

MARTINGALE REPRESENTATION FOR CONTINGENT CLAIMS WITH REGIME SWITCHING

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ABSTRACT. We derive a martingale representation for a contingent claim under a Markov-modulated version of the Black-Scholes economy. The martingale representation for the price of the claim is established with respect to an equivalent martingale measure chosen by the Esscher transform. Under some differentiability conditions for the coefficients of the price processes, we shall identify explicitly the integrands in the martingale representation using stochastic flows. We shall introduce a zero-coupon bond to minimize the residual risk due to incomplete hedging.

1. Introduction

Recently, regime-switching models have played an important role in different branches of modern financial economics. The origin of regime-switching models in econometrics goes back to the original work of Hamilton (1989) in which a discrete-time Markov-switching autoregressive time series models was proposed. Applications of regime-switching models penetrate different areas in modern financial economics. Some works on these applications include Elliott and van der Hoek (1997) for asset allocation, Pliska (1997), and Elliott, Hunter and Jamieson (2001) for short rate models, Elliott and Hinz (2002) for portfolio analysis and chart analysis, Niak (1993), Guo (2001) and Buffington and Elliott (2002a,b) for option valuation and Elliott, Malcolm and Tsoi (2003) for volatility estimation.

The representation of martingales as stochastic integrals and the identification of the integrand in the representation are important topics in stochastic calculus and its applications. In mathematical finance, the integrand in the martingale representation of the discounted price process of a contingent claim may be used to construct a hedging policy for the claim. Colwell, Elliott and Kopp (1991) use the martingale representation theorem and stochastic flows to identify a hedging policy for a contingent claim in the context of a multi-dimensional diffusion model. Colwell and Elliott (1993) use a martingale representation result to construct the local risk-minimizing strategy explicitly. Their result provides an alternative motive for the concept of the minimal martingale measure.

In this paper, we shall derive a martingale representation for a contingent claim written on a risky asset under a Markov-modulated version of the Black-Scholes economy. We assume that the market parameters, including the market interest

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rate of a bank account, the drift and the volatility of the underlying risky asset, switch over time according to the state of a continuous-time Markov chain. The state of the Markov chain represents the state of an economy. One key feature of the Markov-modulated Black-Scholes economy is that the market is not complete. Here, we employ the Esscher transform to determine an equivalent martingale measure for pricing. The Esscher transform is a well-known tool in actuarial science. It has been adopted to develop premium rules and approximate aggregate claim distributions. The seminal work of Gerber and Shiu (1994) pioneers the use of the Esscher transform for option valuation in an incomplete market. Their work highlights the interplay between financial and actuarial pricing, which is a key topic in contemporary actuarial research, as pointed by Bühlmann et al. (1996). The martingale representation for the price of the claim is then established based on the equivalent martingale measure chosen by the Esscher transform. The price of the claim is represented as a sum of two stochastic integrals, one with respect to the standard Brownian motion driving the price process of the underlying risky asset and the other with respect to the martingale component of the Markov chain. Under some differentiability conditions for the coefficients of the price processes we shall explicitly identify the integrands in the martingale representation of the claim's price using the concept of stochastic flows. Due to the incompleteness of the market, there is no perfect hedging strategy for the claim. Here, we shall introduce a zero-coupon bond to minimize the residual risk due to incomplete hedging.

This paper is structured as follows. Section two presents the price dynamics of the model and the use of the Esscher transform to determine an equivalent martingale measure. The main result for the martingale representation and its derivation are presented in section three. We shall use a zero-coupon bond to minimize the risk due to incomplete hedging. The final section summarizes the paper.

2. The Dynamics and Change of Measures

Consider a financial market with a bank account B and a risky asset S , that are tradable continuously. Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where \mathcal{P} is a real-world probability. Denote by \mathcal{T} the interval $[0, T]$, where $T < \infty$. Let $X := \{X_t\}_{t \in \mathcal{T}}$ be a continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathcal{P})$ taking values in a finite state space $\mathcal{X} := \{x_1, x_2, \dots, x_N\}$. We interpret the state of X as the state of the economy. Without loss of generality, we can identify the state space of X with a finite set of unit vectors $\mathcal{E} := \{e_1, e_2, \dots, e_N\}$, where $e_i = (0, \dots, 1, \dots, 0)^T \in \mathfrak{R}^N$ and y^T denotes the transpose of a vector y . This can be done by introducing a bijection which maps \mathcal{X} into \mathcal{E} , (see Elliott (1993) for details). The set \mathcal{E} is called the canonical representation of \mathcal{X} . Let A denote the generator, or the rate matrix, $[a_{ij}(t)]_{i,j=1,2,\dots,N}$ of X . Then, with the canonical representation of \mathcal{X} , Elliott (1993) and Elliott et al (1994) provide the following semi-martingale decomposition for X :

$$X_t = X_0 + \int_0^t AX_s ds + M_t .$$

Here $\{M_t\}_{t \in \mathcal{T}}$ is an \mathfrak{R}^N -valued martingale with respect to the filtration generated by X under \mathcal{P} .

Let r_t denote the instantaneous market interest rate of the bank account B at time t . We suppose that

$$r_t := r(t, X_t) = \langle r, X_t \rangle ,$$

where $r := (r_1, r_2, \dots, r_N)^T \in \mathfrak{R}^N$ with $r_i > 0$, for each $i = 1, 2, \dots, N$.

