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LOCAL APPROXIMATIONS FOR BRANCHING PARTICLE SYSTEMS

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ABSTRACT. We derive different local approximations along with estimates of the remainders for the total mass processes of two branching particle systems. To this end, we investigate the corresponding classes of integer-valued variables. One of them is comprised of Pólya-Aeppli distributions, while members of the other class are the convolutions of a zero-modified geometric law. We obtain the closed-form representation for the probability function of the latter convolutions and derive its asymptotics. Properties of the limiting Poisson-exponential laws are also described. Our techniques involve a Poisson mixture representation, Laplace's method and upper estimates in the local Poisson theorem. The parallels with Gnedenko's method of accompanying infinitely divisible laws are established.

1. Introduction

This work concerns subtle properties of the total mass process of two branching-fluctuating particle systems (or *BPS*'s). The consideration of these systems is partly motivated by the specific models studied in author's joint papers Dawson and Vinogradov [9]; Dawson et al. [7]–[8], which contain their '*high-density*' approximations by virtue of *Dawson-Watanabe processes*. In addition, this paper is related to author's studies Vinogradov [30]–[32] on properties of Pólya-Aeppli, zero-modified geometric and Poisson-exponential probability laws.

Our main result is Theorem 2.9. It reveals the difference between the local versions of a famous (common) approximation for the cluster structure of two BPS's. This result seems to be of a particular value. The reader is referred to Dawson and Vinogradov [9, Proof of Proposition 1.10 and formula (1.16')] for the background information on this approximation. Our local counterparts to the above-quoted results as well as the methods used for their derivation provide a better understanding of this *'high-density'* approximation. Moreover, they establish parallels between imposing the *Poisson initial condition* and *Gnedenko's method of accompanying infinitely divisible laws*. The latter method is reviewed in Refs. 14, Section 24 and 13, Section 48, Theorem 1. See also Remark 2.8. In short, it becomes possible to approximate the original BPS, for which the distribution of the total number of surviving particles can be represented as n-fold convolution of a zero-modified geometric law, by virtue of an auxiliary BPS. The latter one possesses a simpler structure having all its univariate distributions to belong to the Pólya-Aeppli additive exponential dispersion model (compare to Vinogradov [31, Section 3]). At the same time, the difference between the distributions of these two BPS's can be evaluated

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with a reasonable accuracy by employing sharp upper estimates in the local Poisson limit theorem.

In addition, the techniques used involve a Poisson mixture representation and Laplace's method. They enable one to obtain an upper estimate for the accuracy of the local approximation of scaled Pólya-Aeppli distributions by means of a Poisson-exponential law. Herewith, Theorem 2.9.i refines Theorem 3.2 and Corollary 3.1 of Vinogradov [31]. The author elected to include a review of the properties of the limiting Poisson-exponential distributions in Section 2. This exposition can be useful to those readers who want to familiarize themselves with this class of mixed probability laws. Its members emerge in numerous areas of the theory of stochastic processes, and their systematic studies are thus long overdue.

Recall that the total number of particles of the original BPS which are alive at an arbitrary fixed time instant can be represented as n-fold convolution of a zero-modified geometric law. It appears that comprehensive studies of these convolutions were overlooked. Therefore, in the same Section 2 we derive the closed-form representation for the probability function of such convolutions and investigate its asymptotic properties (see Theorem 2.3 and Propositions 2.4–2.5).

For the sake of brevity, we frequently refer to our earlier articles quoted above. For convenience of the reader, the technical proofs are deferred to the concluding Section 3.

2. The Main and Auxuliary Results

In this section, we first introduce zero-modified geometric laws and their n-fold convolutions relating them to a branching particle system. This will enable the reader to get a deeper insight into the structure of various models that emerge in the theory of branching and related processes. Subsequently, we will proceed with the derivation of asymptotic properties of these and related probability laws. Also, we will present some properties of the limit distributions and draw parallels between the techniques employed and the classical method developed by Gnedenko.

Now, let us proceed with the rigorous statements. Hereinafter, $\mathbf{Z}_+$ and $\mathbf{N}$ denote the sets of all non-negative and all positive integers, respectively.

