda\right)\left(2\gamma\right)^{-1}p_y^2\right\} \\
&\quad\times\exp\left\{i\left[\left(y'-y''\right)-\tan\left(\gamma\Lambda\right)x'\right]p_y\right\}dp_y.
\end{aligned}
\tag{4.18}
$$

The Gaussian integration in Eq. (4.18) over dp_x and dp_y can be performed and we obtain,

$$\begin{aligned} K\left(x'',y'';x',y'\right) &= \frac{\gamma}{2\pi i\sin\left(\gamma\Lambda\right)}\exp\left\{\frac{i}{2}\gamma\cot\left(\gamma\Lambda\right)\left[\left(x'-x''\right)^2+\left(y'-y''\right)^2\right]\right\} \\ &\quad\times\exp\left\{i\gamma\left(x'y''-x''y'\right)\right\}. \end{aligned} \tag{4.19}$$

In terms of the Hermite polynomials H_n, Eq.(4.19) becomes,

$$\begin{aligned} K\left(x'',y'';x',y'\right) &= \left(\gamma/\pi\right)\exp\left\{i\gamma\left(x'y''-x''y'\right)\right\} \\ &\quad\times\exp\left\{\left(-\gamma/2\right)\left[\left(x''-x'\right)^2+\left(y''-y'\right)^2\right]\right\} \\ &\quad\times\sum_{n=0}^{\infty}\sum_{q=0}^{n}\frac{\left(-1\right)^{-n}2^{-2n}}{q!\left(n-q\right)!} \\ &\quad\times H_{2q}\left(\sqrt{\gamma}\left(x''-x'\right)\right)H_{2(n-q)}\left(\sqrt{\gamma}\left(y''-y\right)\right) \\ &\quad\times\exp\left[-i\left(n+\frac{1}{2}\right)2\gamma\Lambda\right]. \end{aligned} \tag{4.20}$$

Using Eqs. (3.7) and (4.20), we can now write Eq. (3.6) as,

$$\begin{aligned} &\left\langle x''y''z''\right|\exp\left(-iH_s\Lambda\right)\left|x'y'z'\right\rangle \\ =\ &\left(\gamma/2\pi^2\right)\exp\left\{i\gamma\left(x'y''-x''y'\right)\right\} \\ &\times\exp\left\{\left(-\gamma/2\right)\left[\left(x''-x'\right)^2+\left(y''-y'\right)^2\right]\right\} \\ &\times\int dk_z\exp\left[ik_z\left(z''-z'\right)\right]\sum_{n=0}^{\infty}\sum_{q=0}^{n}\frac{\left(-1\right)^{-n}2^{-2n}}{q!\left(n-q\right)!} \\ &\times H_{2q}\left(\sqrt{\gamma}\left(x''-x'\right)\right)H_{2(n-q)}\left(\sqrt{\gamma}\left(y''-y'\right)\right) \\ &\times\exp\left[-i\left(k_z^2-b^2\right)\Lambda/2m\right], \end{aligned} \tag{4.21}$$

where $b^2/2m = \left(k_0^2/2m\right) + \left[s_i - \left(n+\frac{1}{2}\right)\right]\left(2\gamma/m\right)$ (putting back the mass m). From Eqs. (3.2) and (4.21), the Green function for the iterated Dirac equation (2.8), appears as,

$$\begin{aligned} g\left(\mathbf{r}'',\mathbf{r}'\right) &= \left[i\gamma/m\left(2\pi\right)^2\int dk_z\sum_{s_i}\eta_{s_i}\eta_{s_i}^{+}\exp\left[ik_z\left(z''-z'\right)\right]\right] \\ &\quad\times\exp\left\{i\gamma\left(x'y''-x''y'\right)\right\}\exp\left\{\left(-\gamma/2\right)\right. \\ &\quad\times\left.\left[\left(x''-x'\right)^2+\left(y''-y'\right)^2\right]\right\}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\left[\frac{\left(-1\right)^{-n}2^{-2n}}{q!\left(n-q\right)!}\right] \\ &\quad\times H_{2q}\left(\sqrt{\gamma}\left(x''-x'\right)\right)H_{2(n-q)}\left(\sqrt{\gamma}\left(y''-y'\right)\right) \\ &\quad\times\int_0^{\infty}\exp\left[-i\left(k_z^2-b^2\right)\Lambda/2m\right]d\Lambda. \end{aligned} \tag{4.22}$$

