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INFINITE DIMENSIONAL LAPLACIANS ON A LÉVY–GEL'FAND TRIPLE

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Abstract. We introduce a new Hilbert space H_L, called *Lévy–Hilbert space*, arising from the Lévy Laplacian. This Hilbert space is used to construct a new Gel'fand triple called *Lévy–Gel'fand triple*, which exhibits the essentially infinite dimensional character of the Lévy Laplacian. By using the fact that the usual trace and the Lévy trace (being associated with an appropriate orthonormal system) coincide on the Lévy–Hilbert space, we prove a theorem asserting that the Lévy Laplacian and the Gross Laplacian coincide on the space of *Lévy test functions* as continuous linear operators.

1. Introduction

In 1922 P. Lévy [13] introduced an infinite dimensional Laplacian Δ_L on the space $L^2(0,1)$. Since then this Laplacian has attracted much attention because of its peculiar and unexpected properties, which only appear essentially in infinite dimensional analysis. In 1967 Gross [8] introduced another Laplacian Δ_G on an abstract Wiener space as a natural infinite dimensional analogue of the finite dimensional Laplacian and studied potential theory associated with Δ_G. These two infinite dimensional Laplacians have been further studied by many authors in various aspects, see e.g. [2, 3, 5, 9, 12, 15] and references cited therein.

Within the white noise framework, the Gross Laplacian Δ_G has been formulated by Kuo in [11] as a continuous linear operator acting on test white noise functions, while the Lévy Laplacian Δ_L acts on a domain in the space of generalized white noise functionals and vanishes on the domain of the Gross Laplacian. Moreover, the Gross Laplacian is independent of the choice of an orthonormal basis for a Hilbert space. But the Lévy Laplacian depends on a given orthonormal basis and is usually defined with respect to an equally dense and uniformly bounded orthonormal basis. This drastic difference between Δ_G and Δ_L proves the necessity of investigating a deep relationship between them. Some results in this direction were first obtained in Kuo et al. [12], see also [9] and [11].

Recently, Accardi et al. [1] developed a quantum approach to Laplace operators based on the von Neumann algebras arising from representations of the CCR. One of the completely new achievement of this work is the unification, in the quantum domain, of the Gross and Lévy Laplacians. It is proved that the difference between them is reduced to the choice of the "multiplicity space" of the white noise which

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is a Hilbert space on distributions. Being Motivated by these new findings, our goal in this paper is to investigate this relationship between infinite dimensional Laplacians in analytical context. In Section 2, we summarize basic definitions and results on Gel'fand triples. In Section 3, we construct the *Lévy–Hilbert space* and give some properties. In particular, we prove that the usual trace and the Lévy trace coincide on this Hilbert space. In Section 4, we study the Lévy Laplacian and prove a somewhat unexpected theorem stating that it coincides with the Gross Laplacian on the space of Lévy test functions.

2. Basic Gel'fand Triples

Let H be an infinite dimensional real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|_0$. Let D be an operator on H such that

$$De_n = \lambda_n e_n, \quad n = 1, 2, \ldots, \tag{2.1}$$

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for H and $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$.

For each $p \in \mathbb{R}$ define

$$|\xi|_p^2 = \sum_{n=1}^{\infty} \langle \xi, e_n \rangle^2 \lambda_n^{2p} = |D^p \xi|_0^2 \ , \quad \xi \in H.$$

For $p \geq 0$, let X_p be the Hilbert space consisting of all $\xi \in H$ with $|\xi|_p < \infty$ and let X_{-p} be the completion of H with respect to the norm $|\cdot|_{-p}$. Since D^{-1} is of Hilbert–Schmidt type, identifying H with its dual space, we come to the real standard Gel'fand triple

$$X := \operatorname*{proj-lim}_{p \to \infty} X_p \ \subset \ H \ \subset \ \operatorname*{ind-lim}_{p \to \infty} X_{-p} =: X'. \tag{2.2}$$