Then, the price dynamics of the bank account $\{B_t\}_{t \in \mathcal{T}}$ are:

$$B_t = \exp \left(\int_0^t r_u du \right) , \quad B_0 = 1 .$$

Let $\mu(\cdot, \cdot, \cdot) : \mathcal{T} \times \mathfrak{R}^+ \times \mathcal{E} \rightarrow \mathfrak{R}$ and $\sigma(\cdot, \cdot, \cdot) : \mathcal{T} \times \mathfrak{R}^+ \times \mathcal{E} \rightarrow \mathfrak{R}^+$ denote measurable functions of (t, S, X) such that $\mu(t, S, e_i)$ and $\sigma(t, S, e_i)$ are three times differentiable in S and which, together with their derivatives, are bounded, for each $t \in \mathcal{T}$ and each $i = 1, 2, \dots, N$.

Let $W := \{W_t\}_{t \in \mathcal{T}}$ denote a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to $\mathcal{F}^W := \{\mathcal{F}_t^W\}_{t \in \mathcal{T}}$, the \mathcal{P} -augmentation of the natural filtration generated by the W . Then, the price dynamics of the risky asset S are assumed to be governed by the following Markov-modulated stochastic differential equation:

$$dS_t = S_t(\mu(t, S_t, X_t)dt + \sigma(t, S_t, X_t)dW_t) .$$

Denote the discounted price as $\xi := \{\xi_t\}_{t \in \mathcal{T}}$, where $\xi_t := e^{-\int_0^t r_u du} S_t$. Let $R_t := e^{\int_0^t r_u du}$, for each $t \in \mathcal{T}$. Then, the discounted price process is governed by:

$$d\xi_t = (\mu(t, R_t \xi_t, X_t) - r_t)\xi_t dt + \sigma(t, R_t \xi_t, X_t)\xi_t dW_t . \quad (2.1)$$

Define $f(\cdot, \cdot, \cdot) : \mathcal{T} \times \mathfrak{R}^+ \times \mathcal{E} \rightarrow \mathfrak{R}$ and $\sigma(\cdot, \cdot, \cdot) : \mathcal{T} \times \mathfrak{R}^+ \times \mathcal{E} \rightarrow \mathfrak{R}^+$:

$$f(t, \xi, X) := (\mu(t, R\xi, X) - r_t)\xi ,$$

and

$$\Sigma(t, \xi, X) := \sigma(t, R\xi, X)\xi .$$

Note that f and σ are measurable functions that are three times differentiable in ξ , and which, together with their derivatives, are bounded, for each $t \in \mathcal{T}$ and $X \in \mathcal{E}$. Then, we can write the discounted price process as follows:

$$d\xi_t = f(t, \xi_t, X_t)dt + \Sigma(t, \xi_t, X_t)dW_t .$$

The market considered here is incomplete in general due to an additional source of uncertainty described by the switching regimes. Since the market is incomplete, there is more than one equivalent martingale measure. Here, we shall employ the regime-switching Esscher transform as in Elliott, Chan and Siu (2005) to determine an equivalent martingale measure.

Let $\{\mathcal{F}_t^X\}_{t \in \mathcal{T}}$ denote the complete, right-continuous filtration generated by X . Define \mathcal{G}_t to be the σ -algebra $\mathcal{F}_t^X \vee \mathcal{F}_t^W$, for each $t \in \mathcal{T}$. Let θ_t denote a regime-switching Esscher parameter at time t . We suppose that

$$\theta_t := \theta(t, X_t) = \langle \theta, X_t \rangle ,$$

where $\theta := (\theta_1, \theta_2, \dots, \theta_N)^T \in \mathfrak{R}^N$.

Define a \mathcal{G} -adapted process $\Lambda^\theta := \{\Lambda_t^\theta\}_{t \in \mathcal{T}}$ as below:

$$\Lambda_t^\theta := \frac{\exp(\int_0^t \theta_s dW_s)}{E[\exp(\int_0^t \theta_s dW_s) | \mathcal{F}_t^X]},$$

where $E[\cdot]$ denotes an expectation with respect to the measure \mathcal{P} . Note that

$$E\left[\exp\left(\int_0^t \theta_s dW_s\right) \middle| \mathcal{F}_t^X\right] = \exp\left(\frac{1}{2} \int_0^t \theta_s^2 ds\right).$$

Then,

$$\Lambda_t^\theta = \exp\left(\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds\right),$$

or equivalently,

$$d\Lambda_t^\theta = \Lambda_t^\theta \theta_s dW_s.$$

Hence, Λ^θ is a local-martingale with respect to $(\mathcal{G}, \mathcal{P})$. We suppose that Λ^θ is a $(\mathcal{G}, \mathcal{P})$ -martingale.