The integration over Λ in Eq. (4.22) makes use of Eq. (2.5) which yields the result [12],

$$\begin{aligned} g\left(\mathbf{r}'', \mathbf{r}'\right) &= \left[i\gamma/m\left(2\pi\right)^2\right]\int dk_z \sum_{s_i}\eta_{s_i}\eta_{s_i}^{+}\exp\left[ik_z\left(z''-z'\right)\right] \\ &\times\exp\left\{i\gamma\left(x'y''-x''y'\right)\right\}\exp\left\{(-\gamma/2)\right. \\ &\times\left.\left[\left(x''-x'\right)^2+\left(y''-y'\right)^2\right]\right\}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\left[\frac{(-1)^{-n}\,2^{-2n}}{q!\,(n-q)!}\right] \\ &\times H_{2q}\left(\sqrt{\gamma}\left(x''-x'\right)\right)\ H_{2(n-q)}\left(\sqrt{\gamma}\left(y''-y'\right)\right) \\ &\times\left\{(1/2m)\left(E^2-m^2-k_z^2\right)+\left[s_i-\left(n+\frac{1}{2}\right)\right](2\gamma/m)+i\varepsilon\right\}^{-1}, \end{aligned} \tag{4.23}$$

where $\varepsilon \to 0$. The discrete energy spectrum [17] can be obtained from the poles of the Green function and is given by, $E^2 = m^2 + k_z^2 + \left[(2n+1) - 2s_i\right] e\mathcal{B}$.

The Green function $G\left(\mathbf{r}'', \mathbf{r}'\right)$ for the first-order Dirac equation can be obtained with the help of Eqs. (2.3) and (4.23). We can rewrite $\widehat{M}$ in Eq. (2.2) with $V = 0$ as

$$\begin{aligned} \widehat{M} &= -\beta\alpha_x\left(-i\partial/\partial x''+e\mathcal{B}y''/2\right)-\beta\alpha_y\left(-i\partial/\partial y''-e\mathcal{B}x''/2\right) \\ &\quad+\left(i\beta\alpha_z\partial/\partial z''\right)+\beta E \end{aligned}$$

and using the recursion relations of the Hermite polynomials

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x),$$

we obtain,

$$\begin{aligned} G\left(\mathbf{r}'', \mathbf{r}'\right) &= \left\{m-\left(\frac{e\mathcal{B}}{2}\right)\beta\alpha_{+}\left[\left(y''-y'\right)+i\left(x''-x'\right)\right]+\beta E-k_z\beta\alpha_z\right\} \\ &\quad\times g\left(\mathbf{r}'', \mathbf{r}'\right)+i\beta\alpha_x\bar{g}\sqrt{\frac{e\mathcal{B}}{2}}4q \\ &\quad\times H_{2q-1}\left(\sqrt{\frac{e\mathcal{B}}{2}}\left(x''-x'\right)\right)\ H_{2(n-q)}\left(\sqrt{\frac{e\mathcal{B}}{2}}\left(y''-y'\right)\right) \\ &\quad+i\beta\alpha_y\bar{g}\sqrt{\frac{e\mathcal{B}}{2}}4\left(n-q\right) \\ &\quad\times H_{2(n-q)-1}\left(\sqrt{\frac{e\mathcal{B}}{2}}\left(y''-y'\right)\right)\ H_{2q}\left(\sqrt{\frac{e\mathcal{B}}{2}}\left(x''-x'\right)\right), \end{aligned}$$

where $\alpha_{+} = \alpha_y + i\alpha_x$, and

$$\bar{g} = \left[ie\mathcal{B}/2m\left(2\pi\right)^2\right]\int dk_z\sum_{s_i}\eta_{s_i}\eta_{s_i}^{+}\exp\left[ik_z\left(z''-z'\right)\right]$$

$$\times \exp\left\{ (ie\mathcal{B}/2)\,(x'y'' - x''y') - (e\mathcal{B}/4)\left[(x''-x')^2 + (y''-y')^2 \right] \right\}$$
$$\times \sum_{n=0}^{\infty} \sum_{q=0}^{n} \left[(-1)^{-n}\, 2^{-2n}/q!\,(n-q)! \right]$$
$$\times \left\{ (1/2m)\left(E^2 - m^2 - k_z^2\right) + \left[s_i - \left(n + \frac{1}{2}\right) \right] (e\mathcal{B}/m) + i\varepsilon \right\}^{-1},$$

as $\varepsilon \to 0$. We note here that an early attempt to provide a path integral treatment of a Dirac particle in a uniform magnetic field can be found in reference [24].

5. Conclusion

In this paper, we demonstrated how white noise calculus may be applied to solve relativistic quantum problems. By iterating the Dirac equation and expressing the corresponding Green function in terms of a path integral, the form of the effective Lagrangian enables one to take advantage of results available in the nonrelativistic case. This similarity with the nonrelativistic problem is, in fact, the feature which allows us to extend the applicability of the Hida-Streit white noise path integral to solve quantum relativistic systems. Using the procedure presented in this paper, other relativistic systems can likewise be handled. Examples would be a Dirac particle on a circle, as well as a relativistic particle in an Aharonov-Bohm (AB) potential, and combinations thereof, such as a Dirac particle in the presence of a uniform magnetic field plus the AB potential [4]. We also note that, since we made use of the iterated Dirac equation, the procedure discussed in this paper can likewise be applied to treat spin 0 Klein-Gordon particles.

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