

QUANTUM STOCHASTIC PROCESS ASSOCIATED WITH QUANTUM LÉVY LAPLACIAN

UN CIG JI*, HABIB OUERDIANE, AND KIMIYAKI SAITÔ**

ABSTRACT. A noncommutative extension of the Lévy Laplacian, called the quantum Lévy Laplacian, is introduced and its relevant properties are studied, in particular, a relation between the classical and quantum Lévy Laplacians is studied. We construct a semigroup and a quantum stochastic process generated by the quantum Lévy Laplacian.

1. Introduction

Since an infinite dimensional analogue of the usual Laplacian on an Euclidean space has been introduced and studied by Lévy in his famous book [15] and so called the Lévy Laplacian, the Lévy Laplacian has been studied in [5, 8, 21]. On the other hand, the Lévy Laplacian acting on white noise functionals has been introduced by Hida and studied by many authors within the framework of white noise theory, see [6, 11, 12] and the references cited therein. In recent years the Lévy Laplacian has afforded us much interest for its newly discovered relations with certain stochastic processes [2, 22, 25], Yang–Mills equations [3, 14], Gross Laplacian [9, 12], infinite dimensional rotation group [16], quadratic quantum white noise [19, 20], Poisson noise functionals [24] and so on. In recent papers [1, 4, 5, 10], noncommutative generalizations of the Lévy Laplacian, called the *quantum Lévy Laplacian*, acting on operators have been introduced and studied. In particular, in [1], the authors studied the quantum extension of the time shift of the Brownian motion to give a positive answer to the Meyer’s problem. Then the generator of the Markov semigroup generated by the quantum extension of the time shift is a quantum Laplacian. If we consider the Cesàro Hilbert space as a state space, then the generator is called the quantum Lévy Laplacian.

In this paper, we introduce a new type of noncommutative generalization of the Lévy Laplacian as follows: Consider a space $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ of white noise operators within white noise Gelfand triple $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$. Then by the kernel theorem there is a topological isomorphism

$$U : \mathcal{L}((\mathcal{S}), (\mathcal{S})^*) \longrightarrow (\mathcal{S})^* \otimes (\mathcal{S})^*$$

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since (\mathcal{S}) is a nuclear space. The classical Lévy Laplacian Δ_L acting on generalized white noise functionals is extended, denoted by $\Delta_L \otimes I + I \otimes \Delta_L$, to a subspace of $(\mathcal{S})^* \otimes (\mathcal{S})^*$ as a Lévy Laplacian acting on white noise functionals in two variables, where I is the identity operator on $(\mathcal{S})^*$. Then for Ξ belonging to a certain class of white noise operators, the quantum Lévy Laplacian Δ_{QL} is given by

$$\Delta_{QL}\Xi = \mathcal{U}^{-1}(\Delta_L \otimes I + I \otimes \Delta_L)\mathcal{U}\Xi$$

using the topological isomorphism \mathcal{U} which can be easily understood through the (classical) Lévy Laplacian and the kernel theorem in the white noise theory. Therefore, the difficulty of domain problem for the quantum Lévy Laplacian comes back to the problem for the (classical) Lévy Laplacian. This point of view is one of our main purpose of this paper. This representation implies a natural quantum-classical correspondence (Theorem 4.8). Moreover, the quantum Lévy Laplacian in [1] can be represented by the quantum Lévy Laplacian in our approach. Note that the noncommutative extension of the Lévy Laplacian introduced by Ji–Obata–Ouerdiane [10] is slightly different from our quantum Lévy Laplacian. In [10], the quantum Lévy Laplacian has been introduced by using the Wick symbols of white noise operators. On the other hand, in this paper we use the symbols of white noise operators for the quantum Lévy Laplacian. We study relevant properties of the quantum Lévy Laplacian, and construct a semigroup and a quantum stochastic process generated by the quantum Lévy Laplacian.

The paper is organized as follows. In Section 2 we recall the spaces of white noise functionals. In Section 3 we remind the theory of white noise operators with study of the basic topological isomorphism \mathcal{U} . In Section 4 we introduce the quantum Lévy Laplacian and study its properties with the natural quantum-classical correspondence. In Section 5 we study a domain of the quantum Lévy Laplacian. In Section 6 we construct an one parameter semigroup generated by the quantum Lévy Laplacian. In Section 7 we construct a quantum stochastic process associated with the quantum Lévy Laplacian.

2. White Noise Functionals

Let $H_{\mathbb{R}} = L_{\mathbb{R}}^2(\mathbb{R}, dt)$ be the real Hilbert space of square integrable functions with respect to the Lebesgue measure dt and let $A = 1 + t^2 - d^2/dt^2$ be the harmonic oscillator. The norm on $H_{\mathbb{R}}$ is denoted by $|\cdot|_0$. Then for each $p \geq 0$,

$$\mathcal{S}_{p,\mathbb{R}} = \{\xi \in H_{\mathbb{R}}; |\xi|_p \equiv |A^p \xi|_0 < \infty\}$$

becomes a Hilbert space and we have a Gelfand triple:

$$\mathcal{S}_{\mathbb{R}} = \text{proj} \lim_{p \rightarrow \infty} \mathcal{S}_{p,\mathbb{R}} \subset H_{\mathbb{R}} \subset \mathcal{S}_{\mathbb{R}}^* \cong \text{ind} \lim_{p \rightarrow \infty} \mathcal{S}_{-p,\mathbb{R}},$$

where \cong stands the topological isomorphism and for each $p > 0$, $\mathcal{S}_{-p,\mathbb{R}}$ is the completion of the Hilbert space $H_{\mathbb{R}}$ with respect to the Hilbertian norm $|\cdot|_{-p} = |A^{-p} \cdot|_0$. Then $\mathcal{S}_{\mathbb{R}}$ becomes a nuclear space and coincides with the Schwartz space. The complexification of a real locally convex space $\mathfrak{X}_{\mathbb{R}}$ is denoted by $\mathfrak{X}_{\mathbb{C}}$. If there is no confusion, we use the symbol \mathfrak{X} for $\mathfrak{X}_{\mathbb{C}}$.

The Boson Fock space over the (complex) Hilbert space $H \equiv H_{\mathbb{C}}$, denoted by $\Gamma(H)$, is the Hilbert space consisting of sequences $(f_n)_{n=0}^{\infty}$, where $f_n \in H^{\otimes n}$ (the

n -fold symmetric tensor product of H) and $\sum_{n=0}^{\infty} n! |f_n|_0^2 < \infty$, where $|\cdot|_0$ is the norm on $H^{\otimes n} \cong L^2_{\mathbb{C}}(\mathbb{R}^n)$ for each n .