Being compatible to the inner product of H, the canonical bilinear form on $X' \times X$ is also denoted by $\langle \cdot, \cdot \rangle$. An important example of such triple is given by the following white noise triple

$$X \equiv S(\mathbb{R}) \ \subset \ H \equiv L^2(\mathbb{R}, dt) \ \subset \ S'(\mathbb{R}) \equiv X', \tag{2.3}$$

where $S(\mathbb{R})$ is the space of rapidly decreasing functions on $\mathbb{R}$, $L^2(\mathbb{R}, dt)$ is the Hilbert space of $\mathbb{R}$-valued square-integrable functions on $\mathbb{R}$, and $S'(\mathbb{R})$ is the space of tempered distributions. In this case, D is taken to be the operator $D = -\frac{d^2}{dt^2} + t^2 + 1$ and e_n's are the Hermite functions given by

$$e_n(t) = \frac{1}{\sqrt{\sqrt{\pi} 2^{n-1}(n-1)!}} H_{n-1}(t) e^{-t^2/2}, \quad n = 1, 2, \ldots,$$

where $H_n(t) = (-1)^n e^{t^2} \left(\frac{d}{dt}\right)^n e^{-t^2}$ is the Hermite polynomial of degree n. Then

$$De_n = 2n e_n, \quad n = 1, 2, \ldots,$$

i.e., $\lambda_n = 2n$, $n = 1, 2, \ldots$ Moreover, for every $\xi \in L^2(\mathbb{R}, dt)$, its Hilbertian norm is given by

$$|\xi|_p = \left(\sum_{n=1}^{\infty} (2n)^{2p} \langle \xi, e_n \rangle^2 \right)^{\frac{1}{2}}.$$

Let $N = X + iX$ and $N_p = X_p + iX_p$, $p \in \mathbb{R}_+$, be the complexifications of X and X_p, respectively. For $n \in \mathbb{N}_0$ we denote by $N^{\widehat{\otimes} n}$ the n-fold symmetric tensor product of N equipped with the π-topology and by $N_p^{\widehat{\otimes} n}$ the n-fold symmetric Hilbertian tensor product of N_p. We will preserve the notation $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $N_p^{\widehat{\otimes} n}$ and $N_{-p}^{\widehat{\otimes} n}$, respectively.

Let θ be a Young function, namely, it is a continuous, convex, and increasing function defined on $\mathbb{R}_+$ such that $\theta(0) = 0$ and $\lim_{x\to\infty} \frac{\theta(x)}{x} = \infty$. We define the conjugate function θ^* of θ by

$$\theta^*(x) = \sup_{t\geq 0} \big(tx - \theta(t)\big), \quad x \geq 0.$$

For a Young function θ, we denote by $\mathcal{F}_\theta(N')$ the space of holomorphic functions on N' with exponential growth of order θ and of minimal type. Similarly, let $\mathcal{G}_\theta(N)$ denote the space of holomorphic functions on N with exponential growth of order θ and of arbitrary type. Moreover, for each $p \in \mathbb{R}$ and $m > 0$, define $\mathrm{Exp}(N_p, \theta, m)$ to be the space of entire functions f on N_p satisfying the condition:

$$\|f\|_{\theta,p,m} = \sup_{x\in N_p} |f(x)| e^{-\theta(m|x|_p)} < \infty.$$

Then the spaces $\mathcal{F}_\theta(N')$ and $\mathcal{G}_\theta(N)$ can be represented as

$$\mathcal{F}_\theta(N') \;=\; \bigcap_{p\in\mathbb{N}_0,\, m>0} \mathrm{Exp}(N_{-p}, \theta, m),$$

$$\mathcal{G}_\theta(N) \;=\; \bigcup_{p\in\mathbb{N}_0,\, m>0} \mathrm{Exp}(N_p, \theta, m),$$

equipped with the projective limit and the inductive limit topologies, respectively. The space $\mathcal{F}_\theta(N')$ is called the space of *test functions* on N'. Its dual space $\mathcal{F}'_\theta(N')$ equipped with the strong topology is called the space of *distributions* on N'.

For $p \in \mathbb{R}_+$ and $m > 0$, we define the Hilbert spaces

$$F_{\theta,m}(N_p) = \Big\{\vec{\varphi} = (\varphi_n)_{n=0}^\infty \,;\, \varphi_n \in N_p^{\widehat{\otimes} n},\, \sum_{n=0}^\infty \theta_n^{-2} m^{-n} |\varphi_n|_p^2 < \infty\Big\},$$

$$G_{\theta,m}(N_{-p}) = \Big\{\vec{\Phi} = (\Phi_n)_{n=0}^\infty \,;\, \Phi_n \in N_{-p}^{\widehat{\otimes} n},\, \sum_{n=0}^\infty (n!\theta_n)^2 m^n |\Phi_n|_{-p}^2 < \infty\Big\},$$

