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GENERATORS OF DYNAMICAL SYSTEMS ON HILBERT MODULES

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Abstract. We characterize the generators of dynamical systems on Hilbert modules as those generators of one-parameter groups of Banach space isometries which are ternary derivations. We investigate in how far a similar condition can be expressed in terms of generalized derivations.

1. Introduction

Let E be a Hilbert module over a C^*–algebra $\mathcal{B}$. A ***generalized unitary*** on E is a surjection u on E that fulfills

$$\langle ux, uy \rangle \ = \ \varphi(\langle x, y \rangle), \quad x, y \in E \tag{GU}$$

for some automorphism φ of $\mathcal{B}$. We will also say u is a φ–***unitary***. A ***generalized derivation*** of E is a densely defined linear map $\delta \colon E \supset \mathsf{dom}(\delta) \to E$ that fulfills

$$\delta(xb) \ = \ \delta(x)b + xd(b), \quad x \in \mathsf{dom}(\delta), b \in \mathsf{dom}(d) \tag{GD}$$

for some derivation $d \colon \mathcal{B} \supset \mathsf{dom}(d) \to \mathcal{B}$ of $\mathcal{B}$, in such a way that $\mathsf{dom}(\delta)$ is a right $\mathsf{dom}(d)$–module. We will also say δ is a d–***derivation***. A ***dynamical system*** on a Hilbert $\mathcal{B}$–module E is a strongly continuous one-parameter group $u = \big(u_t\big)_{t\in\mathbb{R}}$ of generalized unitaries. Abbaspour, Moslehian and Niknam [2, 1] defined dynamical systems on Hilbert modules and started the investigation of their generators. They showed that the generator of a dynamical system on a full Hilbert $\mathcal{B}$–module is a generalized derivation (of course, with respect to the generator of the associated C^*–dynamical system $\varphi = \big(\varphi_t\big)_{t\in\mathbb{R}}$ on $\mathcal{B}$ such that every u_t is a φ_t–unitary). However, a bounded generalized derivation on E with respect to a bounded $*$–derivation on $\mathcal{B}$ need not generate a dynamical system on E; see Example 3.14. Even if we require that a closed and densely defined linear map δ generates a group of Banach space isometries on E, the condition that δ be a generalized derivation with respect to a $*$–derivation d of $\mathcal{B}$ turns out to be sufficient, only under algebraic *and* analytic conditions on the domain of d; see Theorems 3.12 and 3.18. These conditions depend manifestly on the domain of δ and, therefore, cannot be understood in terms of d alone.

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It is the main scope of these notes to find a better algebraic condition that depends only on δ. This condition will be in terms of *ternary* maps. Ternary maps have the advantage that they their definition refers exclusively to the module E, not to the C^*–algebra $\mathcal{B}$.

A ***ternary automorphism*** of E is a bijection u on E that fulfills

$$u(x\langle y,z\rangle) \;=\; (ux)\langle uy,uz\rangle, \quad x,y,z\in E. \tag{TU}$$

In Section 2 we show that the generalized isometries from a *full* Hilbert $\mathcal{B}$–module E to a Hilbert $\mathcal{C}$–module F are exactly the ternary homomorphims. As a special case this includes the statement that the generalized unitaries on a full Hilbert $\mathcal{B}$–module are exactly its ternary automorphisms. This frees the discussion from worrying about existence of an automorphism φ of $\mathcal{B}$. In fact, the main problem in the proof is to show that existence of such an automorphism is automatic. Consequently, the dynamical systems on a full Hilbert module E are exactly the strongly continuous one-parameter groups of ternary automorphisms.

A ***ternary derivation*** of E is a densely defined linear map $\delta\colon E\supset \mathsf{dom}(\delta)\to E$ that fulfills

$$\delta(x\langle y,z\rangle) \;=\; \delta(x)\langle y,z\rangle + x\langle\delta(y),z\rangle + x\langle y,\delta(z)\rangle, \quad x,y,z\in E \tag{TD}$$

where $\mathsf{dom}(\delta)\left\langle \mathsf{dom}(\delta)\,,\,\mathsf{dom}(\delta)\right\rangle\subset\mathsf{dom}(\delta)$, that is, $\mathsf{dom}(\delta)$ is invariant under the ***ternary product*** $(x,y,z)\mapsto x\langle y,z\rangle$. In Section 3 we show that every ternary derivation of a full Hilbert module is a generalized derivation, while the converse fails. Generators of dynamical systems are always ternary derivations. We show also a sort of converse: If a linear densely defined map on E is the generator of a C_0–***group*** (that is, a strongly continuous one-parameter group of Banach space isometries) on E, then this group is a dynamical system, if and only if δ is a ternary derivation. This reduces the problem of characterizing the generators to the well-known general analytic criteria based on Hille-Yosida theory that state when δ is the generator of a C_0–group, and the purely algebraic question whether δ is a ternary derivation. We see that we have a satisfactory description of generators of dynamical systems on Hilbert modules in terms of ternary derivations, while the larger part of Section 3 illustrates that similar statements in terms of generalized derivations are possible only under rather hard analytical hypothesis.

We note, too, that the condition that the Hilbert $\mathcal{B}$–module E be full is not critical as long as we speak about ternary maps. Restrictions that arise in the case of generalized unitaries *on* a Hilbert module E when $\mathcal{B}$ is not chosen minimal have been analyzed in Skeide [8].

The scope of these notes is to characterize the generators of dynamical systems on Hilbert modules, and that scope does not depend on potential applications. We prefer to outline a view of these potential applications mainly for stochastic analysis, in particular for stochastic analysis in quantum probability, in Section 4.

Conventions and notations. A ***pre-Hilbert*** $\mathcal{B}$***–module*** is a right module E over a (pre-)C^*–algebra, with a sesquilinear inner product $\langle\bullet,\bullet\rangle\colon E\times E\to\mathcal{B}$ that satisfies $\langle x,yb\rangle=\langle x,y\rangle b$ $(x,y\in E; b\in\mathcal{B})$, $\langle x,x\rangle\geq 0$ and $\langle x,x\rangle=0\Rightarrow x=0$ $(x\in E)$. A ***Hilbert*** $\mathcal{B}$***–module*** is a pre-Hilbert $\mathcal{B}$–module that is complete in the ***norm*** $\|x\| := \sqrt{\langle x, x\rangle}$. A pre-Hilbert $\mathcal{B}$–module E is ***full***, if the ***range ideal*** $\mathcal{B}_E := \mathsf{span}\langle E, E\rangle$ is dense in $\mathcal{B}$.

By $\mathcal{B}^a(E)$ and $\mathcal{K}(E)$ we denote the C^*–algebras of all ***adjointable*** operators and of all ***compact*** operators, respectively, on E, where $\mathcal{K}(E)$ is the completion of the pre-C^*–algebra $\mathcal{F}(E)$ of ***finite-rank*** operators which is spanned by the ***rank-one*** operators $xy^*\colon z \mapsto x\langle y, z\rangle$.