Then, the regime switching Esscher transform $\mathcal{Q}_\theta \sim \mathcal{P}$ on \mathcal{G}_T with respect to a family of parameters $\{\theta_t\}_{t \in \mathcal{T}}$ is defined as:

$$\left. \frac{d\mathcal{Q}_\theta}{d\mathcal{P}} \right|_{\mathcal{G}_T} = \Lambda_T^\theta.$$

The fundamental theorem of asset pricing establishes the relationship between the absence of arbitrage opportunities and the existence of an equivalent martingale measure and in a basic framework was proved by Harrison and Kreps (1979), Harrison and Pliska (1981, 1983). The modern version of this theorem, established by Delbaen and Schachermayer (2004), states that the absence of arbitrage opportunities is “essentially” equivalent to the existence of an equivalent martingale measure under which the discounted stock price process is a martingale. Let $\{\eta_t\}_{t \in \mathcal{T}}$ denote a family of risk-neutral regime-switching Esscher parameters. Write $\tilde{\mathcal{G}}_t := \mathcal{F}_T^X \vee \mathcal{F}_t^W$, for each $t \in \mathcal{T}$. In our setting, the martingale condition is given by considering an enlarged filtration:

$$\xi_s = E^\eta[\xi_t | \tilde{\mathcal{G}}_s], \quad \text{for any } t, s \in \mathcal{T} \text{ with } s \leq t,$$

where $E^\eta[\cdot]$ represents an expectation with respect to \mathcal{Q}_η .

Proposition 2.1. *The martingale condition is satisfied if and only if*

$$\eta_t := \eta(t, S, X) = \frac{r_t - \mu(t, S, X)}{\sigma(t, S, X)}, \quad t \in \mathcal{T}.$$

Proof. By the Bayes' rule,

$$\begin{aligned} E^\eta[\xi_t | \tilde{\mathcal{G}}_s] &= \frac{E[\Lambda_t^\eta \xi_t | \tilde{\mathcal{G}}_s]}{E[\Lambda_t^\eta | \tilde{\mathcal{G}}_s]} = E\left[\frac{\Lambda_t^\eta}{\Lambda_s^\eta} \xi_t \middle| \tilde{\mathcal{G}}_s\right] \\ &= \xi_s E\left\{\exp\left[\int_s^t \left(\mu_u - \frac{1}{2}\sigma_u^2 - \frac{1}{2}\eta_u^2 - r_u\right) du + \int_s^t (\eta_u + \sigma_u) dW_u\right] \middle| \tilde{\mathcal{G}}_s\right\} \\ &= \exp\left[\int_s^t (\mu_u - r_u + \eta_u \sigma_u) du\right]. \end{aligned}$$

The martingale condition is satisfied if and only if

$$\int_s^t (\mu_u - r_u + \eta_u \sigma_u) du = 0, \quad \text{for any } t, s \in \mathcal{T} \text{ with } s \leq t.$$

That is,

$$\mu_t - r_t + \eta_t \sigma_t = 0, \quad \text{for all } t \in \mathcal{T}.$$

Hence, the result follows. \square

Note that $\eta(t, S, X)$ is also a measurable function that satisfies the differentiability and linear growth conditions similar to those of f , for each $t \in \mathcal{T}$ and $X \in \mathcal{E}$.

Define the stochastic exponential $\mathcal{M}_{s,t}$ as below:

$$\mathcal{M}_{s,t}(z) := 1 + \int_s^t \mathcal{M}_{s,r}(z) \eta(r, R_r \xi_{s,r}(z), X_r) dW_r. \quad (2.2)$$

Then,

$$\mathcal{M}_{0,t}(z_0) = \exp \left(\int_0^t \eta_u dW_u - \frac{1}{2} \int_0^t \eta_u^2 du \right).$$

Note that $\{\mathcal{M}_{0,t}(z_0)\}_{t \in \mathcal{T}}$ is a local-martingale with respect to $(\mathcal{G}, \mathcal{P})$ with

$$E[\mathcal{M}_{0,t}(z_0)] = 1.$$

Define a probability measure $\mathcal{Q}^\eta \sim \mathcal{P}$ on \mathcal{G}_T as follows:

$$\frac{d\mathcal{Q}^\eta}{d\mathcal{P}} := \mathcal{M}_{0,T}(z_0).$$

By Girsanov's theorem,

$$\tilde{W}_t := W_t - \int_0^t \eta_u du$$

is a standard Brownian motion with respect to $(\mathcal{G}, \mathcal{Q}^\eta)$.

Under the risk-neutral regime-switching Esscher transform \mathcal{Q}^η ,

$$d\xi_t = \sigma_t \xi_t d\tilde{W}_t.$$

This means that the discounted price processes are local martingales with respect to $(\mathcal{G}, \mathcal{Q}^\eta)$. We assume that they are $(\mathcal{G}, \mathcal{Q}^\eta)$ -martingales.

3. A Martingale Representation

In this section, we shall derive a martingale representation result for a European-style contingent claim. More specifically, the price process of the claim is represented as a sum of two stochastic integrals, one with respect to a standard Brownian motion driving the price process of the underlying risky asset and another with respect to the martingale component of the Markov chain X under a risk-neutral probability measure. Note that the integrators of the two stochastic integrals are orthogonal. The integrands of the martingale representation of the claim are then identified explicitly using the concept of stochastic flows. Since the market is incomplete in general, there is no perfect hedging strategy for the claim. We shall

introduce a zero-coupon bond to minimize the residual risk due to incomplete hedging.