where $\{\theta_n\}_{n=0}^\infty$ is the sequence defined by

$$\theta_n = \inf_{r>0} e^{\theta(r)}/r^n, \quad n \in \mathbb{N}_0. \tag{2.4}$$

Put

$$F_\theta(N) \;=\; \bigcap_{p\in\mathbb{N}_0, m>0} F_{\theta,m}(N_p),$$

$$G_\theta(N') \;=\; \bigcup_{p\in\mathbb{N}_0, m>0} G_{\theta,m}(N_{-p}).$$

The space $F_\theta(N)$ equipped with the projective limit topology is a nuclear Frechét space [6]. The space $G_\theta(N')$ carries the dual topology of $F_\theta(N)$ with respect to the $\mathbb{C}$-bilinear pairing given by

$$\langle\langle \vec{\Phi}, \vec{\varphi} \rangle\rangle = \sum_{n=0}^{\infty} n! \langle \Phi_n, \varphi_n \rangle,$$

where $\vec{\Phi} = (\Phi_n)_{n=0}^{\infty} \in G_\theta(N')$ and $\vec{\varphi} = (\varphi_n)_{n=0}^{\infty} \in F_\theta(N)$. It was proved in [6] that the following *Taylor map*

$$\mathcal{T}\colon\ \varphi \longmapsto \Big(\frac{1}{n!}\varphi^{(n)}(0)\Big)_{n=0}^{\infty}$$

is a topological isomorphism from $\mathcal{F}_\theta(N')$ onto $F_\theta(N)$. The Taylor map $\mathcal{T}$ is also a topological isomorphism from $\mathcal{G}_{\theta^*}(N)$ onto $G_\theta(N')$. The action of a distribution $\Phi \in \mathcal{F}_\theta'(N')$ on a test function $\varphi \in \mathcal{F}_\theta(N')$ can be expressed in terms of the Taylor map as follows:

$$\langle\langle \Phi, \varphi \rangle\rangle = \langle\langle \vec{\Phi}, \vec{\varphi} \rangle\rangle,$$

where $\vec{\Phi} = (\mathcal{T}^*)^{-1}\Phi$ and $\vec{\varphi} = \mathcal{T}\varphi$.

It is easy to see that for each $\xi \in N$, the exponential function

$$e_\xi(z) = e^{\langle z, \xi \rangle}, \quad z \in N',$$

is a test function in the space $\mathcal{F}_\theta(N')$ for any Young function θ. Thus we can define the *Laplace transform* of a distribution $\Phi \in \mathcal{F}_\theta'(N')$ by

$$(\mathcal{L}\Phi)(\xi) \equiv \widehat{\Phi}(\xi) = \langle\langle \Phi, e_\xi \rangle\rangle, \quad \xi \in N.$$

From the paper [6], we have the duality theorem which says that the Laplace transform is a topological isomorphism from $\mathcal{F}_\theta'(N')$ onto $\mathcal{G}_{\theta^*}(N)$.

3. Lévy–Gel'fand Triple

Consider the Gel'fand triple in Equation (2.3). Let $\mathcal{A}_L$ denote the subset of all $x \in X'$ such that

$$\|x\|_L^2 := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \langle x, e_k \rangle^2 < \infty,$$

where $\{e_k\}_{k=1}^{\infty} \subset X$ is the orthonormal basis for H given by Equation (2.1). Obviously, $\mathcal{A}_L$ depends on $\{e_k\}_{k=1}^{\infty}$. Moreover, observe that $\mathcal{A}_L$ is not a vector space since it is not closed under the addition. Thus the bilinear form

$$\langle x, y \rangle_L := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \langle x, e_k \rangle \, \langle y, e_k \rangle \tag{3.1}$$

does not define an inner product on $\mathcal{A}_L$.

Proposition 3.1. *For any $0 \le \alpha \le \frac{1}{2}$, we have $X_{-\alpha} \subset \mathcal{A}_L$. Moreover, there is no $\alpha > \frac{1}{2}$ such that $X_{-\alpha} \subset \mathcal{A}_L$.*

Proof. Let $\alpha \in \left[0, \frac{1}{2}\right]$ be given. By direct calculation, we can show that

$$\sup_{n\geq 1}\left(\frac{(2n)^{2\alpha}}{n}\right) = 2^{2\alpha}.$$

Hence, for $x \in X_{-\alpha}$, we have

$$\begin{aligned}
\|x\|_L^2 &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\langle x, e_k\rangle^2 \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{(2k)^{2\alpha}}{k}\right)\left(\frac{k}{(2k)^{2\alpha}}\right)\langle x, e_k\rangle^2 \\
&\leq 2^{2\alpha}\sum_{k=1}^{\infty}(2k)^{-2\alpha}\langle x, e_k\rangle^2 = 2^{2\alpha}|x|_{-\alpha}^2,
\end{aligned}$$

which proves the first assertion. Next let $\alpha > \frac{1}{2}$, $\beta \in \mathbb{R}$ and $x \in X'$ be such that

$$\langle x, e_k\rangle^2 = \frac{(2k)^{2\alpha}}{k^\beta}, \quad \forall\, k = 1, 2, \ldots .$$