2. Generalized isometries *versus* ternary homomorphisms

Unitaries on or between Hilbert modules are inner product preserving surjections. For isometries, surjectivity is missing. For generalized unitaries on a Hilbert module in (GU) the condition that the surjection preserves inner products is modified to that it preserves inner products up to a fixed automorphism of the algebra. When the unitary is between different Hilbert modules, it is not even necessary that these are modules over the same algebra. In this section we investigate generalized isometries between Hilbert modules.

Let E be a Hilbert $\mathcal{B}$–module and let F be a Hilbert $\mathcal{C}$–module. A ***generalized isometry*** from E to F is a map $u\colon E \to F$ that fulfills (GU) for some homomorphism $\varphi\colon \mathcal{B} \to \mathcal{C}$. We will also say u is a φ–***isometry***. Calculating the norm of $ux + (uy)\varphi(b) - u(x + yb)$, we find that every φ–isometry u is φ–***linear***, that is, $ux + (uy)\varphi(b) = u(x + yb)$ $(x, y \in E; b \in \mathcal{B})$. Inserting scalar multiples of an approximate unit, we see that φ–linearity implies $\mathbb{C}$–linearity. Obviously, a φ–isometry has norm 1, unless $\varphi \restriction \mathcal{B}_E$ is 0.

A ***ternary homomorphism*** from E to F is a map $u\colon E \to F$ that fulfills (TU). Obviously, every φ–isometry is a ternary homomorphism. It is our scope to show that, at least if E is full, then every ternary homomorphism is also a φ–isometry.

Theorem 2.1. *For a map u from a full Hilbert $\mathcal{B}$–module E to a Hilbert $\mathcal{C}$–module F the following statements are equivalent:*

(1) *u is a generalized isometry.*
(2) *u is a ternary homomorphism.*

Proof. Given a ternary homomorphism u from a full Hilbert $\mathcal{B}$–module E to a Hilbert $\mathcal{C}$–module F, it is our job to find a homomorphism $\varphi\colon \mathcal{B} \to \mathcal{C}$ such that u fulfills (GU). First, we observe that for full E such a homomorphism is determined uniquely by (GU). The attempt to define the homomorphism φ first on the pre-C^*–algebra $\mathcal{B}_E$ by $\langle x, y\rangle \mapsto \langle ux, uy\rangle$ and then to show that it is bounded by appealing to Muhly, Skeide and Solel [6, Corollary 1.20] has been put into practise in [1] under the assumption that u is linear. Here we follow a different road.

Let us observe that if a suitable φ exists, then u must be φ–linear. So, necessarily we must have $(ux)\varphi(b) = u(xb)$ for all $x \in E$. We use this property to define a left action $\varphi(b)$ of $b \in \mathcal{B}$ on the pre-C^*–algebra $\mathcal{C}_{uE} := \mathsf{span}\langle uE, uE\rangle$ considered pre-Hilbert module over itself in the usual way, that is, with inner product $\langle c, c'\rangle = c^*c'$ and right action simply by multiplication. We put

$$\varphi(b)\langle ux, uy\rangle \ := \ \langle u(xb^*), uy\rangle$$

and we must show, in a first step, that this well-defines a homomorphism into $\mathcal{B}^a(\overline{\mathcal{C}_{uE}})$. As, clearly, $\varphi(b)\varphi(b')\langle ux, uy\rangle = \varphi(bb')\langle ux, uy\rangle$ (so that, once well-defined, φ is multiplicative), it is sufficient to show that $\varphi(b^*)$ is a formal adjoint of $\varphi(b)$ on the spanning subset of elements of the form $\langle ux, uy\rangle$. From this follow both that $\varphi(b)$ is well-defined and that $\varphi(b^*) = \varphi(b)^*$. We start by observing that

$$\big\langle c, \langle ux, uy\rangle\big\rangle \;=\; c^*\langle ux, uy\rangle \;=\; \langle (ux)c, uy\rangle$$

for all $c \in \mathcal{C}_{uE}$. Using this two times, we find

$$\begin{aligned}\Big\langle \langle ux, uy\rangle, \varphi(b)\langle ux', uy'\rangle\Big\rangle \;&=\; \Big\langle \langle ux, uy\rangle, \langle u(x'b^*), uy'\rangle\Big\rangle\\ &=\; \big\langle u(x'b^*)\langle ux, uy\rangle, uy'\big\rangle \;=\; \big\langle u(x'b^*\langle x, y\rangle), uy'\big\rangle\\ &=\; \big\langle u(x'\langle xb, y\rangle), uy'\big\rangle \;=\; \big\langle (ux')\langle u(xb), uy\rangle), uy'\big\rangle\\ &=\; \Big\langle \langle u(xb), uy\rangle, \langle ux', uy'\rangle\Big\rangle \;=\; \Big\langle \varphi(b^*)\langle ux, uy\rangle, \langle ux', uy'\rangle\Big\rangle.\end{aligned}$$

Like every homomorphism from a C^*–algebra into the adjointable operators on a pre-Hilbert module, φ maps into the bounded operators, and like every homomorphism from a C^*–algebra into a pre-C^*–algebra, φ is a contraction.

Next we observe that $\varphi(\langle x, y\rangle)$ acts on the element $\langle ux', uy'\rangle$ of $\mathcal{C}_{uE}$ simply by multiplication from the left with the element $\langle ux, uy\rangle$. The subalgebra $\varphi(\mathcal{B}_E)$ of $\mathcal{B}^a(\overline{\mathcal{C}_{uE}})$ is nothing but $\mathcal{C}_{uE}$, which, of course, is faithfully contained in $\mathcal{B}^a(\overline{\mathcal{C}_{uE}})$. In other words, φ is the unique continuous extension from $\mathcal{B}_E$ to $\mathcal{B} = \overline{\mathcal{B}_E}$ of $\varphi \restriction \mathcal{B}_E$ and, therefore, maps into $\overline{\mathcal{C}_{uE}} \subset \mathcal{C}$. Clearly, φ turns u into a φ–isometry. □

Corollary 2.2. *Every ternary homomorphism is linear and contractive.*

Proof. The only thing that remains is to remark that if E is not full, then we may simply turn E into a full Hilbert module by restricting to $\overline{\mathcal{B}_E}$. □

Observation 2.3. Note that a ternary homomorphism is injective, if and only if the homomorphism $\varphi\colon \overline{\mathcal{B}_E} \to \mathcal{C}$ that turns it into a φ–isometry is injective. (Every surjective but noninjective endomorphism of $\mathcal{B}$ is an example for a surjective ternary homomorphism that is not injective.) This shows, in particular, that the φ induced by a ternary automorphism on a full Hilbert module is itself an automorphism.

Corollary 2.4. *The group of generalized unitaries on a full Hilbert module E coincides with the group of ternary automorphisms of E. Therefore, the dynamical systems on a Hilbert module E are exactly the C_0–groups of ternary automorphisms.*

Remark 2.5. By the construction in the proof of Theorem 2.1 every C_0–group $u = \big(u_t\big)_{t\in\mathbb{R}}$ of ternary automorphisms of a Hilbert $\mathcal{B}$–module comes along with a (unique) family of automorphisms φ_t of $\overline{\mathcal{B}_E}$ and, obviously, the φ_t form a C^*–dynamical system. These automorphisms φ_t do, in general, not necessarily extend to automorphisms of $\mathcal{B}$; see [8]. Therefore, for not necessarily full E there are, in general, more groups of ternary automorphisms than groups of generalized unitaries. In the general case, it seems, therefore, advisable to define a dynamical system on a Hilbert module as a C_0–group of ternary automorphisms.