Definition 2.1. (*zero-modified geometric law*). R.v. $Y_{\gamma,r}$ with the topological support $\mathbf{Z}_+$ is said to have *zero-modified geometric* law with real-valued parameters $\gamma \in (0,1)$ and $r \in (-(1-\gamma)/\gamma, 1)$ if

$$\mathbf{P}\{Y_{\gamma,r} = 0\} = \gamma, \tag{2.1}$$

and $\forall k \in \mathbf{N}$,

$$\mathbf{P}\{Y_{\gamma,r} = k\} = \gamma \cdot (1-\gamma) \cdot (1-r) \cdot \{1-\gamma+\gamma \cdot r\}^{k-1}. \tag{2.2}$$

The proof of the fact that the collection (2.1)–(2.2) stipulates a proper probability law on $\mathbf{Z}_+$ is straightforward.

Remark 2.2. **(i)** The standard geometric distribution with parameter $1-\gamma$ corresponds to $r = 0$. Here, we adapt the terminology developed in Sections 5.2 and 9.7 of the reference book on univariate discrete distributions by Johnson et al. [19]. In contrast, a slight modification of this law which emerges in formulas (3.10)–(3.11), is related to Bernoulli trials, and has range $\mathbf{N}$ is hereinafter referred to as the *shifted geometric* distribution (compare

to Johnson et al. [19, p. 410]). Our preference is mainly justified by better structural properties of geometric laws whose topological support is $\mathbf{Z}_+$.
(ii) The fulfillment of condition $r > 0$ in formula (2.2) corresponds to the *inflation of zeros* as compared to *standard* geometric law, while a value of $r < 0$ is interpreted as their *deflation*. The infinite divisibility of this class and its other properties are addressed in Vinogradov [32].

It was already said that convolutions of zero-modified geometric laws had naturally appeared in the original derivation of the 'high-density' limit for a certain branching-fluctuating population. Although this approach goes back to the seminal papers authored by Jirina, Lamperti, Silverstein and Watanabe, but our presentation relies to a large extent on the ideas and methods developed by Dawson. See Dawson [5] for a comprehensive treatment.

Consider BPS $\Xi_t^{(\eta)}$ which is originated from η identical and independent particles located at the origin at the initial time instant $t = 0$. In the sequel, parameter η will approach infinity. Each particle has the same mass $1/\eta$ and performs an independent spatial motion. The lifetime of each particle is exponentially distributed with mean $1/\eta$. At the end of its birth epoch, the particle either dies out or splits into two offspring. Each of these two events are assumed to have equal probabilities $1/2$. This mechanism is usually termed the *critical binary branching*. Every newly born particle is an identical replicate of its parent and immediately starts to perform the same spatial motion. The motions, lifetimes and branchings of all particles are assumed to be independent of each other and of everything else.

At this stage, let us concentrate on the evolution of an individual (or tagged) particle from the initial set of η particles. It is convenient to introduce the quantity

$$Q_t^{(\eta)} = 2/(\eta \cdot t + 2). \tag{2.3}$$

An application of scaling arguments to the results by Kolmogorov and Dmitriev [23, p. 308] stipulates that expression (2.3) constitutes the *probability of survival* of descendents of this (tagged) particle by time t (see also Dawson and Vinogradov [9, formula (1.12)]). By analogy to Dawson and Vinogradov [9], we say that this (tagged) particle gives rise to a *cluster of its offspring*, which are alive at time t. Of course, such cluster can be empty with probability $1 - Q_t^{(\eta)}$. Let us denote the size of this cluster by $S_t^{(\eta)}(i)$, where $1 \le i \le \eta$ stands for the index of this tagged particle.