For $p \in \mathbb{R}$, we put

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n! |f_n|_p^2, \quad \phi = (f_n) \in \Gamma(H),$$

where $|f_n|_p = |(A^{\otimes n})^p f_n|_0$. We set

$$(\mathcal{S}) = \left\{ \phi \in \Gamma(H); \|\phi\|_p < \infty \text{ for all } p \in \mathbb{R} \right\}$$

which becomes a countable Hilbert nuclear space with the topology induced from the norms $\{\|\cdot\|_p; p \in \mathbb{R}\}$. We thus come to a complex Gelfand triple:

$$(\mathcal{S}) \subset \Gamma(H) \subset (\mathcal{S})^*.$$

Let μ be the standard Gaussian measure on $\mathcal{S}_{\mathbb{R}}^*$ of which the characteristic functional is given by

$$\int_{\mathcal{S}_{\mathbb{R}}^*} \exp\{i \langle x, \xi \rangle\} d\mu(x) = \exp\left\{-\frac{1}{2} |\xi|_0^2\right\}, \quad \xi \in \mathcal{S}_{\mathbb{R}}.$$

Let $(L^2) \equiv L^2(\mathcal{S}_{\mathbb{R}}^*, \mu)$ be the Hilbert space of \mathbb{C} -valued square integrable functions on $\mathcal{S}_{\mathbb{R}}^*$. By the Wiener-Itô decomposition theorem, each $\phi \in (L^2)$ admits an expression

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n} :, f_n \rangle, \quad x \in \mathcal{S}_{\mathbb{R}}^*, \tag{2.1}$$

where $f_n \in H^{\widehat{\otimes} n}$ and $:x^{\otimes n} :$ denotes the Wick ordering of $x^{\otimes n}$. Moreover, the (L^2) -norm $\|\phi\|_0$ of ϕ is given by

$$\|\phi\|_0^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2.$$

An element of (L^2) is called a *white noise functional*. The nuclear space of white noise functionals which is corresponding to (\mathcal{S}) under the Wiener-Itô-Segal isomorphism is denoted by the same symbol (\mathcal{S}) . The strong dual space of the nuclear space (\mathcal{S}) is also denoted by the same symbol $(\mathcal{S})^*$. An element of $(\mathcal{S})^*$ is called a *generalized white noise functionals*. Thus we have the following Gelfand triple:

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$$

which is referred to as the *Hida-Kubo-Takenaka space* (see [11, 18]). For each $\Phi \in (\mathcal{S})^*$, there exists a unique sequence $\{F_n\}_{n=0}^{\infty}$, $F_n \in (\mathcal{S}^{\otimes n})_{sym}^*$ such that

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle$$

for all $\phi \in (\mathcal{S})$ given as in (2.1). Moreover,

$$\|\Phi\|_{-p}^2 = \sum_{n=0}^{\infty} n! |F_n|_{-p}^2 < \infty$$

for some $p \geq 0$. In this case, we use a formal expression for $\Phi \in (\mathcal{S})^*$:

$$\Phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n} :, F_n \rangle, \quad x \in \mathcal{S}_{\mathbb{R}}^* \quad \text{or} \quad \Phi = (F_n)_{n=0}^{\infty}.$$

For $p \geq 0$ let $(\mathcal{S})_p$ be the space of all $\phi \in (L^2)$ such that $\|\phi\|_p < \infty$ and $(\mathcal{S})_{-p}$ the completion of (L^2) with respect to the norm $\|\cdot\|_{-p}$. Then, for each $p \in \mathbb{R}$, equipped with the norm $\|\cdot\|_p$, $(\mathcal{S})_p$ becomes a Hilbert space. The space is identified with the inductive limit space

$$(\mathcal{S})^* = \text{ind} \lim_{p \rightarrow \infty} (\mathcal{S})_{-p}.$$

For each $\xi \in \mathcal{S} \equiv \mathcal{S}_{\mathbb{C}}$, an *exponential vector* (or *coherent vector*) ϕ_{ξ} is defined by

$$\phi_{\xi}(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n} :, \frac{\xi^{\otimes n}}{n!} \right\rangle = \exp \left\{ \langle x, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle \right\}$$

which is corresponding to $(\xi^{\otimes n}/n!)_{n=0}^{\infty}$ denoted by the same symbol ϕ_{ξ} . Then for each $z \in \mathcal{S}^*$, the exponential vector ϕ_z is defined as a generalized white noise functional corresponding to $(z^{\otimes n}/n!)_{n=0}^{\infty}$. It is well-known that $\{\phi_{\xi}; \xi \in \mathcal{S}\}$ spans a dense subspace of (\mathcal{S}) and then white noise functional $\Phi \in (\mathcal{S})^*$ is uniquely specified by the *S-transform* $S\Phi$ of Φ defined by

$$S\Phi(\xi) = \langle \langle \Phi, \phi_{\xi} \rangle \rangle, \quad \xi \in \mathcal{S}.$$

3. White Noise Operators

Since every continuous linear operator from (\mathcal{S}) into $(\mathcal{S})^*$ has a Fock expansion [18] which can be considered as a superposition of the quantum white noise, a continuous linear operator from (\mathcal{S}) into $(\mathcal{S})^*$ is called a *white noise operator*. The space of all white noise operators is denoted by $\mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ equipped with the bounded convergence topology.

A white noise operator Ξ is uniquely specified by its *symbol* which is a \mathbb{C} -valued function on $\mathcal{S} \times \mathcal{S}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \langle \langle \Xi \phi_{\xi}, \phi_{\eta} \rangle \rangle, \quad \xi, \eta \in \mathcal{S}.$$

It is straightforward to see that the symbol $\Theta = \widehat{\Xi}$ of a white noise operator $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ possesses the following properties:

- (O1) for any $\xi, \xi_1, \eta, \eta_1 \in \mathcal{S}$ the function $(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$ is entire holomorphic on $\mathbb{C} \times \mathbb{C}$;
- (O2) there exist constant numbers $C \geq 0$ and $p \geq 0$ such that

$$|\Theta(\xi, \eta)| \leq K \exp \{ |\xi|_p^2 + |\eta|_p^2 \}, \quad \xi, \eta \in \mathcal{S}. \tag{3.1}$$

The following characterization of symbols is proved by Obata in [17].

Theorem 3.1. *Let Θ be a \mathbb{C} -valued function defined on $\mathcal{S} \times \mathcal{S}$. Then Θ is the symbol of some $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ if and only if Θ satisfies the conditions (O1) and (O2).*

For each $x \in \mathcal{S}^*$ the *annihilation operator* $a(x) \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ is defined by

$$a(x)\phi_\xi = \langle x, \xi \rangle \phi_\xi, \quad \xi \in \mathcal{S}.$$

and the adjoint $a^*(x)$ is called the *creation operator*. Note that $a(x) \in \mathcal{L}((\mathcal{S}), (\mathcal{S}))$ and $a^*(x) \in \mathcal{L}((\mathcal{S})^*, (\mathcal{S})^*)$.