Remark 2.6. By [8, Observation 1.4] (for instance) we know that every surjective φ–isometry from a Hilbert $\mathcal{B}$–module E to a Hilbert $\mathcal{C}$–module F extends to a homomorphism between the ***extended linking algebras***

$$\Phi = \begin{pmatrix} \varphi & u^* \\ u & \vartheta \end{pmatrix} : \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{C} & F^* \\ F & \mathcal{B}^a(F) \end{pmatrix}$$

that restricts to a homomorphism between the usual ***linking algebras*** $\left(\begin{smallmatrix} \mathcal{B} & E^* \\ E & \mathcal{K}(E) \end{smallmatrix}\right) \rightarrow \left(\begin{smallmatrix} \mathcal{C} & F^* \\ F & \mathcal{K}(F) \end{smallmatrix}\right)$. (Here $u^*(x^*) := (ux)^*$, while $\vartheta(a)$ acts on $y = ux$ in the only possible way, namely, $\vartheta(a)(ux) = u(ax)$. Well-definedness of $\vartheta(a)$ follows in a way paralleling the proof of well-definedness of φ in the proof of Theorem 2.1.) Therefore, generalized isometries and, consequently, also ternary homomorphisms are even completely contractive. (One may obtain this result also as in [1], by showing that every inflation u^n of u is a φ^n–isometry from $M_n(E)$ to $M_n(F)$ and, therefore, a contraction.) This improves on a result on ternary homomorphisms of C^*–algebras by Bracic and Moslehian [3].

A ternary homomorphism η from E into the Hilbert $\mathcal{B}(G)$–module $\mathcal{B}(G, H)$ for two Hilbert spaces G and H is what has been called a ***representation*** of E from G to H in Skeide [7]. The preceding discussion improves also on [7, Theorem A.4] where the extendibility of η to a representation of the linking algebra has been shown under the explicit hypothesis that η be completely bounded. Now we see that this hypothesis is fulfilled automatically.

3. Generalized derivations *versus* ternary derivations

It is easy to see that the generator δ of a dynamical system $u = \left(u_t\right)_{t\in\mathbb{R}}$ on a full Hilbert $\mathcal{B}$–module E is a generalized derivation; see [2] and cf. also Corollary 3.6. A possible choice for the derivation d in (GD) is simply the generator of the C^*–dynamical system $\varphi = \left(\varphi_t\right)_{t\in\mathbb{R}}$ ***associated*** with u, that is determined uniquely by the requirement that every u_t is a φ_t–unitary; see Remark 2.5. But, even if we know that the generator δ of C_0–group on E is a d–derivation, then it is not possible to conclude that δ generates a dynamical system without making further analytical assumptions about d and algebraic assumptions about the domain of d, see Theorem 3.12. These algebraic conditions are *relative* to δ, that is, they cannot be formulated intrinsically in terms of the derivation d of $\mathcal{B}$ alone, but depend explicitly on δ. On the other hand, it is easy to formulate these conditions intrinsically in terms of δ alone: δ must be a ternary derivation.

We study, first, the intrinsic description of the generators of dynamical systems on Hilbert modules as ternary derivations (Theorem 3.1). Then, we explain the relationship between ternary derivations and generalized derivations. We will see that there is a particular derivation d_δ (Theorem 3.5) that allows to formulate Theorem 3.1 in terms of generalized derivations (Theorem 3.18). In Theorem 3.19 we summarize all criteria and add one more in terms of the linking algebra.

Theorem 3.1. *Let $u = \left(u_t\right)_{t\in\mathbb{R}}$ be C_0–group on a Hilbert $\mathcal{B}$–module E. Then u is a dynamical system if and only if its generator δ is a ternary derivation.*

Proof. Recall that the generator of a C_0–group u is defined as

$$\delta(x) \ := \ \lim_{t\to 0}\frac{u_t x - x}{t}$$

for all x for which the limit exists. Further, recall that this domain $\mathsf{dom}(\delta)$ contains a dense ***core*** of ***entire analytic vectors***. That means, the subspace $\mathcal{A}(\delta) \subset \bigcap_{n\in\mathbb{N}} \mathsf{dom}(\delta^n)$ that consists of all vectors x for which for all $t \in \mathbb{R}$ the series

$$\sum_{n=0}^{\infty}\frac{t^n\delta^n}{n!}x$$

converges absolutely to the limit $u_t x$ is dense in E and δ is the closure of $\delta \upharpoonright \mathcal{A}(\delta)$.

Suppose δ is the generator of a dynamical system u. Let $x, y, z \in \mathsf{dom}(\delta)$. By Corollary 2.4 all u_t are ternary automorphisms, so that

$$\begin{aligned}\frac{u_t(x\langle y,z\rangle) - x\langle y,z\rangle}{t} \ &= \ \frac{(u_t x)\langle u_t y, u_t z\rangle - x\langle y,z\rangle}{t} \\ &= \ \frac{u_t x - x}{t}\langle u_t y, u_t z\rangle + x\Big\langle\frac{u_t y - y}{t}, u_t z\Big\rangle + x\Big\langle y, \frac{u_t z - z}{t}\Big\rangle.\end{aligned}$$

As all u_t are contractions, the families $u_t x$ and $u_t y$ are bounded uniformly. So the limit of the preceding expression exists and is equal to $\delta(x)\langle y,z\rangle + x\langle\delta(y),z\rangle + x\langle y,\delta(z)\rangle$. This shows both that $x\langle y,z\rangle \in \mathsf{dom}(\delta)$ and that δ is a ternary derivation.

Conversely, suppose that δ is a ternary derivation, and choose entire analytic elements $x, y, z \in \mathcal{A}(\delta)$. By a routine induction we show the ternary generalized *Leibniz rule*

$$\delta^n(x\langle y,z\rangle) \ = \sum_{\substack{k,\ell,m\in\mathbb{N}_0\\ k+\ell+m=n}}\frac{n!}{k!\ell!m!}\delta^k(x)\langle\delta^\ell(y),\delta^m(z)\rangle.$$

¿From this, one easily concludes that $x\langle y,z\rangle$ is also in $\mathcal{A}(\delta)$ and that u_t fulfills (TU) on the dense subset $\mathcal{A}(\delta)$. By contractivity of u_t, this extends to all of E so that u_t is a ternary automorphism. □

Remark 3.2. For general results about C_0–groups we refer to Bratteli and Robinson [4]. In particular, the problem to decide, whether a linear densely defined map is the generator of C_0–group, we leave entirely to the comprehensive treatment in [4]. But, once we have such a generator, we see that the problem whether the generated group is a dynamical system, is equivalent to the question whether the generator is a ternary derivation. Thus, we have a complete separation into the general analytic criteria of the Banach space theory that determine when δ is a generator (which we do not treat here) and the completely algebraic criterion in Theorem 3.1.