The arguments by Vinogradov [32, Lemma 2.1 and formulas (2.10)–(2.11)] imply that

$$S_t^{(\eta)}(i) \stackrel{\mathrm{d}}{=} Y_{1-Q_t^{(\eta)},1-Q_t^{(\eta)}/(1-Q_t^{(\eta)})}, \tag{2.4}$$

where the latter quantity constitutes a zero-modified geometric r.v. with the specified values of parameters (see Definition 2.1). Hereinafter, the sign '$\stackrel{\mathrm{d}}{=}$' means that the distributions of r.v.'s coincide. Subsequently, we introduce an important r.v. that is hereinafter denoted by $M_t^{(\eta)}$ and is known as the *total mass* of BPS $\Xi_t^{(\eta)}$:

$$M_t^{(\eta)} := \frac{1}{\eta} \cdot \sum_{i=1}^{\eta} S_t^{(\eta)}(i). \tag{2.5}$$

It follows from formulas (2.4)–(2.5) that the total number of particles $\eta \cdot M_t^{(\eta)}$ of BPS $\Xi_t^{(\eta)}$ is represented as η-fold convolution of a particular zero-modified geometric law $Y_{1-Q_t^{(\eta)},1-Q_t^{(\eta)}/(1-Q_t^{(\eta)})}$ (compare to Vinogradov [32, formula (2.12)]). It is important to derive the closed-form expression and the asymptotic properties for the probability function of such laws. To this end, set

$$U_{\gamma,r}(k) := \sum_{i=1}^{k} Y_{\gamma,r}(i). \tag{2.6}$$

Here, $k \geq 1$ is an arbitrary *fixed* integer, and $\{Y_{\gamma,r}(i), 1 \leq i \leq k\}$ are independent zero-modified geometric r.v.'s with the common distribution given by (2.1)–(2.2). Denote the probability function $\mathbf{P}\left\{U_{\gamma,r}(k) = n\right\}$ by $\mathbf{p}_{\gamma,r,k}(n)$, where $n \in \mathbf{Z}_+$. Let $a \vee b$ denote $max(a, b)$.

Theorem 2.3. *For arbitrary fixed admissible values of parameters $k \in \mathbf{N}$, real $\gamma \in (0, 1)$, real $r \in (-(1-\gamma)/\gamma, 1)$, and an arbitrary $n \in \mathbf{Z}_+$,*

$$\begin{aligned} \mathbf{p}_{\gamma,r,k}(n) &= (-\gamma \cdot r)^k \cdot (1-\gamma+\gamma \cdot r)^{n-k} \\ &\cdot \sum_{m=(k-n)\vee 0}^{k} \binom{k}{m} \cdot \binom{m+n-1}{m+n-k} \cdot \left(\frac{1-\gamma+\gamma \cdot r}{-r} \right)^m . \end{aligned} \tag{2.7}$$

Proof of Theorem 2.3 is deferred to Section 3.

It is clear that in the case when $r = 0$, (2.7) degenerates into the formula for the probability function of the negative binomial distribution. In this case, the sum that emerges on the right-hand side of (2.7) is simplified, since all its terms except the last one are equal to zero. (Of course, one first has to cancel out the corresponding powers of $-r$.)

The asymptotics of the tail of this function are not hard to derive. Thus, in view of (2.6), the probability-generating function (or *p.g.f.*) $\psi_{\gamma,r,k}(z)$ of r.v. $U_{\gamma,r}(k)$ equals $\psi_{\gamma,r,1}(z)^k$, whereas function $\psi_{\gamma,r,1}(z)$ is given by Vinogradov [32, formula (2.7)]. Hence,

$$\begin{aligned} \psi_{\gamma,r,k}(z) &= \left(\frac{\gamma \cdot r}{1-\gamma+\gamma \cdot r} + \frac{\gamma \cdot (1 - r/(1-\gamma+\gamma \cdot r))}{1-(1-\gamma+\gamma \cdot r) \cdot z} \right)^k \\ &= \left(\frac{\gamma \cdot (1 - r \cdot z)}{1-(1-\gamma+\gamma \cdot r) \cdot z} \right)^k , \end{aligned} \tag{2.8}$$

where $|z| < R_{\gamma,r} := (1-\gamma+\gamma \cdot r)^{-1}$. Subsequently, (2.8) implies the validity of

Proposition 2.4. *For arbitrary fixed admissible values of parameters $k \in \mathbf{N}$, real $\gamma \in (0, 1)$ and real $r \in (-(1-\gamma)/\gamma, 1)$,*

$$\mathbf{p}_{\gamma,r,k}(n) \sim \frac{n^{k-1}}{(k-1)!} \cdot [\gamma \cdot (1-\gamma) \cdot (1-r)]^k \cdot (1-\gamma+\gamma \cdot r)^{n-k} \tag{2.9}$$

as $n \to \infty$.