Since (\mathcal{S}) is a nuclear space, by the kernel theorem we have the following isomorphism:

$$\mathcal{L}((\mathcal{S}), (\mathcal{S})^*) \cong (\mathcal{S})^* \otimes (\mathcal{S})^*,$$

i.e., for each $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ there exists a unique $\Phi_\Xi \in (\mathcal{S})^* \otimes (\mathcal{S})^*$ such that

$$\langle \langle \Xi \phi, \varphi \rangle \rangle = \langle \langle \Phi_\Xi, \varphi \otimes \phi \rangle \rangle, \quad \phi, \varphi \in (\mathcal{S}).$$

Define a map $\mathcal{U} : \mathcal{L}((\mathcal{S}), (\mathcal{S})^*) \ni \Xi \mapsto \Phi_\Xi \in (\mathcal{S})^* \otimes (\mathcal{S})^*$. Conversely, for each $\phi_{x_1} \otimes \phi_{x_2} \in (\mathcal{S})^* \otimes (\mathcal{S})^*$, there exists a unique $\Xi_{\phi_{x_1} \otimes \phi_{x_2}} \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ such that

$$\langle \langle \Xi_{\phi_{x_1} \otimes \phi_{x_2}} \phi, \varphi \rangle \rangle = \langle \langle \phi_{x_1} \otimes \phi_{x_2}, \varphi \otimes \phi \rangle \rangle = \langle \langle \phi_{x_1}, \varphi \rangle \rangle \langle \langle \phi_{x_2}, \phi \rangle \rangle, \quad \phi, \varphi \in (\mathcal{S}).$$

Therefore, for any $x_1, x_2 \in \mathcal{S}^*$ and $\xi, \eta \in \mathcal{S}$ we have

$$\widehat{\Xi}_{\phi_{x_1} \otimes \phi_{x_2}}(\xi, \eta) = e^{\langle x_1, \eta \rangle + \langle x_2, \xi \rangle} = \sum_{l, m=0}^{\infty} \frac{1}{l!m!} \langle x_1^{\otimes l} \otimes x_2^{\otimes m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle. \quad (3.2)$$

4. Classical and Quantum Lévy Laplacians

4.1. Classical Lévy Laplacian. Let $F \in C^2(\mathcal{S})$. Then for each $\xi \in \mathcal{S}$ there exist $F'(\xi) \in \mathcal{S}^*$ and $F''(\xi) \in (\mathcal{S} \otimes \mathcal{S})^*$ such that

$$F(\xi + \eta) = F(\xi) + \langle F'(\xi), \eta \rangle + \frac{1}{2} \langle F''(\xi), \eta \otimes \eta \rangle + o(|\eta|_p^2), \quad \eta \in \mathcal{S}$$

for some $p \geq 0$. Moreover, both maps $\xi \mapsto F'(\xi) \in \mathcal{S}^*$ and $\xi \mapsto F''(\xi) \in (\mathcal{S} \otimes \mathcal{S})^*$ are continuous. For more study, we refer to [7].

Now suppose we are given an infinite sequence $\{e_n\}_{n=1}^{\infty} \subset \mathcal{S}$. The sequence $\{e_n\}_{n=1}^{\infty}$ is not necessarily orthogonal, but in order to ensure some reasonable properties of the Lévy Laplacian, we need to assume some conditions on $\{e_n\}$. We shall do so afterwards, see also the original definition due to Lévy [15].

We consider the Cesàro mean of $\langle F''(\xi), e_n \otimes e_n \rangle$ defined by

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle F''(\xi), e_n \otimes e_n \rangle. \quad (4.1)$$

The limit does not necessarily exist. Moreover, the limit depends not only on the choice of the sequence $\{e_n\}$ but also its arrangement. Let $\mathcal{D}_L \equiv \mathcal{D}_L(\mathcal{S}, \{e_n\})$ be the subspace of $F \in C^2(\mathcal{S})$ for which the limit (4.1) exists for all $\xi \in \mathcal{S}$. For $F \in \mathcal{D}_L$ we define

$$\Delta_L F(\xi) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle F''(\xi), e_n \otimes e_n \rangle, \quad \xi \in \mathcal{S}.$$

The operator Δ_L is called the *Lévy Laplacian* on \mathcal{S} associated with $\{e_n\}$.

We fix a finite interval T on \mathbb{R} . Take an orthogonal basis $\{e_n\}_{n=1}^{\infty} \subset \mathcal{S}$ for $L^2(T)$ satisfying the equally dense and uniform boundedness property (see [11, 12, 16]).

Let $\mathfrak{D}_L \equiv \text{Dom}(\Delta_L)$ denote the domain of the Lévy Laplacian Δ_L consisting of all $\Phi \in (\mathcal{S})^*$ satisfying that $S(\Phi) \in \mathcal{D}_L$ and there exists $\Psi_\Phi \in (\mathcal{S})^*$ such that

$$S(\Psi_\Phi)(\xi) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N S(\Phi)''(\xi)(e_n, e_n)$$

Then the *Lévy Laplacian* Δ_L is defined by

$$\Delta_L \Phi = \Psi_\Phi, \quad \Phi \in \mathfrak{D}_L \subset (\mathcal{S})^*.$$

Let \mathfrak{D}_L^T denote the set of all functionals $\Phi \in \mathfrak{D}_L$ such that $S(\Phi)(\eta) = 0$ for all $\eta \in \mathcal{S}$ with $\text{supp}(\eta) \subset T^C$.

4.2. Quantum Lévy Laplacian. Let $F \in C^2(\mathcal{S} \times \mathcal{S})$. Then for each $\xi_1, \xi_2 \in \mathcal{S}$ there exist $F'_i(\xi_1, \xi_2) \in \mathcal{S}^*$ and $F''_{ij}(\xi_1, \xi_2) \in (\mathcal{S} \otimes \mathcal{S})^*$, $i, j = 1, 2$, such that

$$\begin{aligned} F(\xi_1 + \eta_1, \xi_2 + \eta_2) &= F(\xi_1, \xi_2) + \sum_{i=1}^2 \langle F'_i(\xi_1, \xi_2), \eta_i \rangle \\ &\quad + \frac{1}{2} \sum_{i,j=1}^2 \langle F''_{ij}(\xi_1, \xi_2) \eta_i, \eta_j \rangle + o(|\eta_1|_p^2 + |\eta_2|_p^2) \end{aligned}$$

for some $p \geq 0$ and any $\eta_1, \eta_2 \in \mathcal{S}$.

Let \mathcal{D}_L^Q be the subspace of $F \in C^2(\mathcal{S} \times \mathcal{S})$ for which the limits

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle \widehat{\Xi}''_{ii}(\xi, \eta), e_n \otimes e_n \rangle, \quad i = 1, 2$$

exists for all $\xi, \eta \in \mathcal{S}$. Consider the set consisting of $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ for which $\widehat{\Xi} \in \mathcal{D}_L^Q$ and there exists $\Upsilon_\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ such that

$$\widehat{\Upsilon}_\Xi(\xi, \eta) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^2 \sum_{n=1}^N \langle \widehat{\Xi}''_{ii}(\xi, \eta), e_n \otimes e_n \rangle, \quad \xi, \eta \in \mathcal{S}. \quad (4.2)$$

This set serves as the domain of the operator Δ_{QL} in the next definition and will be denoted by $\text{Dom}(\Delta_{QL})$.

Definition 4.1. For each $\Xi \in \text{Dom}(\Delta_{QL})$, we write

$$\Delta_{QL} \Xi \equiv \Upsilon_\Xi, \quad (4.3)$$

where Υ_Ξ is given as in (4.2). Then Δ_{QL} is called the *quantum Lévy Laplacian*.

Let \mathcal{S}_L^* be the set of all elements $f \in \mathcal{S}^*$ such that the limit

$$\langle f, f \rangle_L = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle f, e_n \rangle^2$$

exists.