Theorem 3.1, in principle, completely settles the problem to characterize the generators of dynamical systems on Hilbert modules. Fullness, is not at all a critical assumption, because if necessary we may always make $\mathcal{B}$ smaller. In the remainder of this section we deal with the problem to find similarly useful criteria in terms of generalized derivations. We start by establishing a connection between ternary derivations and a special sort of generalized derivations on the algebraic level. However, a full correspondence we will obtain only under the assumption that the derivations in question generate C_0–groups. On the level of derivations the assumption of fullness becomes much more vital, as we do not see a possibility to show that the derivation of $\mathcal{B}$ that turns a map δ into a d–derivation restricts to a derivation of $\overline{\mathcal{B}_E}$. The following uniqueness result, depending essentially on fullness, is crucial for all other statements that follow.

Proposition 3.3. *Let $\delta\colon E \supset \mathsf{dom}(\delta) \to E$ be a densely defined linear map on a full Hilbert $\mathcal{B}$–module E. Then for every dense subalgebra $\mathcal{B}_0$ of $\mathcal{B}$, there is at most one derivation d of $\mathcal{B}$ with domain $\mathsf{dom}(d) = \mathcal{B}_0$ that turns δ into a d–derivation.*

Corollary 3.4. *If d_1, d_2 are derivations of $\mathcal{B}$ and if δ is a d_1–derivation **and** a d_2–derivation of a full Hilbert $\mathcal{B}$–module E, then*

$$d_1 \;\subset\; d_2 \quad\Longleftrightarrow\quad \mathsf{dom}(d_1) \;\subset\; \mathsf{dom}(d_2).$$

Proof of Proposition 3.3. If δ is a d–derivation, then we have

$$xd(b) \;=\; \delta(xb) - \delta(x)b$$

for all $x \in \mathsf{dom}(\delta)$ and all $b \in \mathsf{dom}(d)$. Since E is full and $\mathsf{dom}(\delta)$ is dense in E, the preceding equation determines $d(b) \in \mathcal{B}$ uniquely. □

Theorem 3.5. *Every ternary derivation δ of a full Hilbert $\mathcal{B}$–module E is also a generalized derivation. More precisely, there is a unique derivation d_δ of $\mathcal{B}$ on the dense domain $\mathsf{dom}(d_\delta) := \mathsf{span}\langle \mathsf{dom}(\delta), \mathsf{dom}(\delta)\rangle$ that fulfills*

$$d_\delta(\langle x, y\rangle) \;=\; \langle \delta(x), y\rangle + \langle x, \delta(y)\rangle. \tag{3.1}$$

d_δ turns δ into a d_δ–derivation. Moreover, d_δ is a $*$–derivation.

Proof. Suppose we have a derivation d_δ on the given domain, that turns δ into a d_δ–derivation. Then (following the proof of Proposition 3.3) for the uniquely determined values of $d_\delta(\langle x, y\rangle)$ we find

$$x\, d_\delta\big(\langle y, z\rangle\big) \;=\; \delta(x\langle y, z\rangle) - \delta(x)\langle y, z\rangle \;=\; x\Big(\langle \delta(y), z\rangle + \langle y, \delta(z)\rangle\Big) \tag{3.2}$$

for all $x, y, z \in \mathsf{dom}(\delta)$. We see that if a suitable derivation d_δ exists, then it must fulfill (3.1). In particular, d_δ is necessarily a $*$–derivation. So the only remaining questions are, firstly, whether (3.1) always well-defines a linear map

$$d_\delta\colon\ \mathsf{span}\langle \mathsf{dom}(\delta), \mathsf{dom}(\delta)\rangle \;\longrightarrow\; \mathcal{B},$$

and, secondly, whether this map is a (necessarily $*$–) derivation. For the first question, suppose y_i, z_i are finitely many elements of $\mathsf{dom}(\delta)$ fulfilling $\sum_i \langle y_i, z_i\rangle = 0$. Then

$$x\Big(\sum_i \langle \delta(y_i), z_i\rangle + \langle y_i, \delta(z_i)\rangle\Big) \;=\; \delta\Big(x\sum_i \langle y_i, z_i\rangle\Big) - \delta(x)\sum_i \langle y_i, z_i\rangle \;=\; 0$$

for all $x \in \mathsf{dom}(\delta)$, so that d_δ is, indeed, well-defined. For the second question, let us compute the inner product of an element $w \in \mathsf{dom}(\delta)$ with (3.2) and the adjoint of the resulting equation. Using this, we find

$$\begin{aligned}
&d_\delta\big(\langle w,x\rangle\big)\,\langle y,z\rangle+\langle w,x\rangle\,d_\delta\big(\langle y,z\rangle\big)\\
&=\;\Big(\langle\delta(w),x\rangle+\langle w,\delta(x)\rangle\Big)\langle y,z\rangle+\langle w,x\rangle\,\Big(\langle\delta(y),z\rangle+\langle y,\delta(z)\rangle\Big)\\
&=\;\langle\delta(w),x\rangle\langle y,z\rangle+\Big\langle w\,,\,\delta(x)\langle y,z\rangle+x\langle\delta(y),z\rangle+x\langle y,\delta(z)\rangle\Big\rangle\\
&=\;\big\langle\delta(w),x\langle y,z\rangle\big\rangle+\big\langle w\,,\,\delta(x\langle y,z\rangle)\big\rangle\;=\;d_\delta\Big(\big\langle w,x\langle y,z\rangle\big\rangle\Big)\;=\;d_\delta\big(\,\langle w,x\rangle\langle y,z\rangle\,\big).
\end{aligned}$$

By linearity this extends to $d_\delta(b)b'+bd_\delta(b')=d_\delta(bb')$ for all b,b' in the domain $\mathsf{span}\langle\mathsf{dom}(\delta),\mathsf{dom}(\delta)\rangle$. □

Corollary 3.6. [2] *Every generator of a dynamical system on a full Hilbert module is a generalized derivation.*

Corollary 3.7. *Every ternary derivation of a full Hilbert module extends as a $*$–derivation to the linking algebra.*

Proof. Suppose δ is ternary derivation of the full Hilbert $\mathcal{B}$–module. By Theorem 3.5 this determines the $*$–derivation d_δ of $\mathcal{B}$ which is the candidate for how to extend δ to the corner $\mathcal{B}$ of the linking algebra. To find the extension to $\mathcal{K}(E)$, we observe that E^* is a full Hilbert $\mathcal{K}(E)$–module, the ***dual*** module of E, with inner product $\langle x^*,y^*\rangle:=xy^*\in\mathcal{K}(E)$ and the right action $x^*a:=(a^*x)$ of elements $a\in\mathcal{K}(E)$ (or even in $\mathcal{B}^a(E)$). Of course, $\delta^*(x^*):=\delta(x)^*$ defines a ternary derivation of E^* with domain $\mathsf{dom}(\delta^*):=\mathsf{dom}(\delta)^*$, and by Theorem 3.5 there is a unique $*$–derivation d_{δ^*} of $\mathcal{K}(E)$ defined on the domain $\mathsf{dom}(d_{\delta^*})=\mathsf{span}\big(\mathsf{dom}(\delta)\,\mathsf{dom}(\delta)^*\big)$, turning δ^* into a d_{δ^*}–derivation. It is routine to check that $\left(\begin{smallmatrix}b&y^*\\x&a\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix}d_\delta(b)&\delta^*(y^*)\\\delta(x)&d_{\delta^*}(a)\end{smallmatrix}\right)$ defines a $*$–derivation of the linking algebra. □

If δ is a ternary derivation, the derivation d_δ plays a distinguished role as it is related more directly to questions of closability than any other derivation d that turns δ into a d–derivation. The following Proposition 3.9 settles some of these closability questions in the setting of general derivations, while in Theorems 3.12 and 3.18 the assumption that the maps are generators of C_0–groups is crucial. The following task, needed in the proofs of Proposition 3.9(1b) and of Lemma 3.15, is so useful that we prefer to formulate it separately.