First, write $\xi_{s,t}(z)$ for the unique, strong solution of the Markov-modulated stochastic differential equation (2.1) for $t \geq s$ with initial condition $\xi_{s,s}(z) = z \in \mathfrak{R}^+$. By adopting similar arguments in Kunita (1978, 1982), Bismut (1981) and Elliott and Kopp (2004), there exists a flow of diffeomorphisms $z \rightarrow \xi_{s,t}(z)$ associated with this system. Write

$$D_{s,t}(z) := \frac{\partial \xi_{s,t}(z)}{\partial z} ,$$

for the Jacobian of the map $z \rightarrow \xi_{s,t}(z)$. Then, the Jacobian D satisfies the following linearized equation:

$$dD_{s,t}(z) = f_x(t, \xi_t, X_t)D_{s,t}dt + \sigma_x(t, \xi_t, X_t)D_{s,t}dW_t ,$$

with initial condition $D_{s,s}(z) = 1$.

Similarly to the arguments in Kunita (1978, 1982), Bismut (1981), it can be shown that $D_{s,t}^{-1}(z)$ exists. Also, $D_{s,t}^{-1}(z)$ satisfies the following equation:

$$\begin{aligned} dD_{s,t}^{-1}(z) &= -D_{s,t}^{-1}(z)f_x(t, \xi_t, X_t)dt - D_{s,t}^{-1}(z)\sigma_x(t, \xi_t, X_t)dW_t \\ &\quad + D_{s,t}^{-1}(z)(\sigma_x(t, \xi_t, X_t))^2 dt , \end{aligned}$$

with initial condition $D_{s,s}^{-1}(z) = 1$.

Recall that

$$d\xi_t = (\mu_t - r_t)\xi_t dt + \sigma_t \xi_t dW_t .$$

Let $Y_t = \ln(S_t/S_0)$. Then,

$$dY_t = \left(\mu_t - \frac{\sigma_t^2}{2} \right) dt + \sigma_t dW_t .$$

Consider a function $\psi(\cdot) : \mathfrak{R}^+ \rightarrow \mathfrak{R}$ such that $\psi(\cdot)$ is twice differentiable and $\psi(\cdot)$ and ψ_x are of at most linear growth in x . We shall determine the current price at time t of a contingent claim of the form $\psi(S_T)$, which is the payoff of the claim at maturity $T > t$. Now, we shall work with the discounted claim as a function of the discounted stock price as follows:

$$\hat{\psi}(\xi_T) := R_T^{-1}\psi(R_T\xi_T) = R_T^{-1}\psi(S_T) .$$

Note that $\hat{\psi}(\cdot)$ also has linear growth. So, define the square-integrable $(\mathcal{G}, \mathcal{Q}^\eta)$ -martingale $\{V_t\}_{t \in \mathcal{T}}$ as below:

$$V_t := E^\eta[\hat{\psi}(\xi_T)|\mathcal{G}_t] , \quad t \in \mathcal{T} .$$

Proposition 3.1. *$\{V_t\}_{t \in \mathcal{T}}$ has the following martingale representation:*

$$V_t = V_0 + \int_0^t \gamma_s d\tilde{W}_s + \int_0^t \langle \alpha_s, dM_s \rangle , \tag{3.1}$$

where γ_s and α_s are such that $\int_0^T E[\gamma_s^2]ds < \infty$ and $\int_0^T E[|\alpha_s|^2]ds < \infty$ with $\|\cdot\|$ a \mathcal{L}^2 -norm on \mathfrak{R}^N . Furthermore:

$$\begin{aligned} \gamma_t &= E \left[\int_t^T \eta_\xi(s, R_s \xi_{0,s}(z_0), X_s) D_{0,s}(z_0) d\tilde{W}_s \cdot \hat{\psi}(\xi_{0,T}(z_0)) \right. \\ &\quad \left. + \hat{\psi}_\xi(\xi_{0,T}(z_0)) D_{0,T}(z_0) \Big| \mathcal{G}_t \right] D_{0,t}^{-1}(z_0) \xi_{0,t}(z_0) \sigma_t, \end{aligned}$$

and

$$\alpha_t = \mathbf{V}(t, \xi_{0,t}(z_0)) \in \mathfrak{R}^N.$$

Proof. Let $z = \xi_{0,t}(z_0)$, for each $t \in \mathcal{T}$. By the semi-group property of the solution of the stochastic differential equation (2.1),

$$\xi_{0,T}(z_0) = \xi_{t,T}(\xi_{0,t}(z_0)) = \xi_{t,T}(z). \quad (3.2)$$

Differentiating (3.2) with respect to z_0 , we obtain:

$$D_{0,T}(z_0) = D_{t,T}(z) D_{0,t}(z_0).$$

Note also that the stochastic exponential \mathcal{M} satisfies:

$$\mathcal{M}_{0,T}(z_0) = \mathcal{M}_{0,t}(z_0) \mathcal{M}_{t,T}(z).$$

For each $z \in \mathfrak{R}^+$ and $x \in \mathcal{E}$, define

$$V(t, z, x) := E[\mathcal{M}_{t,T}(z) \psi(\xi_{t,T}(z)) | X(t) = x, \xi_{0,t}(z_0) = z].$$