Proposition 4.2. For any $f, g \in \mathcal{S}_L^*$, $\Xi_{\phi_f \otimes \phi_g} \in \text{Dom}(\Delta_{QL})$ and we have

$$\Delta_{QL} \Xi_{\phi_f \otimes \phi_g} = (\langle f, f \rangle_L + \langle g, g \rangle_L) \Xi_{\phi_f \otimes \phi_g}.$$

Proof. The proof is straightforward from (3.2). □

Lemma 4.3 ([10]). *For $\eta, \zeta \in \mathcal{S}$ and $z \in \mathbb{C}$ we have*

$$\phi_{\eta+z\zeta} = \sum_{n=0}^{\infty} \frac{z^n}{n!} a^*(\zeta)^n \phi_\eta,$$

where the right hand side converges in (\mathcal{S}) uniformly in z running over a compact set in \mathbb{C} . Therefore,

$$\left. \frac{d^n}{dz^n} \right|_{z=0} \phi_{\eta+z\zeta} = a^*(\zeta)^n \phi_\eta$$

holds in (\mathcal{S}) .

Lemma 4.4. *Let $\zeta \in \mathcal{S}$ and $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$. Then*

$$\begin{aligned} \widehat{a(\zeta)^2 \Xi}(\xi, \eta) &= \left. \frac{d^2}{dz^2} \right|_{z=0} \widehat{\Xi}(\xi, \eta + z\zeta), \\ \widehat{\Xi a^*(\zeta)^2}(\xi, \eta) &= \left. \frac{d^2}{dz^2} \right|_{z=0} \widehat{\Xi}(\xi + z\zeta, \eta). \end{aligned}$$

Proof. For any $\zeta \in \mathcal{S}$ and $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$, by applying Lemma 4.3 we have

$$\widehat{a(\zeta)^2 \Xi}(\xi, \eta) = \langle\langle \Xi \phi_\xi, a^*(\zeta)^2 \phi_\eta \rangle\rangle = \left\langle\left\langle \Xi \phi_\xi, \left. \frac{d^2}{dz^2} \right|_{z=0} \phi_{\eta+z\zeta} \right\rangle\right\rangle$$

which follows the first identity from the continuity of the canonical bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$. Similarly, the second identity is proved. \square

Theorem 4.5. *For each $\Xi \in \text{Dom}(\Delta_{\text{QL}})$ we have*

$$\langle\langle (\Delta_{\text{QL}} \Xi) \phi_\xi, \phi_\eta \rangle\rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle (a(e_n)^2 \Xi + \Xi a^*(e_n)^2) \phi_\xi, \phi_\eta \rangle\rangle, \quad \xi, \eta \in \mathcal{S}. \quad (4.4)$$

Proof. By Lemma 4.4 we see that

$$\begin{aligned} \langle\langle (a(e_n)^2 \Xi) \phi_\xi, \phi_\eta \rangle\rangle &= \widehat{a(e_n)^2 \Xi}(\xi, \eta) = \left. \frac{d^2}{dz^2} \right|_{z=0} \widehat{\Xi}(\xi, \eta + ze_n), \\ \langle\langle (\Xi a^*(e_n)^2) \phi_\xi, \phi_\eta \rangle\rangle &= \widehat{\Xi a^*(e_n)^2}(\xi, \eta) = \left. \frac{d^2}{dz^2} \right|_{z=0} \widehat{\Xi}(\xi + ze_n, \eta). \end{aligned}$$

If $\Xi \in \text{Dom}(\Delta_{\text{QL}})$, the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left\{ \left. \frac{d^2}{dz^2} \right|_{z=0} \widehat{\Xi}(\xi, \eta + ze_n) + \left. \frac{d^2}{dz^2} \right|_{z=0} \widehat{\Xi}(\xi + ze_n, \eta) \right\}$$

exists and coincides with $\widehat{\Delta_{\text{QL}} \Xi}(\xi, \eta)$. Hence

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle (a(e_n)^2 \Xi + \Xi a^*(e_n)^2) \phi_\xi, \phi_\eta \rangle\rangle = \widehat{\Delta_{\text{QL}} \Xi}(\xi, \eta),$$

from which (4.4) follows. \square

Theorem 4.6. *For each $\Xi \in \text{Dom}(\Delta_{\text{QL}})$, we have*

$$\begin{aligned} \langle\langle (\Delta_{\text{QL}}\Xi)\phi_\xi, \phi_\eta \rangle\rangle &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle [\mathcal{U}^{-1}(I \otimes a(e_n)^2)\mathcal{U}\Xi] \phi_\xi, \phi_\eta \rangle\rangle, \\ &+ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle [\mathcal{U}^{-1}(a(e_n)^2 \otimes I)\mathcal{U}\Xi] \phi_\xi, \phi_\eta \rangle\rangle, \quad \xi, \eta \in \mathcal{S}. \end{aligned}$$

Proof. For any $\xi, \eta \in \mathcal{S}$, we have

$$\begin{aligned} \langle\langle \Xi a^*(e_n)^2 \phi_\xi, \phi_\eta \rangle\rangle &= \langle\langle \mathcal{U}\Xi, \phi_\eta \otimes (a^*(e_n)^2 \phi_\xi) \rangle\rangle \\ &= \langle\langle (I \otimes a(e_n)^2)\mathcal{U}\Xi, \phi_\eta \otimes \phi_\xi \rangle\rangle \\ &= \langle\langle [\mathcal{U}^{-1}(I \otimes a(e_n)^2)\mathcal{U}\Xi] \phi_\xi, \phi_\eta \rangle\rangle. \end{aligned}$$

Similarly, we prove that for any $\xi, \eta \in \mathcal{S}$

$$\langle\langle a(e_n)^2 \Xi \phi_\xi, \phi_\eta \rangle\rangle = \langle\langle [\mathcal{U}^{-1}(a(e_n)^2 \otimes I)\mathcal{U}\Xi] \phi_\xi, \phi_\eta \rangle\rangle.$$

Then the proof is completed by applying Theorem 4.5. \square

Remark 4.7. It is well-known in various contexts that

$$\Delta_{\text{L}} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(e_n)^2 \quad \text{on } \text{Dom}(\Delta_{\text{L}}).$$

In fact, the right hand side is given a meaning as following: for $\Phi \in \text{Dom}(\Delta_{\text{L}})$ we have

$$\langle\langle \Delta_{\text{L}}\Phi, \phi_\xi \rangle\rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle a(e_n)^2 \Phi, \phi_\xi \rangle\rangle, \quad \xi \in \mathcal{S}. \quad (4.5)$$

Therefore, from Theorem 4.6 we can write

$$\Delta_{\text{QL}} = \mathcal{U}^{-1}(I \otimes \Delta_{\text{L}})\mathcal{U} + \mathcal{U}^{-1}(\Delta_{\text{L}} \otimes I)\mathcal{U}. \quad (4.6)$$

4.3. Quantum–Classical Correspondence. For each $\phi, \psi \in (\mathcal{S})$, we write $\phi\psi$ for the pointwise multiplication. It is well-known that the pointwise multiplication yields a continuous bilinear map from $(\mathcal{S}) \times (\mathcal{S})$ into (\mathcal{S}) , see [11, 18]. For $\Phi \in (\mathcal{S})^*$ and $\phi \in (\mathcal{S})$ we define $\Phi\phi = \phi\Phi \in (\mathcal{S})^*$ by

$$\langle\langle \Phi\phi, \psi \rangle\rangle = \langle\langle \Phi, \phi\psi \rangle\rangle, \quad \psi \in (\mathcal{S}).$$

Obviously, the map $(\Phi, \phi) \mapsto \Phi\phi$ is a separately continuous bilinear map. In particular, each $\Phi \in (\mathcal{S})^*$ gives rise to a multiplication operator $M_\Phi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*) \cong (\mathcal{S})^* \otimes (\mathcal{S})^*$ defined by $M_\Phi\phi = \Phi\phi$. With this we have a continuous injection $(\mathcal{S})^* \hookrightarrow \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$. Note also that $(M_\Phi)^* = M_\Phi$.