Lemma 3.8. *Suppose the elements of the Hilbert $\mathcal{B}$–module E* ***separate the points*** *of $\mathcal{B}$, that is, $xb=0$ for all $x\in E$ implies $b=0$. (For instance, suppose E is full.) Then*

$$\|b\|\;=\;\sup_{\|x\|\leq 1}\|xb\|\,.$$

Proof. By setting $bx^*:=(xb^*)^*$ we define a representation of $\mathcal{B}$ by adjointable (and, therefore, bounded; see the proof of Theorem 2.1) operators on the dual module E^* (see the proof of Corollary 3.7). By hypothesis, this representation is faithful and, therefore, isometric. In other words, the operator norm of the action of $b\in\mathcal{B}$ as operator on E^* coincides with the norm of b as element of $\mathcal{B}$. Observing that $\|xb\|=\|(xb)^*\|$, this is precisely the statement of the lemma. □

Proposition 3.9. *Let E be a full Hilbert $\mathcal{B}$–module.*

(1) *Let δ be a d–derivation of E.*
   (a) *If δ is closable, then so is d.*
   (b) *If δ is bounded, then so is d.*

(2) *Let δ be a ternary derivation of E. Then δ is closable, if and only if d_δ is closable.*

(3) *Let δ be a ternary derivation* ***and*** *a d–derivation of E. If d_δ is closable, then so is d.*

Proof. (1a) Suppose that δ is a closable d–derivation. Let $b_n \to 0$ be a sequence in $\mathsf{dom}(d)$ such that $d(b_n) \to b \in \mathcal{B}$. Then for every $x \in \mathsf{dom}(\delta)$ we find

$$\delta(xb_n) \;=\; \delta(x)b_n + xd(b_n) \;\longrightarrow\; 0 + xb.$$

As $xb_n \to 0$ and δ is closable, it follows that $\delta(xb_n) \to 0$, so that $xb = 0$ for all $x \in \mathsf{dom}(\delta)$. As E is full, this implies $b = 0$. So, d is closable.

(1b) Suppose that δ is a bounded d–derivation. By Lemma 3.8, for every $b \in \mathsf{dom}(d)$ we find an x in the unit ball of E such that $\|xd(b)\| \geq \frac{1}{2}\|d(b)\|$. So, $\|d(b)\| \leq 2\|xd(b)\| \leq 2\big(\|\delta(xb)\| + \|\delta(x)b\|\big) \leq 4\|\delta\|\,\|b\|$.

(2) Suppose now that δ is a ternary derivation such that d_δ is closable. Let $x_n \to 0$ be a sequence in $\mathsf{dom}(\delta)$ such that $\delta(x_n) \to x \in E$. Then for every $y \in \mathsf{dom}(\delta)$ we find

$$d_\delta(\langle y, x_n\rangle) \;=\; \langle y, \delta(x_n)\rangle + \langle \delta(y), x_n\rangle \;\longrightarrow\; \langle y, x\rangle + 0.$$

As $\langle y, x_n\rangle \to 0$ and d_δ is closeable, it follows that $d_\delta(\langle y, x_n\rangle) \to 0$, so that $\langle y, x\rangle = 0$ for all $y \in \mathsf{dom}(\delta)$ and, therefore, $x = 0$. So, δ is closable. If E is full, then, by Part (1a), also the converse is true.

(3) If d_δ is closable, then, by (2), δ is closable so that, by (1a), d is closable. □

Remark 3.10. Boundedness of d is not sufficient for boundedness of δ. In fact, every generator of a unitary C_0–group on a Hilbert module that is not uniformly continuous is an unbounded ternary derivation and a 0–derivation for the trivial derivation $0\colon b \mapsto 0$.

Observation 3.11. If, in (3), $\mathsf{dom}(d)$ does not contain $\mathsf{dom}(d_\delta)$, then we may easily replace d by the derivation d' defined on $\mathsf{alg}^*\big(\mathsf{dom}(d_\delta), \mathsf{dom}(d)\big)$, the $*$–algebra generated by $\mathsf{dom}(d_\delta)$ and $\mathsf{dom}(d)$, that is determined uniquely (see Proposition 3.3!) by the requirement that δ be a d'–derivation. (If such a d' exists, then, again by Proposition 3.3, this implies also that d' is the unique extension as a derivation of d and d_δ to the new domain.) Let us first define d' on the domain $\mathsf{dom}(d) \cup \mathsf{dom}(d_\delta)$ as $d'(b) := d(b)$ for $b \in \mathsf{dom}(d)$ and $d'(b) := d_\delta(b)$ for $b \in \mathsf{dom}(d_\delta)$. (Once more, by the proof of Proposition 3.3, this is well-defined as d and d_δ coincide on the intersection of their domains.) By induction we show that for every choice of elements $b_1, \ldots, b_n$ from $\mathsf{dom}(d) \cup \mathsf{dom}(d_\delta)$ and for all $x \in \mathsf{dom}(\delta)$ (so that also $xb_1 \ldots b_n$ is in $\mathsf{dom}(\delta)$)

$$\delta(xb_1 \ldots b_n) \;-\; \delta(x)b_1 \ldots b_n \;=\; x\Big(\, d'(b_1)b_2 \ldots b_n \;+\; \ldots \;+\; b_1 \ldots b_{n-1}d'(b_n)\,\Big).$$

This shows that for every b in the new domain there is a uniqe $b' \in \mathcal{B}$ satisfying $xb' = \delta(xb) - \delta(x)b$ and that the map $d'\colon b \mapsto b'$ is linear. Clearly, d' is a derivation and δ is a d'–derivation. By Parts (2) and (3), d' is closable, if and only if d_δ or, equivalently, if δ is closable. In other words, every derivation d that turns a closable ternary derivation δ of a full Hilbert $\mathcal{B}$–module into a d–derivation admits a unique minimal closed extension $\overline{d'} \supset d_\delta$, and δ is also a $\overline{d'}$–derivation.