Proof. It is easily derived by combining the fact that p.g.f. (2.8) is a rational function with a slight correction of Feller [12, v. 1, Exercise XI.25]. □

A similar method enables one to derive the asymptotics of the tail of d.f. of this r.v. In order to present this result, set

$$\mathbf{q}_{\gamma,r,k}(n) := \mathbf{P}\left\{U_{\gamma,r}(k) > n\right\}, \tag{2.10}$$

where integer $n \geq 1$ (compare to Ref. 12, v. 1, formula (XI.1.2)). Then the next representation holds.

Proposition 2.5. *For arbitrary fixed admissible values of parameters $k \in \mathbf{N}$, real $\gamma \in (0,1)$ and real $r \in (-(1-\gamma)/\gamma, 1)$,*

$$\mathbf{q}_{\gamma,r,k}(n) \sim \frac{(\gamma \cdot (1-r))^{k-1} \cdot (1-\gamma)^k}{(k-1)!} \cdot n^{k-1} \cdot (1-\gamma+\gamma \cdot r)^{n-k+1} \tag{2.11}$$

as $n \to \infty$.

Proof of Proposition 2.5 is deferred to Section 3.

It had already been said that it is the case when parameter $\eta \to \infty$ (or $k \to \infty$) that is of primary interest for the theory of branching diffusions. In the case when a spatial motion of particles is being taken into account, this leads to a limit which is called a *Dawson-Watanabe process*. It is also relevant that this limit of BPS $\Xi_t^{(\eta)}$ as $\eta \to \infty$ is frequently termed the '*high-density*' limit.

It is of interest that the univariate distributions of the high-density limit of the total mass process $M_t^{(\eta)}$ defined by (2.5) are expressed in terms of *Poisson-exponential* laws. The latter probability laws were considered in Vinogradov [30]–[32]. In particular, the class of Poisson-exponential distributions comprises a *reproductive* exponential dispersion model. Its members belong to the *power-variance family* of probability laws corresponding to the value of the power parameter $p = 3/2$.

The systematic treatment of the properties of this subclass of the family of compound Poisson distributions does not seem to be available. Since members of this class frequently emerge in the theory of stochastic processes, it appears to be useful to present main properties of Poisson-exponential laws as well as to provide further references.

Following Vinogradov [30], hereinafter this class of probability distributions is denoted by $\left\{Tw_{3/2}(\mu,\lambda), \mu \in \mathbf{R}_+^1, \lambda \in \mathbf{R}_+^1\right\}$, where $\mathbf{R}_+^1 := (0,\infty)$. Set

$$\theta_{3/2} := 2 \cdot \lambda/\sqrt{\mu} \tag{2.12}$$

and

$$\phi_{3/2} := \lambda \cdot \sqrt{\mu}, \tag{2.13}$$

which constitute the *exponential tilting* and *shape parameters*, respectively (compare to Vinogradov [30, formulas (1.2) and (1.5)]). By Vinogradov [30, formula (2.4)], non-negative r.v. $Tw_{3/2}(\mu,\lambda)$ can be represented as the Poisson random sum of independent r.v.'s having common exponential distribution with mean $\theta_{3/2}^{-1}$. The corresponding value of the Poisson parameter should be equal to $2 \cdot \phi_{3/2}$.