Theorem 4.8. *Let $\Phi \in \text{Dom}(\Delta_{\text{L}})$. Then $M_\Phi \in \text{Dom}(\Delta_{\text{QL}})$ and*

$$\frac{1}{2}(\Delta_{\text{QL}}M_\Phi)\phi_0 = \Delta_{\text{L}}\Phi, \quad (4.7)$$

where ϕ_0 is the vacuum vector.

Proof. For any $\xi, \eta \in \mathcal{S}$, by the derivation property of $a(\zeta)$ we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle a(e_n)^2 (\Phi \phi_\xi), \phi_\eta \rangle\rangle \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle (a(e_n)^2 \Phi) \phi_\xi, \phi_\eta \rangle\rangle \\ & \quad + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N [2 \langle\langle (a(e_n) \Phi) (a(e_n) \phi_\xi), \phi_\eta \rangle\rangle + \langle\langle \Phi (a(e_n)^2 \phi_\xi), \phi_\eta \rangle\rangle]. \end{aligned}$$

On the other hand, we can easily see that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N [2 \langle\langle (a(e_n) \Phi) (a(e_n) \phi_\xi), \phi_\eta \rangle\rangle + \langle\langle \Phi (a(e_n)^2 \phi_\xi), \phi_\eta \rangle\rangle] = 0.$$

Therefore, for any $\xi, \eta \in \mathcal{S}$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle a(e_n)^2 (\Phi \phi_\xi), \phi_\eta \rangle\rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle (a(e_n)^2 \Phi) \phi_\xi, \phi_\eta \rangle\rangle$$

Similarly, we prove that for any $\xi, \eta \in \mathcal{S}$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle \Phi a^*(e_n)^2 \phi_\xi, \phi_\eta \rangle\rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle\langle (a(e_n)^2 \Phi) \phi_\xi, \phi_\eta \rangle\rangle.$$

Therefore, by applying Theorem 4.5 with $\Xi = M_\Phi$, $M_\Phi \in \text{Dom}(\Delta_{\text{QL}})$ and we have

$$\langle\langle (\Delta_{\text{QL}} M_\Phi) \phi_0, \phi_\eta \rangle\rangle = 2 \langle\langle \Delta_L \Phi, \phi_\eta \rangle\rangle, \quad \eta \in \mathcal{S} \tag{4.8}$$

which implies (4.7). □

5. New Domains of Lévy Laplacians

Given $f \in L^1_{\mathbb{C}}(\mathbb{R})^{\widehat{\otimes} l} \cap L^2_{\mathbb{C}}(\mathbb{R})^{\widehat{\otimes} l}$ and $\alpha_k \in \mathbb{R}$, $k = 1, 2, \dots, l$, we consider a generalized white noise functional Φ in \mathfrak{D}_L^T of the form:

$$\Phi = \int_{T^l} f(s_1, \dots, s_l) : e^{i\alpha_1 x(s_1)} \dots e^{i\alpha_l x(s_l)} : d\mathbf{s}, \tag{5.1}$$

where $:\cdot:$ is the Wick ordering. The S -transform of Φ is given by

$$S(\Phi)(\xi) = \int_{T^l} f(s_1, \dots, s_l) e^{i\alpha_1 \xi(s_1)} \dots e^{i\alpha_l \xi(s_l)} d\mathbf{s}.$$

By direct computation we can see that

$$\Delta_L \Phi = \left(-\frac{1}{|T|} \sum_{k=1}^l \alpha_k^2 \right) \Phi.$$

We fix a continuous function $\gamma : \mathbb{R} \rightarrow [0, \infty)$ such that there exists a stochastic process $\{X_t; t \geq 0\}$ with $\mathbf{E}[e^{izX_t}] = e^{-t\gamma(z)}$ for each $t \geq 0$. For each $n \in \mathbb{N}$, let \mathbf{D}_n^γ

be the linear space spanned by generalized white noise functionals of the form in Eq. (5.1), i.e.,

$$\Phi = \int_{T^n} f(\mathbf{s}) : \prod_{k=1}^n e^{i\alpha_k x(s_k)} : d\mathbf{s},$$

where $f \in L^1_{\mathbb{C}}(\mathbb{R})^{\widehat{\otimes} n} \cap L^2_{\mathbb{C}}(\mathbb{R})^{\widehat{\otimes} n}$ and $\alpha_k \in \mathbb{R} \setminus \{0\}, k = 1, 2, \dots, n$ satisfy the condition:

$$\alpha_1 + \dots + \alpha_n = \sqrt{|T|}n, \quad \alpha_1^2 + \dots + \alpha_n^2 = |T|\gamma(n).$$

We also put $\mathbf{D}_0^\gamma = \mathbb{C}$. Then \mathbf{D}_n^γ is a linear subspace of $(\mathcal{S})_{-p}$ for any $p > \frac{5}{12}$ (see [13]) and Δ_L is a linear operator from \mathbf{D}_n^γ into itself such that

$$\|\Delta_L \Phi\|_{-p} = \gamma(n)\|\Phi\|_{-p}, \quad \Phi \in \mathbf{D}_n^\gamma.$$

Let $\overline{\mathbf{D}}_n^\gamma$ be the completion of \mathbf{D}_n^γ in $(\mathcal{S})_{-p}$ with respect to $\|\cdot\|_{-p}$. Then for each $n \in \mathbb{N} \cup \{0\}$, $\overline{\mathbf{D}}_n^\gamma$ becomes a Hilbert space with the inner product of $(\mathcal{S})_{-p}$. For each $n \in \mathbb{N} \cup \{0\}$, the operator Δ_L can be extended to a continuous linear operator $\overline{\Delta}_L$ from $\overline{\mathbf{D}}_n^\gamma$ into itself satisfying

$$\|\overline{\Delta}_L \Phi\|_{-p} = \gamma(n)\|\Phi\|_{-p}, \quad \Phi \in \overline{\mathbf{D}}_n^\gamma.$$

The operator $\overline{\Delta}_L$ is a self-adjoint operator on $\overline{\mathbf{D}}_n^\gamma$ for each $n \in \mathbb{N} \cup \{0\}$.