Theorem 3.12. *Suppose δ is the generator of a dynamical system u on a full Hilbert $\mathcal{B}$–module E and a d–derivation for some (by Theorem 3.1 and Proposition 3.9(3), necessarily closable) derivation d of $\mathcal{B}$. Denote by d_φ the generator of the C^*–dynamical system φ associated with u.*

(1) *The unique minimal closed extension $\overline{d'} \supset d_\delta$ of d (see Observation 3.11) is the generator of φ, if and only if $d \subset d_\varphi$.*
(2) *If $d \subset d_\varphi$, then for $\overline{d} = d_\varphi$ it is necessary and sufficient that $\overline{d} \supset d_\delta$.*

Proof. As in the proof of Theorem 3.1 we see that $\mathsf{dom}(d_\delta) \subset \mathsf{dom}(d_\varphi)$ and that the span of $\langle \mathcal{A}(\delta), \mathcal{A}(\delta)\rangle$ is a dense subspace of entire analytic elements of $\mathsf{dom}(d_\varphi)$. Therefore, every subspace D with

$$\langle \mathcal{A}(\delta), \mathcal{A}(\delta)\rangle \;\subset\; D \;\subset\; \mathsf{dom}(d_\varphi)$$

is a core for d_φ. In particular, $\mathsf{dom}(d_\delta)$ is a core for d_φ.

(1) If $d \not\subset d_\varphi$, then $d \subset d' \subset \overline{d'} \not\subset d_\varphi$. Conversely, if $d \subset d_\varphi$ then also $d' \subset d_\varphi$ (because $d_\delta \subset d_\varphi$ and, therefore, $\mathsf{alg}^*(\mathsf{dom}(d), \mathsf{dom}(d_\delta)) \subset \mathsf{dom}(d_\varphi)$), so that $\overline{d'} \subset \overline{d_\varphi} = d_\varphi$.

(2) If $\overline{d} \not\supset d_\delta$, then $\overline{d} \not\supset \overline{d_\delta} = d_\varphi$. Conversely, if $\overline{d} \supset d_\delta$ so that $\overline{d} \supset \overline{d_\delta} = d_\varphi$, then $d_\varphi = \overline{d_\varphi} \supset \overline{d} \supset d_\varphi$. □

Corollary 3.13. *If δ is the generator of a dynamical system u on a full Hilbert $\mathcal{B}$–module, then δ is a $\overline{d_\delta}$–derivation and $\overline{d_\delta}$ is the generator of the C^*–dynamical system associated with u.*

In general, a d–derivation (even bounded) of a full Hilbert $\mathcal{B}$–module for some derivation d of $\mathcal{B}$ need not be a ternary derivation, not even if d is a bounded $*$–derivation.

Example 3.14. The so-called ***inner*** generalized derivations of a Hilbert $\mathcal{B}$–module E are the mappings that can be written in the form

$$\delta(x) \;=\; \alpha x - x\beta$$

form some $\alpha \in \mathcal{B}^a(E)$ and $\beta \in \mathcal{B}$. From

$$\begin{aligned}\delta(x)\langle y,z\rangle + x\langle \delta(y),z\rangle + x\langle y,\delta(z)\rangle \;&=\; (\alpha x - x\beta)\langle y,z\rangle + x\langle \alpha y - y\beta, z\rangle + x\langle y, \alpha z - z\beta\rangle \\ &=\; \alpha x\langle y,z\rangle - x\beta\langle y,z\rangle + x\langle \alpha y,z\rangle - x\langle y\beta,z\rangle + x\langle y,\alpha z\rangle - x\langle y, z\beta\rangle \\ &=\; \delta(x\langle y,z\rangle) - x\langle y(\beta+\beta^*),z\rangle + x\langle y,(\alpha+\alpha^*)z\rangle\end{aligned}$$

we see that δ is a ternary derivation, if and only if $(\beta+\beta^*)\langle y,z\rangle = \langle y,(\alpha+\alpha^*)z\rangle$ for all $y,z \in E$. Inserting yb for y and computing $\langle yb,(\alpha+\alpha^*)z\rangle = b^*\langle y,(\alpha+\alpha^*)z\rangle$, one may check that $\beta+\beta^*$ must be in the center of $\mathcal{B}$. Further, the element $\alpha+\alpha^* \in \mathcal{B}^a(E)$ is given simply as multiplication from the right with the central element $\beta+\beta^*$. Therefore, δ is a ternary derivation, if and only if the real parts of

α and β may be removed without changing δ, or, in other words, if $\delta(x) = \alpha x - x\beta$ for skew-adjoint elements α and β.

Notice, further, that δ is the generator of the uniformly continuous one-parameter group $u_t(x) = e^{t\alpha} x e^{-t\beta}$ on E. It follows that this group is a dynamical system, if and only the groups $e^{t\alpha}$ and $e^{-t\beta}$ are unitary. So, even if β is skew-adjoint (so that d is a $*$–derivation and the generator of a C^*–dynamical system) δ does not generate a dynamical system, unless also α is skew-adjoint. On the other hand, if, in this case, α is not skew-adjoint, then u_t is not a C_0–group.

We will see in a moment that the last statement of the preceding example is typical in the sense that, if a C_0–group u consists of φ_t–linear maps u_t, then u is a dynamical system. But, we think that the following preparatory result inspired very much by Lance [5, Theorem 3.5] is worth to be stated separately.

Lemma 3.15. *Let E be a Hilbert $\mathcal{B}$–module, let F be a Hilbert $\mathcal{C}$–module and suppose $u\colon E \to F$ is a Banach space isometry onto a $\mathcal{C}$–submodule of F. If u is φ–linear for some homomorphism $\varphi\colon \mathcal{B} \to \mathcal{C}$ such that $\varphi(\mathcal{B}) \supset \langle u(F), u(F)\rangle$, then u is a φ–isometry.*

Proof. For $\mathcal{C} = \mathcal{B}$, $\varphi = \mathsf{id}_{\mathcal{B}}$ and surjective u the statement is exactly [5, Theorem 3.5]. We shall prove the statement exactly along the lines of the proof of [5, Theorem 3.5] by appealing to [5, Lemma 3.4] which states

$$b_1 \geq 0, b_2 \geq 0, \text{ and } \|b_1 b\| = \|b_2 b\| \ \forall\, b \in \mathcal{B} \quad \Longrightarrow \quad b_1 = b_2.$$

First, we compute

$$\|ux\|\,\|\varphi(b)\| \ \geq\ \|(ux)\varphi(b)\| \ =\ \|u(xb)\| \ =\ \|xb\|\,.$$

If $0 \neq b \in \overline{\mathcal{B}_E}$ then there exists $x \in E$ such that $xb \neq 0$. By Lemma 3.8, it follows that $\|\varphi(b)\| = \|b\|$ for all $b \in \overline{\mathcal{B}_E}$. Next, for all $b \in \mathcal{B}$ and for all $x \in E$ we have

$$\begin{aligned} \|\varphi(b^*)\langle ux, ux\rangle\varphi(b)\| \ &=\ \|\langle u(xb), u(xb)\rangle\| \ =\ \|u(xb)\|^2 \ =\ \|xb\|^2 \\ &=\ \|b^*\langle x, x\rangle b\| \ =\ \|\varphi(b^*)\varphi(\langle x, x\rangle)\varphi(b)\|\,, \end{aligned}$$

where the last equality follows from $b^*\langle x, x\rangle b \in \mathcal{B}_E$ and the first step. In other words, we have $\left\|\sqrt{\langle ux, ux\rangle}c\right\| = \left\|\sqrt{\varphi(\langle x, x\rangle)}c\right\|$ for all elements $c \in \varphi(\mathcal{B})$. Since by assumption $\langle ux, ux\rangle \in \varphi(\mathcal{B})$ so that also $\sqrt{\langle ux, ux\rangle} \in \varphi(\mathcal{B})$, it follows by [5, Lemma 3.4] that $\sqrt{\langle ux, ux\rangle} = \sqrt{\varphi(\langle x, x\rangle)}$, hence, $\langle ux, ux\rangle = \varphi(\langle x, x\rangle)$ and, finally, by polarization $\langle ux, uy\rangle = \varphi(\langle x, y\rangle)$ for all $x, y \in E$. In other words, u is a φ–isometry. □