The r.v. $Tw_{3/2}(\mu,\lambda)$ has a point mass at the origin:

$$\mathbf{P}\{Tw_{3/2}(\mu,\lambda) = 0\} = \exp\{-2 \cdot \phi_{3/2}\}. \tag{2.14}$$

In addition, its absolutely continuous component has the density hereinafter denoted by

$$\begin{aligned} f_{3/2,\mu,\lambda}(x) &:= x^{-1} \cdot \exp\left\{-\theta_{3/2} \cdot (x+\mu)\right\} \cdot \sum_{k=1}^{\infty} \frac{(4 \cdot \lambda^2 \cdot x)^k}{k! \cdot \Gamma(k)} \\ &= \frac{2 \cdot \lambda}{\sqrt{x}} \cdot \exp\left\{-\theta_{3/2} \cdot (x+\mu)\right\} \cdot I_1(4 \cdot \lambda \cdot \sqrt{x}). \end{aligned} \tag{2.15}$$

Here, $x \in \mathbf{R}^1_+$, whereas $\Gamma(\cdot)$ and $I_1(\cdot)$ denote the gamma function and the *modified Bessel function of the first kind*, respectively. The probabilistic meaning of parameters μ and λ is clarified in Refs. 30, formula (1.6) and 31, Section 2. In particular,

$$\mathbf{Var}(Tw_{3/2}(\mu,\lambda)) = \lambda^{-1} \cdot \mathbf{E}(Tw_{3/2}(\mu,\lambda))^{3/2} = \lambda^{-1} \cdot \mu^{3/2}. \tag{2.16}$$

It is natural to interpret formula (2.16) as a *variance-to-mean relation of the power type*. This is because the variance is propotional to the mean raised to the power $3/2$, whereas coefficient λ is frequently termed the *scaling parameter*.

The Poisson-exponential laws emerge as the limits for quite wide families of distributions rather than just for the total mass processes of two specific BPS's considered in this section (see formulas (2.21) and (2.26) below). Thus, Zhou [33] considered a super-Brownian motion $\mathcal{X}_t$ of dimension 1 with *Lebesgue* initial measure. Given constant $\mathcal{C} \in \mathbf{R}^1_+$, he applied Dawson and Fleischmann [6, Theorem 3.1] to demonstrate that $t^{-1} \cdot \mathcal{X}_t([-\mathcal{C} \cdot t, \mathcal{C} \cdot t])$ converges weakly to a certain Poisson-exponential r.v. as $t \to \infty$.

Other references include Refs. 21, Theorem 4.5; 31, Theorem 2.1; 32, Theorem 3.3. In fact, the resuts of those papers can be reformulated in terms of weak convergence of the corresponding exponential families of Lévy processes (The reader can find the background on such families of stochastic processes in Küchler and Sørensen [24].)

All the members of the Poisson-exponential class are *infinitely divisible*. Thus, it can be shown that the cumulant-generating function (or *c.g.f.*) $\zeta_{3/2,\mu,\lambda}(\cdot)$ of r.v. $Tw_{3/2}(\mu,\lambda)$ admits the following Lévy representation:

$$\begin{aligned} &\zeta_{3/2,\mu,\lambda}(s) := \log\, \mathbf{E} \exp\left\{s \cdot Tw_{3/2}(\mu,\lambda)\right\} \\ &= 2 \cdot \phi_{3/2} \cdot s \cdot (\theta_{3/2} - s)^{-1} = \int_{0+}^{\infty} (e^{s \cdot x} - 1) \cdot \nu_{3/2,\mu,\lambda}(dx). \end{aligned} \tag{2.17}$$

Here, parameters $\theta_{3/2}$ and $\phi_{3/2}$ are defined by formulas (2.12) and (2.13), respectively, and argument $s < \theta_{3/2}$. The Lévy measure $\nu_{3/2,\mu,\lambda}(\cdot)$ that emerges on the right-hand side of formula (2.17) is such that $\forall y \in \mathbf{R}^1_+$,

$$\nu_{3/2,\mu,\lambda}\{[y,\infty)\} = 2 \cdot \phi_{3/2} \cdot \exp\{-\theta_{3/2} \cdot y\}$$

(cf., e.g., Vinogradov [30, formula (1.17) and the erratum]). In addition, (2.17) implies that given constant $\mathcal{C} \in \mathbf{R}^1_+$,

$$\mathcal{C} \cdot Tw_{3/2}(\mu,\lambda) \stackrel{\mathrm{d}}{=} Tw_{3/2}(\mathcal{C} \cdot \mu, \lambda/\sqrt{\mathcal{C}})$$

(compare to Vinogradov [30, formula (1.1)]).