Then, by the Bayes' rule and the Markov property,

$$\begin{aligned} V_t &= E^\eta[\psi(\xi_{0,T}(z_0)) | \mathcal{G}_t] \\ &= \frac{E[\mathcal{M}_{0,T}(z_0) \psi(\xi_{0,T}(z_0)) | \mathcal{G}_t]}{E[\mathcal{M}_{0,T}(z_0) | \mathcal{G}_t]} \\ &= E[\mathcal{M}_{t,T}(z) \psi(\xi_{t,T}(z)) | X(t) = x, \xi_{0,t}(z_0) = z] \\ &= V(t, z, x). \end{aligned}$$

Note that under \mathcal{Q}^η ,

$$d\xi_{0,t}(z_0) = \xi_t \sigma_t d\tilde{W}_t.$$

Write

$$\mathbf{V}(s, \xi_{0,s}(z_0)) := (V(s, \xi_{0,s}(z_0), e_1), V(s, \xi_{0,s}(z_0), e_2), \dots, V(s, \xi_{0,s}(z_0), e_N)).$$

Then, we expand $V(t, z, x) := V(t, \xi_{0,t}(z_0), x)$ by Itô's differentiation rule and obtain:

$$\begin{aligned} V(t, \xi_{0,t}(z_0), x) &= V_t \\ &= V(0, z_0, X_0) + \int_0^t \left[\frac{\partial V}{\partial t}(s, \xi_{0,s}(z_0), x) + \frac{1}{2} \sigma_s^2 \frac{\partial^2 V}{\partial z^2}(s, \xi_{0,s}(z_0), x) \right] ds \\ &\quad + \int_0^t \frac{\partial V}{\partial z}(s, \xi_{0,s}(z_0), x) \xi_s \sigma_s d\tilde{W}_s + \int_0^t \langle \mathbf{V}(s, \xi_{0,s}(z_0)), A X_s \rangle ds \\ &\quad + \int_0^t \langle \mathbf{V}(s, \xi_{0,s}(z_0)), dM_s \rangle. \end{aligned} \quad (3.3)$$

Note that a special semi-martingale can be represented uniquely as the sum of a (local) martingale and a predictable process of (locally) integrable variation and that such a decomposition is unique. Since V_t is a special semi-martingale, the decompositions (3.1) and (3.3) must be identical. Note that there is no bounded variation terms, which are not martingales, in (3.1). So,

$$\frac{\partial V}{\partial t}(t, \xi_{0,t}(z_0), x) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 V}{\partial z^2}(t, \xi_{0,t}(z_0), x) + \langle \mathbf{V}(s, \xi_{0,t}(z_0)), AX_t \rangle = 0 ,$$

with $V(T, z, x) = \hat{\psi}(z)$. Also, $\gamma_t = \frac{\partial V}{\partial z}(t, \xi_{0,t}(z_0), x)\xi_t\sigma_t$ and $\alpha_t = \mathbf{V}(t, \xi_{0,t}(z_0))$.

Recall that $\xi_{t,T}(z) = \xi_{0,T}(z_0)$. So, from the differentiability and linear growth conditions of η ,

$$\frac{\partial V}{\partial z}(t, z, x) = E \left[\frac{\partial \mathcal{M}_{t,T}(z)}{\partial z} \hat{\psi}(\xi_{0,T}(z_0)) + \mathcal{M}_{t,T}(z) \frac{\partial \hat{\psi}}{\partial z}(\xi_{t,T}(z)) \right] .$$

Using the differentiability of solutions of stochastic differential equations,

$$\begin{aligned} \frac{\partial \mathcal{M}_{t,T}(z)}{\partial z} &= \int_t^T \eta(s, R_s \xi_{t,s}(z), X_s) \frac{\partial \mathcal{M}_{t,s}(z)}{\partial z} dW_s \\ &\quad + \int_t^T \eta_\xi(s, R_s \xi_{t,s}(z), X_s) \frac{\partial \xi_{t,s}(z)}{\partial z} \mathcal{M}_{t,s}(z) dW_s . \end{aligned} \tag{3.4}$$

By the variation of constants,

$$\frac{\partial \mathcal{M}_{t,T}(z)}{\partial z} = \mathcal{M}_{t,T}(z) \int_t^T \eta_\xi(s, R_s \xi_{t,s}(z), X_s) D_{t,s}(z) d\tilde{W}_s . \tag{3.5}$$

This result can be verified by differentiation, since (3.4) has a unique solution.

Applying Itô's differentiation rule to the product

$$\mathcal{M}_{t,T}(z) \int_t^T \eta_\xi(s, R_s \xi_{t,s}(z), X_s) D_{t,s}(z) d\tilde{W}_s$$