Proposition 5.1 ([23]). *Let $\Phi = \sum_{n=0}^\infty \Phi_n, \Psi = \sum_{n=0}^\infty \Psi_n$ be generalized white noise functionals such that $\Phi_n, \Psi_n \in \overline{\mathbf{D}}_n^\gamma$ for each $n \in \mathbb{N} \cup \{0\}$. If $\Phi = \Psi$ in $(\mathcal{S})^*$, then $\Phi_n = \Psi_n$ for each $n \in \mathbb{N} \cup \{0\}$.*

From now on, $p > \frac{5}{12}$ will be an arbitrarily fixed number. For each $N \in \mathbb{N} \cup \{0\}$, we put

$$\mathbf{S}_{-p,N}^\gamma = \left\{ \sum_{n=0}^\infty \Phi_n \in (\mathcal{S})^*; \sum_{k=0}^N \sum_{n=0}^\infty \|\overline{\Delta}_L^k \Phi_n\|_{-p}^2 < \infty, \quad \Phi_n \in \overline{\mathbf{D}}_n^\gamma, n = 0, 1, 2, \dots \right\}.$$

By the Schwarz inequality we see that for any $N \in \mathbb{N}$, $\mathbf{S}_{-p,N}^\gamma$ is contained in $(\mathcal{S})_{-p}$ and is a Hilbert space equipped with the new norm $\|\cdot\|_{-p,N}$ given by

$$\|\Phi\|_{-p,N}^2 = \sum_{k=0}^N \sum_{n=0}^\infty \|\overline{\Delta}_L^k \Phi_n\|_{-p}^2, \quad \Phi = \sum_{n=0}^\infty \Phi_n \in \mathbf{S}_{-p,N}^\gamma.$$

Moreover, in view of the inclusion relations:

$$\dots \subset \dots \subset \mathbf{S}_{-p,N+1}^\gamma \subset \mathbf{S}_{-p,N}^\gamma \subset \dots \subset \mathbf{S}_{-p,1}^\gamma \subset (\mathcal{S})_{-p},$$

we define

$$\mathbf{S}_{-p,\infty}^\gamma = \text{proj} \lim_{N \rightarrow \infty} \mathbf{S}_{-p,N}^\gamma = \bigcap_{N=1}^\infty \mathbf{S}_{-p,N}^\gamma.$$

Note that the space $\mathbf{S}_{-p,\infty}^\gamma$ includes $\overline{\mathbf{D}}_n^\gamma$ for any $n \in \mathbb{N} \cup \{0\}$. The operator $\overline{\Delta}_L$ can be extended to a continuous linear operator from $\mathbf{S}_{-p,2}^\gamma$ into $\mathbf{S}_{-p,1}^\gamma$, denoted by the same notation $\overline{\Delta}_L$, satisfying

$$\|\overline{\Delta}_L \Phi\|_{-p,N} \leq \|\Phi\|_{-p,N+1}, \quad \Phi \in \mathbf{S}_{-p,\infty}^\gamma, \quad N = 1, 2, \dots$$

Therefore, $\overline{\Delta}_L$ is a continuous linear operator from $\mathbf{S}_{-p,\infty}^\gamma$ into itself.

Theorem 5.2 ([23]). *The operator $\overline{\Delta}_L$ is a self-adjoint operator densely defined on $\mathbf{S}_{-p,N}^\gamma$ for each $N \in \mathbb{N}$.*

Put

$$\mathbf{S}_{-p,\infty}^\gamma \otimes \mathbf{S}_{-p,\infty}^\gamma = \text{proj} \lim_{N \rightarrow \infty} \mathbf{S}_{-p,N}^\gamma \otimes \mathbf{S}_{-p,N}^\gamma \subset (\mathcal{S})_{-p} \otimes (\mathcal{S})_{-p}$$

and

$$\mathbf{L}_{-p,\infty}^\gamma = \mathcal{U}^{-1} (\mathbf{S}_{-p,\infty}^\gamma \otimes \mathbf{S}_{-p,\infty}^\gamma).$$

For any $\Phi, \Psi \in \mathbf{D}_n^\gamma$, by (4.6) we have

$$(\mathcal{U}\Delta_{\text{QL}}\mathcal{U}^{-1}) \Phi \otimes \Psi = (\Delta_L \otimes I + I \otimes \Delta_L) \Phi \otimes \Psi.$$

Therefore, the following result is straightforward.

Theorem 5.3. *The quantum Lévy Laplacian can be extended to $\mathbf{L}_{-p,\infty}^\gamma$ as a continuous linear operator, denoted by the notation $\overline{\Delta}_{\text{QL}}$. In this case, we have*

$$\overline{\Delta}_{\text{QL}} = \mathcal{U}^{-1} (\overline{\Delta}_L \otimes I + I \otimes \overline{\Delta}_L) \mathcal{U}.$$

From now on we write

$$\mathcal{L}_Q = \overline{\Delta}_L \otimes I + I \otimes \overline{\Delta}_L.$$

Then the following result is straightforward from Theorem 5.2.

Theorem 5.4. *The operator \mathcal{L}_Q is a continuous linear operator from $\mathbf{S}_{-p,\infty}^\gamma \otimes \mathbf{S}_{-p,\infty}^\gamma$ into itself. Moreover, \mathcal{L}_Q is a self-adjoint operator densely defined on $\mathbf{S}_{-p,N}^\gamma \otimes \mathbf{S}_{-p,N}^\gamma$ for each $N \in \mathbb{N}$.*

6. Semigroups Generated by Lévy Laplacians

For each $t \geq 0$ we define G_t on $\mathbf{S}_{-p,\infty}^\gamma$ by

$$G_t \Phi = \sum_{n=0}^{\infty} e^{-\gamma(n)t} \Phi_n, \quad \Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{S}_{-p,\infty}^\gamma, \quad \Phi_n \in \overline{\mathbf{D}}_n^\gamma, \quad n = 0, 1, 2, \dots$$

Then for any $N \in \mathbb{N}$ we have

$$\begin{aligned} \|G_t \Phi\|_{-p,N}^2 &= \sum_{k=1}^N \sum_{n=0}^{\infty} \gamma(n)^{2k} e^{-2\gamma(n)t} \|\Phi_n\|_{-p}^2 \\ &\leq \|\Phi\|_{-p,N}^2. \end{aligned}$$

Therefore, $\{G_t\}_{t \geq 0}$ is equi-continuous in t . On the other hand, for each $t, t_0 \geq 0$ and any $N \in \mathbb{N}$, we obtain that

$$\begin{aligned} \|G_t \Phi - G_{t_0} \Phi\|_{-p,N}^2 &= \sum_{k=1}^N \sum_{n=0}^{\infty} \gamma(n)^{2k} \left| e^{-\gamma(n)t} - e^{-\gamma(n)t_0} \right|^2 \|\Phi_n\|_{-p}^2 \\ &\leq 4 \|\Phi\|_{-p,N}^2. \end{aligned}$$

Therefore, by the Lebesgue dominated convergence theorem, we see that

$$\lim_{t \rightarrow t_0} G_t \Phi = G_{t_0} \Phi \quad \text{in } \mathbf{S}_{-p,\infty}.$$

It is easily checked the semigroup properties that $G_0 = I$ and $G_t G_s = G_{t+s}$ for any $s, t \geq 0$. Hence $\{G_t\}_{t \geq 0}$ is an equi-continuous C_0 -semigroup. For each $t \geq 0$, we put

$$\mathcal{G}_t^F = G_t \otimes G_t \quad \text{on} \quad \mathbf{S}_{-p, \infty}^\gamma \otimes \mathbf{S}_{-p, \infty}^\gamma.$$

Then it is straightforward that $\{\mathcal{G}_t^F\}_{t \geq 0}$ is an equi-continuous C_0 -semigroup on $\mathbf{S}_{-p, \infty}^\gamma \otimes \mathbf{S}_{-p, \infty}^\gamma$.