There is a decomposition criterion for the Poisson-exponential class of probability distributions $\{Tw_{3/2}(\mu,\lambda), \mu \in \mathbf{R}^1_+, \lambda \in \mathbf{R}^1_+\}$ (see Vinogradov [30, Theorem 2.1]). It is instructive to present a corollary to this result, which stipulates the useful *additivity* property (2.18) of the shape parameter.

Proposition 2.6. *Fix arbitrary real $c_i > 0$, where $1 \le i \le n$. Consider independent r.v.'s $\{U_i,\ 1 \le i \le n\}$ such that $U_i \stackrel{\mathrm{d}}{=} Tw_{3/2}(\mu_i,\lambda_i)$. Here, μ_i's and λ_i's, $1 \le i \le n$, are certain constants which belong to $\mathbf{R}^1_+$. Set $\theta^{(i)}_{3/2} := 2 \cdot \lambda_i/\sqrt{\mu_i}$ and assume that*

$$\theta^{(1)}_{3/2}/c_1 = \ldots = \theta^{(n)}_{3/2}/c_n.$$

Then r.v. $U := \sum_{i=1}^{n} c_i \cdot U_i$ *is such that* $U \stackrel{\mathrm{d}}{=} Tw_{3/2}(\mu, \lambda)$ *with* $\mu = \sum_{i=1}^{n} c_i \cdot \mu_i$ *and* $\lambda = \left(\sum_{i=1}^{n} \lambda_i^2/c_i\right)^{1/2}$. *In addition,*

$$\phi_{3/2} \ (:= \lambda/\sqrt{\mu}) = \sum_{i=1}^{n} \phi_{3/2}^{(i)} \,, \tag{2.18}$$

where $\phi_{3/2}^{(i)}$*'s are expressed in terms of* λ_i*'s and* μ_i*'s according to formula (2.13).*

Proof. It follows from Vinogradov [30, Proposition 2.1]. □

It seems that the additivity property (2.18) is related to the *multiplicative property of Dawson-Watanabe processes* (cf., e.g., Ref. 5, formula (5.1.1)). Also, it appears that Proposition 2.6 has its counterparts in the theory of Bessel processes. In order to clarify this, note that each non-central gamma r.v. can be decomposed into the sum of two independent components which are central gamma and Poisson-exponential distributed, respectively (cf., e.g., Ref. 18, formula (29.5a–b)). The corresponding Poisson-exponential r.v. thus determines the *non-centrality* (Note in passing that the study of non-central gamma distributions is required for the evaluation of the power of important statistical tests including the analysis of variance (cf., e.g., Ref. 20, Chapter 2).)

Subsequently, the additivity property of the non-central gamma family given in Ref. 16, p. 509 easily follows from the combination of Proposition 2.6 with Cochran's theorem. But this property is analogous to the additivity of squared Bessel processes established by Shiga and Watanabe [28]. Moreover, the Poisson-exponential laws, which are frequently regarded as non-central gamma distribtuions with zero degree of freedom, correspond to particular squared Bessel processes which have an absorbing state. The interested reader is referred to Ref. 10, pp. 40–42 for the distribution theory background on squared Bessel processes.

Also, compound Poisson processes with Poisson-exponential marginals emerged in various stochastic models of Property and Casualty (or *P & C*) Insurance (cf., e.g., Panjer and Willmot [27]). Note that in the theory of stochastic models of P & C Insurance, a *positive mass* at zero is usually interpreted as a reasonably high likelihood of the absence of claims within a certain time period. At the same time, in Section 8.3, Case III and Section 8.4 of Ref. 4, this process was employed as a model for wear. It is relevant that the distribution of the first passage time for the compound Poisson-exponential process was derived therein. Lamperti [25] considered a similar compound Poisson-exponential process in the study of statistics of extremes (see p. 218 therein).