and using (2.2), we get

$$\begin{aligned} &\mathcal{M}_{t,T}(z) \int_t^T \eta_\xi(s, R_s \xi_{t,s}(z), X_s) D_{t,s}(z) d\tilde{W}_s \\ &= \int_t^T \mathcal{M}_{t,s}(z) \eta_\xi(s, R_s \xi_{t,s}(z), X_s) D_{t,s}(z) (dW_s - \eta(s, R_s \xi_{t,s}(z), X_s) ds) \\ &\quad + \int_t^T \left(\mathcal{M}_{t,s}(z) \int_t^s \eta_\xi(u, R_u \xi_{t,u}(z), X_u) D_{t,u}(z) d\tilde{W}_u \right) \eta(s, R_s \xi_{t,s}(z), X_s) dW_s \\ &\quad + \int_t^T \mathcal{M}_{t,s}(z) \eta_\xi(s, R_s \xi_{t,s}(z), X_s) D_{t,s}(z) \eta(s, R_s \xi_{t,s}(z), X_s) ds \\ &= \int_t^T \mathcal{M}_{t,s}(z) \eta_\xi(s, R_s \xi_{t,s}(z), X_s) D_{t,s}(z) dW_s \\ &\quad + \int_t^T \left(\mathcal{M}_{t,s}(z) \int_t^s \eta_\xi(u, R_u \xi_{t,u}(z), X_u) D_{t,u}(z) d\tilde{W}_u \right) \eta(s, R_s \xi_{t,s}(z), X_s) dW_s . \end{aligned}$$

From (3.5),

$$\mathcal{M}_{t,s}(z) \int_t^s \eta_\xi(u, R_u \xi_{t,u}(z), X_u) D_{t,u}(z) d\tilde{W}_u = \frac{\partial \mathcal{M}_{t,s}(z)}{\partial z} .$$

Hence,

$$\begin{aligned} \mathcal{M}_{t,T}(z) & \int_t^T \eta_\xi(s, R_s \xi_{t,s}(z), X_s) D_{t,s}(z) d\tilde{W}_s \\ & = \int_t^T \mathcal{M}_{t,s}(z) \eta_\xi(s, R_s \xi_{t,s}(z), X_s) D_{t,s}(z) dW_s \\ & \quad + \int_t^T \frac{\partial \mathcal{M}_{t,s}(z)}{\partial z} \eta(s, R_s \xi_{t,s}(z), X_s) dW_s . \end{aligned}$$

This verifies (3.5). By noticing that $z = \xi_{0,t}(z_0)$,

$$\begin{aligned} \frac{\partial V(t, z, x)}{\partial z} & = E \left\{ \mathcal{M}_{t,T}(z) \left[\int_t^T \eta_\xi(s, R_s \xi_{t,s}(z), X_s) D_{t,s}(z) d\tilde{W}_s \cdot \hat{\psi}(\xi_{0,T}(z_0)) \right. \right. \\ & \quad \left. \left. + \hat{\psi}_\xi(\xi_{t,T}(z)) D_{t,T}(z) \right] \right\} \\ & = E^\eta \left[\int_t^T \eta_\xi(s, R_s \xi_{0,s}(z_0), X_s) D_{0,s}(z_0) d\tilde{W}_s \cdot \hat{\psi}(\xi_{0,T}(z_0)) \right. \\ & \quad \left. + \hat{\psi}_\xi(\xi_{0,T}(z_0)) D_{0,T}(z_0) \Big| \mathcal{G}_t \right] D_{0,t}^{-1}(z_0) . \end{aligned}$$

Hence, the result follows. \square

Consider a zero-coupon bond with maturity T and face value one. Then, the price of the bond at time t is given by:

$$P(t, T, X_t) := E^\eta \left[\exp \left(- \int_t^T r_u du \right) \Big| \mathcal{F}_t^X \right] .$$

Let $\phi^i(t) = P(t, T, e_i)$, for each $i = 1, 2, \dots, N$ and for each $t \in \mathcal{T}$, and write $\phi(t) := (\phi_1(t), \phi_2(t), \dots, \phi_N(t))^T \in \mathfrak{R}^N$. Then,

$$P(t, T, X_t) = \langle \phi(t), X_t \rangle .$$

Let $B := \text{diag}(r) - A^T$, where $\text{diag}(r)$ is an $(N \times N)$ -diagonal matrix with elements given by the vector r . Then, Elliott and Kopp (2004) show that

$$\frac{d\phi(t)}{dt} = B\phi(t) ,$$

where $\phi(T) = \mathbf{1}_N = (1, 1, \dots, 1)^T \in \mathfrak{R}^N$.

Now, the discounted bond price

$$\begin{aligned}\tilde{P}(t, T, X_t) &:= \exp\left(-\int_0^t r_u du\right) P(t, T, X_t) \\ &= \exp\left(-\int_0^t r_u du\right) \langle \phi(t), X_t \rangle \\ &= E^\eta \left[\exp\left(-\int_0^T r_u du\right) \Big| \mathcal{F}_t^X \right],\end{aligned}$$

is a $(\mathcal{F}^X, \mathcal{Q}^\eta)$ -martingale.