Choose $\beta > 1$ so that $x \in X_{-\alpha}$. Then

$$\|x\|_L^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\langle x, e_k\rangle^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(2k)^{2\alpha}}{k^\beta}.$$

Then choose n large enough to ensure that

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\frac{(2k)^{2\alpha}}{k^\beta} &= \frac{2^{2\alpha}}{n}\sum_{k=1}^{n}k^{2\alpha-\beta} \approx \frac{2^{2\alpha}}{n}\int_1^n t^{2\alpha-\beta}dt \\
&= \frac{2^{2\alpha}}{2\alpha-\beta+1}\left(n^{2\alpha-\beta} - \frac{1}{n}\right) \longrightarrow \infty
\end{aligned}$$

as $n \to \infty$ for any choice of $\beta \in (1, 2\alpha)$. This proves that $x \notin \mathcal{A}_L$. □

From the above Lemma one can see that $\alpha = \frac{1}{2}$ is the largest $\alpha \geq 0$ such that $X_{-\alpha} \subset \mathcal{A}_L$. Therefore, the minimal linear vector space E_L such that $X_{-1/2} \subset E_L \subset \mathcal{A}_L$ is $E_L = X_{-1/2}$. Equipped with the bilinear form in Equation (3.1), E_L is a real pre-Hilbert space.

Definition 3.2. The completion of the factor space $E_L/_{\|\cdot\|_L}$ with respect to $\|\cdot\|_{L,0} \equiv \|\cdot\|_L$ is called the *Lévy–Hilbert space* associated with an orthonormal basis $\{e_k\}_{k=1}^\infty$. We shall denote this Hilbert space by H_L.

Let D_L be a positive densely defined self-adjoint operator on H_L with Hilbert–Schmidt inverse and discreet spectrum so that inf-spec$(D_L) > 1$. As in Section 2, we can consider an orthonormal basis $\{\varepsilon_k\}_{k=1}^\infty$ for H_L and a sequence $\{\Lambda_{L,k}\}_{k=1}^\infty$ of real numbers such that

$$\begin{aligned}
&1 < \Lambda_{L,1} < \Lambda_{L,2} < \cdots , \\
&D_L\, \varepsilon_k = \Lambda_{L,k}\, \varepsilon_k\, , \ \ k = 1, 2, \ldots .
\end{aligned}$$

Applying the standard construction in Section 2, we get a real Gel'fand triple

$$X_L := \operatorname*{proj-lim}_{p\to\infty} X_{L,p} \subset H_L \subset \operatorname*{ind-lim}_{p\to\infty} X_{L,-p} =: X'_L, \tag{3.2}$$

which is called a *Lévy–Gel'fand triple*. Its complexification is given by

$$N_L := \operatorname*{proj-lim}_{p\to\infty} N_{L,p} \subset K_L \subset \operatorname*{ind-lim}_{p\to\infty} N_{L,-p} =: N'_L. \tag{3.3}$$

For each $p \in \mathbb{R}$, define a norm on N_L by

$$\|z\|_{L,p} := \|D_L^p z\|_{L,0} = \left(\sum_{k=1}^{\infty} \Lambda_{L,k}^{2p} |\langle z, \varepsilon_k\rangle|^2\right)^{1/2}, \quad z \in N_L.$$

The canonical $\mathbb{C}$-bilinear form on $N'_L \times N_L$ will also be denoted $\langle \cdot, \cdot\rangle_L$. As in Section 2, we construct the nuclear spaces $\mathcal{F}_\theta(N'_L)$, $\mathcal{G}_{\theta^*}(N_L)$, $F_\theta(N_L)$, and $G_\theta(N'_L)$. Moreover, we keep the same notation for the Laplace transform $\mathcal{L}$ and the Taylor transform $\mathcal{T}$. In particular, the topological isomorphisms induced by $\mathcal{L}$ and $\mathcal{T}$ remain valid.