It is of interest that compound Poisson processes with Poisson-exponential marginals emerge in the studies of local times for Lévy processes (cf., e.g., Ref. 1, p. 27 for a wide class of Lévy processes and Ref. 17, Section 2.8, Problem 4 for the standard Brownian motion).

Let us employ representation (2.15) in the case when $\mu = 1$ and $\lambda = 1/t$. The corresponding r.v. has the same distribution as the total mass M_t ($\stackrel{d}{=} Tw_{3/2}(1, 1/t)$) of the limiting Dawson-Watanabe process (see formulas (2.21) and (2.26) below). By (2.16), its expectation and variance are equal to 1 and t, respectively. The stochastic process $\{M_t,\ t \geq 0\}$ is a continuous martingale that satisfies the next stochastic differential equation:

$$dM_1(t) = \sqrt{M_1(t)} \cdot d\mathcal{W}_t$$

(cf., e.g., Dawson [5, p. 20]). Here, $\mathcal{W}_t$ denotes the standard univariate Brownian motion. The process M_t is a time-homogeneous Markov diffusion process with generator $\mathcal{B}g(x) := (x/2) \cdot g''(x)$ (compare to Dawson and Vinogradov [9, p. 230]). It is frequently called the *Feller diffusion* (with zero drift).

It is relevant that an application of the well-known properties of function $I_1(\cdot)$ implies that $\forall t \in \mathbf{R}^1_+$,

$$\lim_{x \downarrow 0} f_{3/2,1,1/t}(x) = \frac{4}{t^2} \cdot e^{-2/t} =: f_{3/2,1,1/t}(0). \tag{2.19}$$

At the same time, it follows from (2.14) that

$$\mathbf{P}\{Tw_{3/2}(1,\ 1/t) = 0\} = e^{-2/t}. \tag{2.20}$$

Next, it is well known that

$$M_t^{(\eta)} \xrightarrow{d} Tw_{3/2}(1,\ 1/t) \tag{2.21}$$

as $\eta \to \infty$ (cf., e.g., Ref. 9, Proof of Proposition 1.10 and formula (1.16')). Hereinafter, the sign '$\xrightarrow{d}$' denotes weak convergence.

Recall that the original BPS $\Xi_t^{(\eta)}$ is assumed to start from a *non-random* number η of particles. At the same time, it was later understood that it is more natural to impose the condition that the initial number of particles of a branching-fluctuating particle system is *random*. Thus, the derivation of the 'high-density' approximation becomes rather elegant under the assumption that the *initial number of particles is Poisson* distributed with mean η (cf., e.g., Ref. 11, pp. 47-49). It is relevant that this *Poisson* (or *quasi-stationary*) initial condition results in the appearance of a modified BPS $\Upsilon_t^{(\eta)}$ that was considered, among others, in Refs. 7 and 8. The idea goes back to Ref. 15, p. 92.

It is known that the univariate distributions of the corresponding total mass process $\widetilde{M}_t^{(\eta)}$ of BPS $\Upsilon_t^{(\eta)}$ follow *Pólya-Aeppli* law (see Vinogradov [31]).

Next, let us introduce this two-parametric class of distributions indexed by real-valued parameters $\delta > 0$ and $Q \in (0, 1)$ by employing their characterization given by Ref. 19, formulas (9.133)–(9.135). To this end, suppose that r.v. $\mathcal{T} \stackrel{\mathrm{d}}{=} Poiss(\delta \cdot (1 - Q))$, and $\{\mathcal{H}_k, k \geq 1\}$ are i.i.d. *shifted* geometric r.v.'s. It is assumed that they do not depend on $\mathcal{T}$ and are such that $\forall n \in \mathbf{N}$, $\mathbf{P}\{\mathcal{H}_k = n\} = Q \cdot (1 - Q)^{n-1}$.