By Itô's differentiation rule,

$$\begin{aligned}& \exp\left(-\int_0^t r_u du\right) \langle \phi(t), X_t \rangle \\ &= P(0, T, X_0) + \int_0^t \left[-r_s \exp\left(-\int_0^s r_u du\right) \right] \langle \phi(s), X_s \rangle ds \\ & \quad + \int_0^t \exp\left(-\int_0^s r_u du\right) \left(\left\langle \frac{d\phi(s)}{ds}, X_s \right\rangle + \langle \phi(s), \Pi X_s \rangle \right) ds \\ & \quad + \int_0^t \exp\left(-\int_0^s r_u du\right) \langle \phi(s), dM_s \rangle.\end{aligned}$$

Since the discounted bond price is an $(\mathcal{F}^X, \mathcal{Q}^\eta)$ -martingale, the bounded variation terms, which are not martingales, in the above stochastic integral representation must be identical to zero. Hence,

$$\begin{aligned}\exp\left(-\int_0^t r_u du\right) \langle \phi(t), X_t \rangle &= P(0, T, X_0) \\ & \quad + \int_0^t \exp\left(-\int_0^s r_u du\right) \langle \phi(s), dM_s \rangle.\end{aligned}$$

We shall now describe a trading portfolio involving the risky asset S and the zero-coupon bond which minimizes the residual risk of incomplete hedging. Let $\{u_t\}_{t \in \mathcal{T}}$ denote a \mathcal{G} -predictable real-valued process such that $\int_0^T E[u_t^2] dt < \infty$. For each $t \in \mathcal{T}$, u_t represents the number of units of the discounted zero-coupon bond in the trading portfolio at time t . Note that $E^\eta[\hat{\psi}(\xi_T)]$ is the fair price of the contingent claim. Let

$$\begin{aligned}\tilde{\gamma}_t &:= E \left[\int_t^T \eta_\xi(s, R_s \xi_{0,s}(z_0), X_s) D_{0,s}(z_0) d\tilde{W}_s \cdot \hat{\psi}(\xi_{0,T}(z_0)) \right. \\ & \quad \left. + \hat{\psi}_\xi(\xi_{0,T}(z_0)) D_{0,T}(z_0) \Big| \mathcal{G}_t \right] D_{0,t}^{-1}(z_0).\end{aligned}$$

Then, write the martingale representation in the following form:

$$\begin{aligned}V_t &= E^\eta[\hat{\psi}(\xi_T)] + \int_0^t \tilde{\gamma}_s d\xi_s + \int_0^t u_t d\tilde{P}(t, T, X) \\ & \quad + \int_0^t \left\langle \alpha_s - u_s \exp\left(-\int_0^s r_u du\right) \phi(s), dM_s \right\rangle.\end{aligned}\tag{3.6}$$

Let

$$\mathcal{R}_t(u) := \int_0^t \left\langle \alpha_s - u_s \exp \left(- \int_0^s r_u du \right) \phi(s), dM_s \right\rangle .$$

Then, $\{\mathcal{R}_t(u)\}_{t \in \mathcal{T}}$ is an $(\mathcal{F}^X, \mathcal{P})$ -martingale. So, $E[\mathcal{R}_T(u)] = 0$ and the variance of $\mathcal{R}_T(u)$ is:

$$\text{Var}[\mathcal{R}_T(u)] = E[(\mathcal{R}_T(u))^2] .$$

Suppose that the residual risk due to incomplete hedging corresponding to the trading portfolio $(\tilde{\gamma}, u)$ is described by $\text{Var}[\mathcal{R}_T(u)]$. Then, the optimal trading policy \hat{u} can be determined by minimizing the residual risk as follows:

$$R_T(\hat{u}) = \min_u E[(\mathcal{R}_T(u))^2] . \quad (3.7)$$

We shall evaluate $E[(\mathcal{R}_T(u))^2]$ in the sequel. First, note that the state vector X_t is one of the $e_i = (0, 0, \dots, 1, \dots, 0)^T \in \mathfrak{R}^N$. So,

$$X_t \otimes X_t = \text{diag}(X_t) , \quad (3.8)$$

where \otimes represents a tensor or Kronecker product.

We shall then evaluate (3.8) in two different ways. First, by applying Itô's differentiation rule on $X_t \otimes X_t$,

$$X_t \otimes X_t = X_0 \otimes X_0 + \int_0^t X_{u-} dX_u^T + \int_0^t dX_u X_{u-}^T + [X, X]_t ,$$

where $[X, X]_t = \sum_{0 < u \leq t} (\Delta X_u)(\Delta X_u)^T \in \mathfrak{R}^{N \times N}$.

Now,

$$[X, X]_t = [M, M]_t = \sum_{0 < u \leq t} (\Delta M_u)(\Delta M_u)^T ,$$

and $\langle M, M \rangle_t$ is the unique predictable process such that $[M, M]_{t-} + \langle M, M \rangle_t$ is a martingale. Then,

$$\begin{aligned} X_t \otimes X_t &= X_0 \otimes X_0 + \int_0^t (X_u \otimes X_u) A^T du + \int_0^t A (X_u \otimes X_u) du \\ &\quad + \int_0^t X_{u-} dM_u^T + \int_0^t dM_u X_{u-}^T + [M, M]_{t-} + \langle M, M \rangle_t + \langle M, M \rangle_t \\ &= \text{diag}(X_0) + \int_0^t \text{diag}(X_u) \cdot A^T du + \int_0^t A \cdot \text{diag}(X_u) du + \int_0^t X_{u-} dM_u^T \\ &\quad + \int_0^t dM_u X_{u-}^T + [M, M]_{t-} + \langle M, M \rangle_t + \langle M, M \rangle_t \in \mathfrak{R}^{N \times N} . \end{aligned} \quad (3.9)$$

Note that

$$\begin{aligned} \text{diag}(X_t) &= \text{diag} \left(X_0 + \int_0^t A X_u du + M_t \right) \\ &= \text{diag}(X_0) + \int_0^t \text{diag}(A X_u) du + \text{diag}(M_t) \in \mathfrak{R}^{N \times N} . \end{aligned} \quad (3.10)$$