The usual trace τ is defined by

$$\langle \tau, x\otimes y\rangle := \langle x, y\rangle_L = \sum_{k=1}^{\infty} \langle x, \varepsilon_k\rangle_L \, \langle y, \varepsilon_k\rangle_L, \quad x, y \in N_L.$$

Lemma 3.3. *For any orthonormal basis $\{\varepsilon_k\}_{k=1}^{\infty}$ for H_L, we have the equality $\tau = \sum_{k=1}^{\infty} \varepsilon_k \otimes \varepsilon_k$ with respect to the strong topology of $(N_L \otimes N_L)'$. Moreover, for any $\alpha \geq \frac{1}{2}$, the following inequality holds*

$$\|\tau\|_{L,-\alpha} \leq \|D_L^{-1}\|_{op}^{2\alpha-1} \, \|D_L^{-1}\|_{HS}. \tag{3.4}$$

Proof. Since N_L is a nuclear space imbedded in K_L, there exists $\beta > 0$ such that the inclusion map $i_{\beta,0} : N_{L,\beta} \to N_{L,0}$ is a Hilbert–Schmidt operator. It follows that the inclusion map

$$i^*_{\beta,0} : N_{L,0} \equiv N'_{L,0} \longrightarrow N_{L,-\beta} \equiv N'_{L,\beta}$$

is also a Hilbert–Schmidt operator. Therefore, we have

$$\|i^*_{\beta,0}\|^2_{HS} = \sum_{k=1}^{\infty} \|\varepsilon_k\|^2_{L,-\beta} < \infty. \tag{3.5}$$

Now for any $n \geq 1$, put

$$\tau_n = \sum_{k=1}^{n} \varepsilon_k \otimes \varepsilon_k,$$

which belongs to $N_{L,0} \otimes N_{L,0}$. Then

$$\langle \tau - \tau_n, \omega\rangle_L = \sum_{k=n+1}^{\infty} \langle \varepsilon_k \otimes \varepsilon_k, \omega\rangle_L, \quad \omega \in N_L \otimes N_L.$$

Hence for any bounded subset B of $N_L \otimes N_L$, we have

$$\begin{aligned}\sup_{\omega\in B}|\langle \tau-\tau_n,\omega\rangle| &\le \sup_{\omega\in B}\sum_{k=n+1}^{\infty}|\langle \varepsilon_k\otimes\varepsilon_k,\omega\rangle_L| \\ &\le \left(\sup_{\omega\in B}\|\omega\|_{L,\beta}\right)\sum_{k=n+1}^{\infty}\|\varepsilon_k\otimes\varepsilon_k\|_{L,-\beta} \\ &= \left(\sup_{\omega\in B}\|\omega\|_{L,\beta}\right)\sum_{k=n+1}^{\infty}\|\varepsilon_k\|_{L,-\beta}^2,\end{aligned}$$

which converges to 0 by Equation (3.5) as $n\to\infty$. This shows that $\lim_{n\to\infty}\tau_n=\tau$ with respect to the strong dual topology of $(N_L\otimes N_L)'$. Next let $\alpha\ge\frac{1}{2}$. Then

$$\begin{aligned}\|\tau\|_{L,-\alpha}^2 &= \sum_{k=1}^{\infty}\|(D_L^{-\alpha}\varepsilon_k)\otimes(D_L^{-\alpha}\varepsilon_k)\|_L^2 \\ &= \sum_{k=1}^{\infty}\|D_L^{-\alpha}\varepsilon_k\|_L^4=\sum_{k=1}^{\infty}\Lambda_{L,k}^{-4\alpha} \\ &\le \Lambda_{L,1}^{-4(\alpha-\frac{1}{2})}\sum_{k=1}^{\infty}\Lambda_{L,k}^{-2}.\end{aligned}$$

This implies Equation (3.4). □

Definition 3.4. The *Lévy trace* τ_L associated with an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for H is defined to be the generalized function satisfying the equality

$$\langle x\otimes y,\tau_L\rangle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\langle x,e_k\rangle\,\langle y,e_k\rangle,\quad x,y\in N_L,$$

whenever the limit exists.

Theorem 3.5. *The following assertions hold:*

(1) *The usual trace τ and the Lévy trace τ_L associated with the orthonormal basis $\{e_k\}_{k=1}^{\infty}$ coincide on K_L.*
(2) *For any $\alpha\ge\frac{1}{2}$, the Lévy trace τ_L belongs to $N_{L,-\alpha}\otimes N_{L,-\alpha}$.*
(3) *For any $\omega\in(N\otimes N)'\cap(N_L\otimes N_L)$, we have*

$$\langle\omega,\tau_L\rangle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\langle\omega,e_k\otimes e_k\rangle. \tag{3.6}$$