Proposition 6.1. *The infinitesimal generator of the semigroup $\{\mathcal{G}_t^F\}_{t \geq 0}$ is \mathcal{L}_Q .*

Proof. With the help of Theorem 5.4, the proof is a simple modification of the proof of Proposition 5 in [23]. \square

For each $t \geq 0$ we put

$$\mathcal{G}_t^Q = \mathcal{U}^{-1} \mathcal{G}_t^F \mathcal{U}.$$

Then $\{\mathcal{G}_t^Q\}_{t \geq 0}$ becomes a semigroup on $\mathbf{L}_{-p, \infty}^\gamma$ and the following result is obvious from Proposition 6.1.

Theorem 6.2. *The infinitesimal generator of the semigroup $\{\mathcal{G}_t^Q\}_{t \geq 0}$ is $\overline{\Delta}_{QL}$.*

7. Stochastic Processes Generated by Lévy Laplacians

Let $\{X_{1,t}\}_{t \geq 0}$ and $\{X_{2,t}\}_{t \geq 0}$ be independent continuous stochastic processes of which the characteristic functions of $X_{1,t}$ and $X_{2,t}$ are given by

$$\mathbf{E} [e^{izX_{1,t}}] = \mathbf{E} [e^{izX_{2,t}}] = e^{-t\gamma(z)}$$

and let ζ_T be a smooth function in \mathcal{S} with $\zeta_T(u) = (1/|T|)^{1/2}$ on T . We use the same notation $\overline{\Delta}_{QL}$ as $\overline{\Delta}_{QL}(\widehat{\Xi}) = \overline{\Delta}_{QL}\Xi$ for any $\Xi \in \mathbf{L}_{-p, \infty}^\gamma$. Then we can construct a quantum stochastic process generated by the quantum Lévy Laplacian as follows.

Theorem 7.1. *Let Θ be the symbol of a white noise operator in $\mathbf{L}_{-p, \infty}^\gamma$. Then it holds that*

$$e^{t\overline{\Delta}_{QL}}\Theta(\xi, \eta) = \mathbf{E}[\Theta(\xi + X_{1,t}\zeta_T, \eta + X_{2,t}\zeta_T)], \quad \xi, \eta \in \mathcal{S}.$$

Proof. Let $l, m \in \mathbb{N}$. Suppose that Θ is given by

$$\Theta(\xi, \eta) = \int_{T^{l+m}} f(s_1, \dots, s_l; t_1, \dots, t_m) \prod_{j=1}^l \left(e^{i\alpha_j \xi(s_j)} \right) \prod_{k=1}^m \left(e^{i\beta_k \eta(t_k)} \right) ds dt \quad (7.1)$$

with $f \in L_{\mathbb{C}}^1(\mathbb{R})^{\widehat{\otimes}(l+m)} \cap L_{\mathbb{C}}^2(\mathbb{R})^{\widehat{\otimes}(l+m)}$, and $\alpha_j \in \mathbb{R} \setminus \{0\}, j = 1, 2, \dots, l, \beta_k \in \mathbb{R} \setminus \{0\}, k = 1, 2, \dots, m$ satisfying the condition:

$$\begin{aligned} \alpha_1 + \dots + \alpha_l &= \sqrt{|T|}l, & \alpha_1^2 + \dots + \alpha_l^2 &= |T|\gamma(l), \\ \beta_1 + \dots + \beta_m &= \sqrt{|T|m}, & \beta_1^2 + \dots + \beta_m^2 &= |T|\gamma(m). \end{aligned} \quad (7.2)$$

Then we have

$$\mathbf{E}[\Theta(\xi + X_{1,t}\zeta_T, \eta + X_{2,t}\zeta_T)] = e^{-t(\gamma(l)+\gamma(m))}\Theta(\xi, \eta) = e^{t\overline{\Delta}_{QL}}\Theta(\xi, \eta), \quad \xi, \eta \in \mathcal{S}.$$

Let $\Theta = \sum_{l,m=0}^{\infty} \Theta_{l,m}$ belong to the set $\widehat{\mathbf{L}}_{-p,\infty}^{\gamma}$ of all symbols of operators in $\mathbf{L}_{-p,\infty}^{\gamma}$. Denote the integral given as in (7.1) by $\mathbf{I}_{\alpha,\beta}(f)(\xi, \eta)$. Then for each $l, m \in \mathbb{N} \cup \{0\}$, $\Theta_{l,m}$ can be expressed as the form:

$$\Theta_{l,m}(\xi, \eta) = \lim_{N \rightarrow \infty} \sum_{\alpha^{[N]}, \beta^{[N]}} \mathbf{I}_{\alpha^{[N]}, \beta^{[N]}}(f_{\alpha^{[N]}, \beta^{[N]}})(\xi, \eta)$$

for a sequence $f_{\alpha^{[N]}, \beta^{[N]}}$, $N = 1, 2, \dots$ of functions in $L_{\mathbb{C}}^1(\mathbb{R})^{\widehat{\otimes}(l+m)} \cap L_{\mathbb{C}}^2(\mathbb{R})^{\widehat{\otimes}(l+m)}$, where $\sum_{\alpha^{[N]}, \beta^{[N]}}$ means a sum of finitely many terms on $\alpha^{[N]} = (\alpha_1^{[N]}, \dots, \alpha_l^{[N]}) \in (\mathbb{R} \setminus \{0\})^l$ and $\beta^{[N]} = (\beta_1^{[N]}, \dots, \beta_m^{[N]}) \in (\mathbb{R} \setminus \{0\})^m$ satisfying (7.2). Therefore, we have

$$\begin{aligned} & \sum_{l,m=0}^{\infty} \mathbf{E} [|\Theta_{l,m}(\xi + X_{1,t}\zeta_T, \eta + X_{2,t}\zeta_T)|] \\ &= \sum_{l,m=0}^{\infty} \mathbf{E} \left[\lim_{N \rightarrow \infty} \left| \sum_{\alpha^{[N]}, \beta^{[N]}} \mathbf{I}_{\alpha^{[N]}, \beta^{[N]}}(f_{\alpha^{[N]}, \beta^{[N]}})(\xi, \eta) e^{ilX_{1,t}} e^{imX_{2,t}} \right| \right] \\ &= \sum_{l,m=0}^{\infty} \lim_{N \rightarrow \infty} \left| \sum_{\alpha^{[N]}, \beta^{[N]}} \mathbf{I}_{\alpha^{[N]}, \beta^{[N]}}(f_{\alpha^{[N]}, \beta^{[N]}})(\xi, \eta) \right| \\ &= \sum_{l,m=0}^{\infty} |\Theta_{l,m}(\xi, \eta)|. \end{aligned}$$