Corollary 3.16. *Every φ–linear, isometric Banach space isomorphism between full Hilbert modules with surjective φ is necessarily a φ–unitary.*

Remark 3.17. We do not know, whether the (necessary) condition $\varphi(\mathcal{B}) \supset \langle u(F), u(F)\rangle$ in Lemma 3.15 (and the corresponding condition φ be surjective of Corollary 3.16) does not, possibly, follow from the remaining hypothesis.

Theorem 3.18. *Suppose that d is a $*$–derivation that is the generator of a C_0–group φ on the C^*–algebra $\mathcal{B}$, and suppose that δ is a d–derivation that is* *the generator of a C_0–group u on the full Hilbert $\mathcal{B}$–module E. Then u is a dynamical system on E and φ is the C^*–dynamical system associated with u. Of course, δ is a ternary derivation and a d–derivation and $\mathsf{dom}(d_\delta)$ is a core for d.*

Proof. For all $x \in \mathcal{A}(\delta)$ and all $b \in \mathcal{A}(d)$ as in the proof of Theorem 3.1 one shows that also $xb \in \mathcal{A}(\delta)$ and that

$$u_t(xb) \;=\; u_t(x)\varphi_t(b).$$

In exactly the same way one shows that φ_t (of course, a $*$–map) is multiplicative. In other words, φ_t is an automorphism of $\mathcal{B}$ and u_t is a surjective and right φ_t–linear Banach space isometry. By Corollary 3.16, u_t is a φ_t–unitary. In other words, u is a dynamical system and φ is the C^*–dynamical system associated with it. □

For the sake of clarity we summarize the criteria provided by Theorem 3.1, Corollary 3.13, and Theorem 3.18. Without the obvious proof, we add a fourth criterion based on the observation (as explained in Remark 2.6) that a dynamical system on E extends to a C^*–dynamical system on the linking algebra.

Theorem 3.19. *Let δ be the generator of a C_0–group u on a full Hilbert $\mathcal{B}$–module. Then the following statements are equivalent:*

(1) *u is a dynamical system.*
(2) *δ is a ternary derivation.*
(3) *There exists a $*$–derivation d that is the generator of a C_0–group on $\mathcal{B}$ (necessarily a C^*–dynamical system) such that δ is a d–derivation.*
(4) *δ extends to the generator of a C^*–dynamical system on the linking algebra of the form $\Delta = \begin{pmatrix} d & \delta^* \\ \delta & D \end{pmatrix}$ with $\delta(x^*)^* := \delta(x)^*$ and d and D being generators of C^*–dynamical systems on $\mathcal{B}$ and $\mathcal{K}(E)$, respectively.*

Remark 3.20. In all criteria where we make explicit reference to a derivation d of the *corner* $\mathcal{B}$, we assume that both δ and d are generators of C_0–groups. We leave open the very interesting question whether the algebraic conditions alone might already be sufficient to conclude from one, δ or d, being a generator, that also the other is a generator.

4. An outline of possible applications

Hilbert modules are a hybrid in between Hilbert spaces and C^*–algebras. Formally, the axioms of a Hilbert module over a C^*–algebra $\mathcal{B}$ generalize the axioms of a Hilbert space in that the $\mathbb{C}$–valued inner product of Hilbert spaces is replaced by an inner product that takes values in $\mathcal{B}$. But, also every C^*–algebra is a Hilbert module over itself. We have also seen that every Hilbert $\mathcal{B}$–module E is a subset of a C^*–algebra, the linking algebra $\begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix}$. Dynamical systems on Hilbert modules and their generators have consequences under all three aspects.

While the notion of Hilbert module, formally, is a generalization of that of Hilbert space, the notion of generalized automorphism is a generalization of that of automorphism of a C^*–algebra. The same is true for generalized derivations. (For Hilbert spaces we recover the known notions of unitary and Stone generator of a unitary one-parameter group, respectively.) An automorphism of a C^*–algebra and a derivation of a C^*–algebra are a generalized automorphism and a generalized derivation, respectively, of that C^*–algebra when the C^*–algebra is considered as a Hilbert module over itself. It is noteworthy that every φ–unitary of the Hilbert $\mathcal{B}$–module $\mathcal{B}$ has the form $x \mapsto u\varphi(x)$, where u is a unitary in $\mathcal{B}$; see [8, Corollary 1.13]. Similarly, if δ is a d–derivation of $\mathcal{B}$, then $\delta - d$ defines a right linear map $\mathsf{dom}(\delta) \cap \mathsf{dom}(d) \to \mathcal{B}$. If δ is defined everywhere, then the results in [1, Section 3.1] assert that δ and d and, therefore, $\delta - d$ are bounded.

We start by thinking of Hilbert modules as a generalization of Hilbert spaces. This means, in particular, we think of a Hilbert module as a space on which an operator algebra can act. The situation is particularly well under control if this algebra acting on E is the algebra of all adjointable operators $\mathcal{B}^a(E)$ itself. An automorphism u of the representation space E gives rise to an automorphism $a \mapsto uau^{-1}$ of the algebra of operators on this representation space. In the case of a Hilbert space H, the automorphisms of H are the unitaries $u \in \mathcal{B}(H)$, the corresponding automorphisms of $\mathcal{B}(H)$ are the inner automorphisms. It is well-known that $\mathcal{B}(H)$ does not have other automorphisms than inner. Of course, also $\mathcal{B}^a(E)$ admits inner automorphisms, but there are more. Every φ–unitary u gives rise to an automorphism $a \mapsto uau^{-1}$ of $\mathcal{B}^a(E)$. This automorphisms is inner, if and only if φ is *quasi inner*; Skeide [8, Corollary 2.3]. But the discussion following [8, Corollary 2.3] also shows that there are more (strict) automorphisms of $\mathcal{B}^a(E)$ than the automorphisms obtained by conjugation with generalized unitaries.

Summarizing, the algebra $\mathcal{B}^a(E)$ of all adjointable operators on a Hilbert modules is more general than $\mathcal{B}(H)$, but it preserves many of its features. The generalized unitaries on E are a class of operators on the Banach space E more general than unitaries, but sharing many properties of the unitaries. The class of automorphisms of $\mathcal{B}^a(E)$ obtained by conjugation with generalized unitaries is wider than the class of inner automorphisms but shares much of the simplicity of the latter. It seems, therefore, appropriate to try to do the whole program of finding the quantum evolution of an interacting system as a cocycle perturbations of the system without interaction for groups or cocycles of generalized unitaries. The most obvious approaches are the following two:

Firstly, on can replace the unperturbed evolution of the system, usually, implement by a unitary evolution (the second quantized time shift on the Fock space, for instance) by evolutions that are implemented by conjugation with generalized unitaries, that is, by conjugation with a dynamical system. These may, then, be perturbed in the usual way by a unitary cocycle obtained via a (still to be constructed!) quantum stochastic calculus. We shall discuss in a minute a possible interpretation of this generalization of the unperturbed setting in term of an interaction picture for algebras $\mathcal{B}$ different from $\mathcal{B}(G)$.