Proof. For $x,y\in K_L$, we have $\langle x\otimes y,\tau_L\rangle=\langle x,y\rangle_L=\langle\tau,x\otimes y\rangle_L$. Hence

$$\tau_L=\tau=\sum_{n=1}^{\infty}\varepsilon_n\otimes\varepsilon_n$$

on K_L. This proves the first assertion. For $\alpha\ge\frac{1}{2}$, we have

$$\|\tau_L\|_{L,-\alpha}^2=\sum_{n=1}^{\infty}\Lambda_{L,n}^{-4\alpha}\,\|\varepsilon_n\otimes\varepsilon_n\|_L^2\le\sum_{n=1}^{\infty}\Lambda_{L,n}^{-2}\,\|\varepsilon_n\otimes\varepsilon_n\|_L^2=\|D_L^{-1}\|_{HS}^2<\infty,$$

which proves the second statement. The third assertion is a simple consequence of the definition of τ_L. □

4. Infinite Dimensional Laplacians

In this section we will assume that the underlying Young function θ satisfies an additional condition that

$$\limsup_{r\to\infty} \frac{\theta(r)}{r^2} < \infty,$$

which is equivalent to

$$\liminf_{r\to\infty} \frac{\theta^*(r)}{r^2} > 0. \tag{4.1}$$

Under this additional condition, we have the following Gel'fand triple

$$\mathcal{F}_\theta(N'_L) \subset L^2(X'_L, \mu) \subset \mathcal{F}'_\theta(N'_L),$$

where μ is the standard Gaussian measure on the space X'_L, i.e., its characteristic function is given by

$$\int_{X'_L} e^{i\langle \omega, z\rangle}\, d\mu(\omega) = e^{-\frac{1}{2}\|z\|^2_{L,0}}, \quad z \in X_L.$$

The existence of such a probability measure is guaranteed by the Minlos theorem [7]. This measure μ is unique.

Let E be a real nuclear space and let $E_{\mathbb{C}} = E + iE$ be its complexification. A function $F : E \to \mathbb{R}$ is said to be *twice differentiable* at $\xi \in E$ if there exist $F'(\xi) \in E'$ and $F''(\xi) \in \mathcal{L}(E, E')$, the space of continuous linear operators from E into E', such that

$$F(\xi + \eta) = F(\xi) + \langle F'(\xi), \eta\rangle + \frac{1}{2}\langle F''(\xi)\eta, \eta\rangle + o(\eta), \quad \eta \in E,$$

where $o(\eta)$ is a quantity such that $\lim_{t\to 0} \frac{o(t\eta)}{t^2} = 0$ for all $\eta \in E$.

We use $C^2(E)$ to denote the space of everywhere twice differentiable functions $F : E \to \mathbb{R}$ such that the derivatives

$$\xi \longmapsto F'(\xi) \in E', \quad \xi \longmapsto F''(\xi) \in \mathcal{L}(E, E')$$

are continuous. In view of the Nuclear Kernel Theorem [7, 16], we can use the common symbol $F''(\xi)$ for all of the following equalities:

$$\langle F''(\xi)\eta, \eta\rangle = \langle F''(\xi), \eta \otimes \eta\rangle = F''(\xi)(\eta, \eta) = D_\eta D_\eta F(\xi),$$

where D_η is the Fréchet derivative in the direction η, i.e.,

$$(D_\eta F)(\xi) = \lim_{\lambda\to 0} \frac{F(\xi + \lambda\eta) - F(\xi)}{\lambda}.$$

A $\mathbb{C}$-valued function F on E is said to belong to $C^2(E)$ if both its real and imaginary parts belong to $C^2(E)$. In that case, we have $F'(\xi) \in E'_{\mathbb{C}}$ and $F''(\xi) \in (E_{\mathbb{C}} \otimes E_{\mathbb{C}})'$.

For a function $\Psi \in C^2(N'_L)$, the Gross Laplacian $\Delta_G\Psi(z)$ of Ψ is defined in [8, 10] to be the function

$$\Delta_G\Psi(z) = \mathrm{trace}_{K_L}\, \Psi''(z), \quad z \in N'_L, \tag{4.2}$$

where K_L is given by Equation (3.3). It is easy to check that

$$\Delta_G \Psi(z) = \sum_{k=1}^{\infty} \langle \Psi''(z), \varepsilon_k \otimes \varepsilon_k \rangle, \quad z \in N_L', \tag{4.3}$$

where $\{\varepsilon_k\}_{k=1}^{\infty} \subset N_L$ is any orthonormal basis for H_L. Note that the Gross Laplacian $\Delta_G \Psi$ is independent of the orthonormal basis $\{\varepsilon_k\}_{k=1}^{\infty}$. Moreover, if the Taylor expansion of $\Psi \in \mathcal{F}_\theta(N_L')$ is given by