Here, $X_t \otimes X_t$ is a special semimartingale. Hence, using the uniqueness of its decomposition into a sum of a predictable process and a martingale, from (3.9) and (3.10),

$$\langle M, M \rangle_t = \int_0^t \text{diag}(A_u X_u) du - \int_0^t \text{diag}(X_u) \cdot A^T du - \int_0^t A \cdot \text{diag}(X_u) du .$$

Suppose

$$\mathcal{R}_T(u) = \int_0^T \langle z_s, dM_s \rangle ,$$

where $z_s := \alpha_s - u_s \exp(-\int_0^s r_u du) \phi(s) \in \mathfrak{R}^N$. Then,

$$\begin{aligned} E[(\mathcal{R}_T(u))^2] &= E\left[\left(\int_0^T \langle z_s, dM_s \rangle\right)^2\right] \\ &= E\left[\int_0^T z_s^T d \langle M_s, M_s \rangle z_s\right] \\ &= E\left[\int_0^T z_s^T (\text{diag}(A_s X_s) - \text{diag}(X_s) \cdot A^T - A \cdot \text{diag}(X_s)) z_s ds\right] . \end{aligned} \quad (3.11)$$

Differentiating (3.11), at each s , we require the u_s such that

$$\begin{aligned} &-\exp\left(-\int_0^s r_u du\right) \phi^T(s) \left(\text{diag}(A_s X_s) - \text{diag}(X_s) \cdot A^T - A \cdot \text{diag}(X_s)\right) \\ &\left(\alpha_s - u_s \exp\left(-\int_0^s r_u du\right) \phi(s)\right) = 0 . \end{aligned} \quad (3.12)$$

This provides the first-order condition for the minimization problem (3.7). Solve (3.12) to get

$$u_s = \frac{\phi^T(s) \left(\text{diag}(A_s X_s) - \text{diag}(X_s) \cdot A^T - A \cdot \text{diag}(X_s)\right) \alpha_s}{\exp\left(-\int_0^s r_u du\right) \phi^T(s) \left(\text{diag}(A_s X_s) - \text{diag}(X_s) \cdot A^T - A \cdot \text{diag}(X_s)\right) \phi(s)} .$$

From (3.6),

$$\begin{aligned} V_t &= E^\eta[\hat{\psi}(\xi_T)] + \int_0^t \tilde{\gamma}_s d\xi_s + \int_0^t u_s d\tilde{P}(s, T, X) \\ &\quad + \int_0^t \left\langle \alpha_s - u_s \exp\left(-\int_0^s r_y dy\right) \phi(s), dM_s \right\rangle \\ &= E^\eta[\hat{\psi}(\xi_T)] + \int_0^t \tilde{\gamma}_s d\xi_s + \int_0^t u_s d\tilde{P}(s, T, X) + \mathcal{R}_t(u) . \end{aligned} \quad (3.13)$$

Here, $E^\eta[\hat{\psi}(\xi_T)]$ is the initial price of the claim determined by the Esscher transform. $\tilde{\gamma}_s$ and u_s represent the amount of the discounted risky asset and the discounted zero-coupon bond, respectively, held at time s . The sum of the first two stochastic integrals in the representation (3.13) is the gain process of a risk-minimization trading strategy $\{(\tilde{\gamma}_t, u_t)\}_{t \in \mathcal{T}}$ involving the hedging strategy

$\{\tilde{\gamma}_t\}_{t \in \mathcal{T}}$. The term $\{\mathcal{R}_t(u)\}_{t \in \mathcal{T}}$ represents the (cumulative) cost process for the strategy $\{(\tilde{\gamma}_t, u_t)\}_{t \in \mathcal{T}}$ and is the extra wealth or capital cost that must be put in. Note that $\{\mathcal{R}_t(u)\}_{t \in \mathcal{T}}$ is a $(\mathcal{F}^X, \mathcal{P})$ -martingale. Hence, $\{(\tilde{\gamma}_t, u_t)\}_{t \in \mathcal{T}}$ is said to be mean-self-financing. This is a generalization of the self-financing case in which $\mathcal{R}_t(u) = \mathcal{R}_0(u) = 0$, for all $t \in \mathcal{T}$. Note that $\mathcal{R}_T(u)$ is random and represents the residual risk of incomplete hedging. Here, we adopt $Var[\mathcal{R}_T(u)]$ to measure or describe this risk and $\{(\tilde{\gamma}_t, u_t)\}_{t \in \mathcal{T}}$ is the trading strategy which minimizes $Var[\mathcal{R}_T(u)]$.

4. Conclusion

We derived a martingale representation result for the price process of a contingent claim written on a risky asset whose price dynamic is governed by a Markov-modulated stochastic differential equation. We adopted the Esscher transform to determine an equivalent martingale measure in the incomplete market setting. The martingale representation was established under the equivalent martingale measure chosen by the Esscher transform. By adopting the concept of stochastic flows and the unique decomposition of a special semi-martingale, we identified the integrands in the martingale representation explicitly. We introduced a zero-coupon bond to develop a trading policy that minimizes the residual risk due to incomplete hedging.

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