Secondly, we may try to perturb the usual dynamics by cocycles of generalized unitaries. For this we refer to the discussion in [8, Remark 3.8] and the considerations about cocycles in the end of [8, Section 1].

For the first proposal a good knowledge the dynamical systems on a Hilbert module is indispensable. For the second proposal it is necessary to find candidates for the stochastic generators of generalized unitary cocycles. We may not hope to understand these stochastic generators without understanding first the generators of generalized unitary cocycles with respect to the identity, in other words, the generators of dynamical systems. These notes provide the necessary theory.

The second possibility how to think of generalized unitaries is that to think of an automorphism of E generalizing the notion of automorphisms of a C^*–algebra. The interpretation in terms of the linking algebra is closest: By Remark 2.6 (based on [8, Observation 1.4]) we know that the generalized unitaries are exactly those maps on E that extend in a "block-wise way" to an automorphism of the linking algebra, while the ternary automorphisms are exactly those maps on E that extend in a "block-wise way" to an automorphism of the ***reduced linking algebra*** ($\mathcal{B}$ replaced with $\overline{\mathcal{B}_E}$). Theorem 2.1 tells us that for full Hilbert modules the two sorts of maps are the same, while in the general case the class of ternary automorphisms is wider.

The extension of u to the corner $\mathcal{B}^a(E)$ of the extended linking algebra is nothing but conjugation with u, $a \mapsto uau^{-1}$. It is not difficult to check that the automorphism of $\overline{\mathcal{B}_E}$ we constructed in the proof of Theorem 2.1 is nothing but conjugation with the ternary automorphism $x^* \mapsto (ux)^*$ of E^*, when restricted to $\overline{\mathcal{B}_E} = \mathcal{K}(E^*) \subset \mathcal{B}^a(E^*)$. (It is a warmly recommended exercise to check all these assertions. The reader might ask, given a ternary automorphisms of E, why we did not immediately define the multiplicative map $a \mapsto uau^{-1}$. It will help to appreciate better the proof of Theorem 2.1, if the reader tries to find out why it is not possible to prove that this defines an automorphism of $\mathcal{B}^a(E)$ without doing something like we did in the proof of Theorem 2.1.)

Let φ by an automorphism of $\mathcal{B}$. We see that finding a φ–unitary u means, first, to lift φ to a "generalized automorphism" u of E and, further, to an automorphism $a \mapsto uau^{-1}$ of $\mathcal{B}^a(E)$. If we describe typical applications of quantum stochastic calculus in terms of Hilbert modules, then E is the ***GNS-correspondence*** of a conditional expectation from $\mathcal{A} \subset \mathcal{B}^a(E)$ onto $\mathcal{B} \subset \mathcal{A}$. That is, E is, acually, a $\mathcal{B}$–bimodule (with left action $\mathcal{B} \to \mathcal{A} \to \mathcal{B}^a(E)$) and there is a cyclic vector $\xi \in E$ giving back the conditional expectation as $a \mapsto \langle \xi, a\xi \rangle$. By the conditional expectation property it follows that $b\xi = \xi b$.

Example 4.1. Let $\mathcal{B} = \mathcal{B}(G)$ (G a Hilbert space) and $E = \mathcal{B}\big(G, G \otimes \Gamma(L^2(\mathbb{R}, K))\big)$ (K some Hilbert space) the symmetric Fock module (with inner product $\langle x, y \rangle = x^*y$). From the left $\mathcal{B}$ acts directly on the factor G of $G \otimes \Gamma(L^2(\mathbb{R}, K))$. Then $\mathcal{B}^a(E) = \mathcal{B}\big(G \otimes \Gamma(L^2(\mathbb{R}, K))\big)$ and $\xi \colon g \mapsto g \otimes \Omega$ generates the usual vacuum conditional expectation $\langle \xi, \bullet \xi \rangle \colon \mathcal{B}\big(G \otimes \Gamma(L^2(\mathbb{R}, K))\big) \to \mathcal{B}(G)$.

If we find a φ–unitary u on E, then $b \mapsto \langle \xi, ubu^{-1}\xi \rangle$ need no longer leave invariant the identity. Indeed, if u is ***left*** φ–***linear***, that is, $v(by) = \varphi(b)(vy)$, then $\langle \xi, ubu^{-1}\xi \rangle = \big\langle \xi, uu^{-1}(\varphi(b)\xi) \big\rangle = \langle \xi, \varphi(b)\xi \rangle = \langle \xi, \xi \rangle \varphi(b) = \varphi(b)$.

Example 4.2. Let F be a correspondence over $\mathcal{B}$ (that is, a Hilbert $\mathcal{B}$–modules with a nondegenerate left action of $\mathcal{B}$). Denote by $E := \mathcal{F}(F) := \bigoplus_{n=0}^{\infty} F^{\odot n}$ the ***full Fock module*** over F, where $F^{\odot 0} = \mathcal{B}$ and $\xi = \mathbf{1} \in \mathcal{B} = F^{\odot 0}$. Let v be a left φ–linear φ–unitary on F. It is not difficult to check that the *second quantization* of v

$$F^{\odot 0} \ni b \ \longmapsto \ \varphi(b) \qquad F^{\odot n} \ni y_n \odot \ldots \odot y_1 \ \longmapsto \ vy_n \odot \ldots \odot vy_1$$

defines a left φ–linear φ–unitary u on E. Of course, every such v corresponds to an ***even*** automorphism (that is, an automorphism that sends creators to creators) of the Pimsner-Toeplitz and the Cuntz-Pimsner algebras acting on $\mathcal{F}(F)$.

That the vacuum expectation in the usual setting is invariant under the free evolution is nothing else but the statement that we are in interaction picture. A left φ–linear φ–unitary changes this behavior. A dynamical system allows to switch from a noninteracting dynamics leaving the subalgebra $\mathcal{B}$ invariant, to a dynamics that restricts to C^*–dynamical system on $\mathcal{B}$. Of course, the can be seen also in the other direction: Given a C^*–dynamical system on $\mathcal{A} \subset \mathcal{B}^a(E)$ that restricts to a C^*–dynamical system on $\mathcal{B} \subset \mathcal{A}$, then finding a dynamical system u on E the the inverse C^*–dynamical system φ^{-1} such that every u_t is also φ_t^{-1}–left linear and fulfilling some cocycle condition with respect to the original dynamics on $\mathcal{A}$, we may switch to a dynamics leaving $\mathcal{B}$ invariant by conjugation with u. We believe that this is the correct way to think of the interacting picture that generalizes the usual setting. The version of calculi we mentioned in the first part of this section should, therefore, furnish a calculus that works also if we are not in the interacting picture.

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