On the other hand, $\Theta_{l,m} = \widehat{\Xi}_{l,m}$ for some $\Xi_{l,m} \in \mathbf{L}_{-p,\infty}^{\gamma}$ and so

$$\sum_{l,m=0}^{\infty} |\Theta_{l,m}(\xi, \eta)| \leq \sum_{l,m=0}^{\infty} \|\mathcal{U}\Xi_{l,m}\|_{-p} \|\phi_{\xi}\|_p \|\phi_{\eta}\|_p < \infty, \quad \xi, \eta \in \mathcal{S}.$$

Therefore, by continuity of $e^{t\overline{\Delta}_{\text{QL}}}$ we obtain that

$$\begin{aligned} \mathbf{E} [\Theta(\xi + X_{1,t}\zeta_T, \eta + X_{2,t}\zeta_T)] &= \sum_{l,m=0}^{\infty} \mathbf{E} [\Theta_{l,m}(\xi + X_{1,t}\zeta_T, \eta + X_{2,t}\zeta_T)] \\ &= \sum_{l,m=0}^{\infty} e^{t\overline{\Delta}_{\text{QL}}} \Theta_{l,m}(\xi, \eta) \\ &= e^{t\overline{\Delta}_{\text{QL}}} \Theta(\xi, \eta) \end{aligned}$$

which proves the assertion. □

For each $\Phi \in (\mathcal{S})^*$ and $\eta \in \mathcal{S}$, the translation $T_{\eta}\Phi$ of Φ is well defined as an element in $(\mathcal{S})^*$ by $T_{\eta}\Phi(\cdot) = \Phi(\cdot + \eta)$ and then $S(T_{\eta}\Phi)(\xi) = S(\Phi)(\xi + \eta)$ for any $\xi \in \mathcal{S}$. Therefore, for any $\eta, \zeta \in \mathcal{S}$ and any operator $\Xi \in \mathcal{L}((\mathcal{S}), (\mathcal{S})^*)$ we define $\mathbf{T}_{\eta,\zeta}\Xi$ by

$$\mathbf{T}_{\eta,\zeta}\Xi = \mathcal{U}^{-1} (T_{\eta} \otimes T_{\zeta}) \mathcal{U}\Xi.$$

With this notation the following corollary is obvious from Theorem 7.1.

Corollary 7.2. *Let $\Xi \in \mathbf{L}_{-p,\infty}^\gamma$. Then it holds that*

$$\mathcal{G}_t^{\mathbf{Q}\Xi} = e^{t\bar{\Delta}_{\mathbf{Q}\mathbf{L}}}\Xi = \mathbf{E} [\mathbf{T}_{X_{1,t}\zeta_T, X_{2,t}\zeta_T}\Xi].$$

Corollary 7.3. *Let $\Phi \in \mathbf{S}_{-p,\infty}^\gamma$. Then it holds that*

$$G_t\Phi = e^{t\bar{\Delta}_{\mathbf{L}}}\Phi = \mathbf{E} [T_{X_{1,t}\zeta_T}\Phi]. \quad (7.3)$$

Proof. For each $\Phi \in \mathbf{S}_{-p,\infty}^\gamma$, $\Xi_{\Phi \otimes \phi_0} = \mathcal{U}^{-1}(\Phi \otimes \phi_0) \in \mathbf{L}_{-p,\infty}^\gamma$ and

$$\mathcal{G}_t^{\mathbf{Q}}(\Xi_{\Phi \otimes \phi_0})\phi_0 = (\Xi_{G_t\Phi \otimes \phi_0})\phi_0 = G_t\Phi.$$

and, by applying Corollary 7.2 and Theorem 5.3, we have

$$\mathcal{G}_t^{\mathbf{Q}}(\Xi_{\Phi \otimes \phi_0})\phi_0 = e^{t\bar{\Delta}_{\mathbf{Q}\mathbf{L}}}(\Xi_{\Phi \otimes \phi_0})\phi_0 = \Xi_{e^{t\bar{\Delta}_{\mathbf{L}}}\Phi \otimes \phi_0}\phi_0 = e^{t\bar{\Delta}_{\mathbf{L}}}\Phi$$

which proves the first identity in (7.3). On the other hand, for any $\eta, \zeta \in \mathcal{S}$

$$\mathbf{T}_{\eta,\zeta}\Xi_{\Phi \otimes \phi_0} = \Xi_{T_\eta\Phi \otimes \phi_0}, \quad (\mathbf{T}_{\eta,\zeta}\Xi_{\Phi \otimes \phi_0})\phi_0 = T_\eta\Phi.$$

Therefore, for any $\eta \in \mathcal{S}$

$$\begin{aligned} \langle\langle \mathbf{E} [\mathbf{T}_{X_{1,t}\zeta_T, X_{2,t}\zeta_T}\Xi_{\Phi \otimes \phi_0}] \phi_0, \phi_\eta \rangle\rangle &= \mathbf{E} [\langle\langle (\mathbf{T}_{X_{1,t}\zeta_T, X_{2,t}\zeta_T}\Xi_{\Phi \otimes \phi_0}) \phi_0, \phi_\eta \rangle\rangle] \\ &= \mathbf{E} [\langle\langle T_{X_{1,t}\zeta_T}\Phi, \phi_\eta \rangle\rangle] \\ &= \langle\langle \mathbf{E} [T_{X_{1,t}\zeta_T}\Phi], \phi_\eta \rangle\rangle \end{aligned}$$

which implies that

$$\mathbf{E} [\mathbf{T}_{X_{1,t}\zeta_T, X_{2,t}\zeta_T}\Xi_{\Phi \otimes \phi_0}] \phi_0 = \mathbf{E} [T_{X_{1,t}\zeta_T}\Phi].$$

Therefore, by Corollary 7.2 we have

$$\begin{aligned} (\mathcal{G}_t^{\mathbf{Q}}\Xi_{\Phi \otimes \phi_0})\phi_0 &= \mathbf{E} [\mathbf{T}_{X_{1,t}\zeta_T, X_{2,t}\zeta_T}\Xi_{\Phi \otimes \phi_0}] \phi_0 \\ &= \mathbf{E} [T_{X_{1,t}\zeta_T}\Phi] \end{aligned}$$

which proves the second identity in (7.3). \square

The results in this paper can be extended to a more general Gelfand triple if we replace the test function (\mathcal{S}) by a test function space $W = F_\theta(N)$ as in [10].

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UN CIG JI: DEPARTMENT OF MATHEMATICS, RESEARCH INSTITUTE OF MATHEMATICAL FINANCE, CHUNGBUK NATIONAL UNIVERSITY, CHEONGJU 361-763, KOREA
E-mail address: uncigji@cbucc.chungbuk.ac.kr

HABIB OUERDIANE: DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DE SCIENCES DE TUNIS, UNIVERSITÉ DE TUNIS EL MANAR, CAMPUS UNIVERSITAIRE, TUNIS, 1060 TUNISIA
E-mail address: habib.ouerdiane@fst.rnu.tn

KIMIAKI SAITÔ: DEPARTMENT OF MATHEMATICS, MEIJO UNIVERSITY, NAGOYA 468-8502, JAPAN
E-mail address: ksaito@ccmfs.meijo-u.ac.jp