$$\Psi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \Psi_n \rangle,$$

then the Gross Laplacian $\Delta_G \Psi(z)$ exists at every $z \in N_L'$. Moreover, it follows from Equation (4.3) that

$$\begin{aligned} \Delta_G \Psi(z) &= \sum_{n=0}^{\infty} (n+2)(n+1) \langle z^{\otimes n}, \tau \widehat{\otimes}_2 \Psi_{n+2} \rangle \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) \langle z^{\otimes n}, \tau_L \widehat{\otimes}_2 \Psi_{n+2} \rangle, \end{aligned} \tag{4.4}$$

where $\widehat{\otimes}_2$ is the symmetric 2-contraction tensor product [14]. For more information on the Gross Laplacian, see [4, 9, 11, 14].

Let $\{e_k\}_{k=1}^{\infty} \subset N$ be an orthonormal basis for H. For a function $\Psi \in C^2(N_L')$, the *Lévy Laplacian* $\Delta_L \Psi(z)$ is defined to be the function

$$\Delta_L \Psi(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \langle \Psi''(z), e_k \otimes e_k \rangle, \quad z \in N_L',$$

when the limit exists.

Theorem 4.1. *The Lévy Laplacian Δ_L, associated with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for H, coincides with the Gross Laplacian Δ_G, as continuous linear operators on $\mathcal{F}_\theta(N_L')$. Moreover, for any $\alpha \geq \frac{1}{2}$ and $m > 0$, there exist $\beta > \alpha$ and $\delta > 0$ such that*

$$\|\Delta_L \Psi\|_{\theta,-\alpha,\delta} \leq C \, \|\tau_L\|_{L,-\alpha} \, \|\Psi\|_{\theta,-\beta,m}, \quad \forall \, \Psi \in \mathcal{F}_\theta(N_L'), \tag{4.5}$$

with the constant C given by

$$C = \sum_{n=0}^{\infty} (n+2)! \, (n+1) \, (n+2) \, \theta_{n+2} \, \theta_n \left(\frac{e\, m}{\delta} \, \|i_{\beta,\alpha}\|_{HS} \right)^{n+2} < \infty, \tag{4.6}$$

where $\{\theta_n\}_{n=0}^{\infty}$ is the sequence defined by Equation (2.4).

Proof. Let $\Psi \in \mathcal{F}_\theta(N_L')$ be represented by $\overrightarrow{\Psi} = (\Psi_n)_{n=0}^{\infty} \in F_\theta(N_L)$. By direct computation, for any $z \in N_L'$ and $k \in \mathbb{N}$, we have

$$\langle \Psi''(z), e_k \otimes e_k \rangle = \sum_{n=0}^{\infty} (n+2)(n+1) \left\langle z^{\otimes n}, (e_k \otimes e_k) \widehat{\otimes}_2 \Psi_{n+2} \right\rangle.$$

Then apply Equation (3.6) to get

$$\begin{aligned}\Delta_L \Psi(z) &= \lim_{p\to\infty} \frac{1}{p}\sum_{k=1}^{p}\sum_{n=0}^{\infty}(n+2)(n+1)\left\langle z^{\otimes n}, (e_k\otimes e_k)\widehat{\otimes}_2\Psi_{n+2}\right\rangle \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)\left\langle z^{\otimes n}, \tau_L\widehat{\otimes}_2\Psi_{n+2}\right\rangle. \qquad (4.7)\end{aligned}$$

Equations (4.4) and (4.7) show that Δ_L and Δ_G coincide on $\mathcal{F}_\theta(N_L')$.

Now let $\alpha \geq \frac{1}{2}$. Then

$$|\Delta_L\Psi(z)| \leq \sum_{n=0}^{\infty}(n+2)(n+1)\|\tau_L\|_{L,-\alpha}\,\|z\|_{L,-\alpha}^n\,\|\Psi_{n+2}\|_{L,\alpha}.$$

By Lemma 1 of the paper [6], for any $n \geq 1$ and $\Psi_n \in N_L^{\widehat{\otimes} n}$, we have the following inequality

$$\|\Psi_n\|_{L,\alpha} \leq e^n n!\,\theta_n m^n \|\Psi\|_{\theta,-\beta,m}\|i_{\beta,\alpha}\|_{HS}^n, \quad \forall\, m\geq 0,\ \beta>\alpha.$$

It follows that

$$\begin{aligned}|\Delta_L\Psi(z)| \;&\leq\; \|\tau_L\|_{L,-\alpha}\|\Psi\|_{\theta,-\beta,m} \\ &\quad\times \sum_{n=0}^{\infty}(n+2)(n+1)\,\|z\|_{L,-\alpha}^n\, e^{n+2}(n+2)!\,\theta_{n+2}m^{n+2}\|i_{\beta,\alpha}\|_{HS}^{n+2}.\end{aligned}$$

Thus for any $\delta > 0$,

$$\begin{aligned}|\Delta_L\Psi(z)\,e^{-\theta(\delta\|z\|_{L,-\alpha})}| \;&\leq\; \|\tau_L\|_{L,-\alpha}\|\Psi\|_{\theta,-\beta,m} \\ &\quad\times \sum_{n=0}^{\infty}(n+2)!(n+1)\,(n+2)\,\theta_{n+2}\,\theta_n\left(\frac{e\,m}{\delta}\,\|i_{\beta,\alpha}\|_{HS}\right)^{n+2},\end{aligned}$$

which implies Equation (4.5) with C given by Equation (4.6). It remains to show that the constant C is a finite number.

Obviously, we have the inequalities

$$\theta_p\,\theta_q \;\leq\; 2^{p+q}\theta_{p+q}, \quad r^n e^{-\theta(r)} \leq \theta_n.$$

By the condition in Equation (4.1), there exist constants $a, b > 0$ such that

$$e^{\theta(r)} \leq a\,e^{br^2}, \quad r>0,$$

which yields that

$$\theta_{n+2} = \inf_{r>0}\left(\frac{e^{\theta(r)}}{r^{n+2}}\right) \leq \inf_{r>0}\left(\frac{a\,e^{br^2}}{r^{n+2}}\right) = a\left(\frac{2be}{n+2}\right)^{\frac{n+2}{2}}.$$

On the other hand, by the Stirling formula,

$$(n+2)! \;\leq\; e\sqrt{n+2}\,\left(\frac{n+2}{e}\right)^{n+2}.$$

Therefore, we have

$$(n+2)!\,\theta_{n+2}\,\theta_n \;\leq\; e\,a^2\theta_2^{-1}\sqrt{n+2}\,(4b)^{n+2}.$$

Thus for $\beta > \alpha$ and $m > 0$ such that $\frac{4bem}{\delta}\|i_{\beta,\alpha}\|_{HS} < 1$, we can conclude that

$$C = \sum_{n=0}^{\infty}(n+2)(n+1)\,(n+2)!\,\theta_{n+2}\,\theta_n\left(\frac{m\,e}{\delta}\,\|i_{\beta,\alpha}\|_{HS}\right)^{n+2} < \infty.$$

This completes the proof of the theorem. □

We need to pint out several important remarks and consequences concerning Theorem 4.1. As shown in the proof of Theorem 4.1, the Lévy Laplacian Δ_L is well-defined on the space $\mathcal{F}_\theta(N'_L)$. On the other hand, recall that from Equation (2.2) we have the Gel'fand triple

$$X \subset H \subset X'$$

and from Equation (3.2) we have the Lévy–Gel'fand triple

$$X_L \subset H_L \subset X'_L.$$

The Lévy Laplacian Δ_L, as a continuous linear operator on $\mathcal{F}_\theta(N'_L)$, depends on an orthonormal basis $\{e_n\}_{n=1}^\infty$ for the Hilbert space H of the Gel'fand triple $X \subset H \subset X'$. However, when we go one level up to the Lévy–Gel'fand triple $X_L \subset H_L \subset X'_L$, then the Lévy Laplacian Δ_L is independent of an orthonormal basis $\{\varepsilon_k\}_{k=1}^\infty$ for the Hilbert space H_L of the Lévy–Gel'fand triple $X_L \subset H_L \subset X'_L$. This is a consequence of the equality $\Delta_L = \Delta_G$ from Theorem 4.1 and the fact that the Gross Laplacian is independent of orthonormal bases in view of the definition in Equation (4.2).

Moreover, observe that the Lévy Laplacian Δ_L, as a continuous linear operator on $\mathcal{F}_\theta(N'_L)$, depends intrinsically on the Lévy–Gel'fand triple $X_L \subset H_L \subset X'_L$. Once the "initial space" H_L is fixed, it is quite natural that the Lévy Laplacian Δ_L is independent of an orthonormal basis for H_L as remarked in the previous paragraph. This fact is consistent with a crucial point for the quantum domain in the paper [1], where it is proved that we can unify quantum Laplacians once the "initial space" arising from the Lévy Laplacian is fixed.

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