Definition 2.7. (*Pólya-Aeppli distributions*). We say that r.v. $W_{\delta,Q}$ has *Pólya-Aeppli* distribution with the values of parameters $\delta \in \mathbf{R}^1_+$ and $Q \in (0, 1)$ if it can be represented as a Poisson sum of i.i.d. *shifted* geometric r.v.'s:

$$W_{\delta,Q} \stackrel{\mathrm{d}}{=} \sum_{k=1}^{\mathcal{T}} \mathcal{H}_k. \tag{2.22}$$

The representation (2.22) is especially convenient for describing the *cluster structure* of the modified BPS $\Upsilon_t^{(\eta)}$ (cf., e.g., Dawson et al. [7] or Ref. 31, formulas (3.8)–(3.10)). In addition, the infinite divisibility of any member of the Pólya-Aeppli family is obtained from (2.22) at no cost. It is of interest that the product $\delta \cdot Q$ remains invariant under *Esscher* (or *exponential tilting*) transformation. Moreover, Ref. 31, Section 3 demonstrates that family (2.22) comprises an *additive exponential dispersion model*. Note in passing that because of the infinite divisibility of the members of Pólya-Aeppli family, the corresponding additive exponential dispersion model can be interpreted as the class of univariate distributions of the exponential family of the Lévy processes, which are constructed starting from these laws. The reader is referred to Küchler and Sørensen [24] for a comprehensive description of such families.

It can be shown that $\forall\, n \in \mathbf{Z}_+$,

$$\mathbf{p}_{\delta,Q}(n) := \mathbf{P}\left\{W_{\delta,Q} = n\right\} = e^{-\delta} \cdot \delta \cdot Q \cdot (1-Q)^n \cdot \ {}_1F_1[n+1;\ 2,\ \delta \cdot Q]. \quad (2.23)$$

Here, ${}_1F_1[a; b, z]$ stands for the *confluent hypergeometric function*:

$${}_1F_1[a; b, z] := 1 + \frac{a}{b} \cdot \frac{z}{1!} + \frac{a \cdot (a+1)}{b \cdot (b+1)} \cdot \frac{z^2}{2!} + ... \quad (2.24)$$

(compare to Ref. 29, formula (1.1.8)).

Consequently, Ref. 31, formula (3.10) implies that for each fixed real $t \in \mathbf{R}^1_+$, the total mass $\widetilde{M}_t^{(\eta)}$ of the modified BPS $\Upsilon_t^{(\eta)}$ has a scaled Pólya-Aeppli law in the sense that

$$\widetilde{M}_t^{(\eta)} \stackrel{\mathrm{d}}{=} \frac{1}{\eta} \ \cdot W_{2/t,Q_t^{(\eta)}} \ . \quad (2.25)$$

Recall that $Q_t^{(\eta)}$ is given by (2.3). In addition, the analogue of (2.21) remains valid:

$$\widetilde{M}_t^{(\eta)} \stackrel{d}{\to} Tw_{3/2}(1, 1/t) \quad (2.26)$$

as $\eta \to \infty$ (cf., e.g., Vinogradov [31, Theorem 3.1 and formula (3.12)]).

It is methodologically important that although the weak convergence results (2.21) and (2.26) were originally derived independently of each other, but in fact, the validity of (2.21) can be established by employing the combination of (2.26) with a slight modification of Gnedenko's method of accompanying infinitely divisible laws. (The reader is referred to the Introduction for the bibliography pertinent to this method.) It is relevant that a similar approach can also be developed by estimating the difference between $M_t^{(\eta)}$ and $\widetilde{M}_t^{(\eta)}$ with the use of the compound Poisson approximation bounds. Such bounds are given by Le Cam [26, Theorem 1] and Chen [3, formula (1.3)] implying that

$$\parallel \eta \cdot M_t^{(\eta)} - \eta \cdot \widetilde{M}_t^{(\eta)} \parallel \leq\ 2\eta \cdot (Q_t^{(\eta)})^2 \ \sim\ (8/t^2) \cdot \eta^{-1} \quad (2.27)$$

as $\eta \to \infty$. Hereinafter, $\parallel \mathcal{G} \parallel$ denotes the norm of a (generic) finite signed measure $\mathcal{G}$. Note in passing that our *local* approximations, which are derived by using formulas (3.10), (3.12)–(3.14) below, are of the same spirit as the *integal* bound (2.27) but more subtle. See the comment below Theorem 2.9 and formula (2.34), which address the issue of sharpness of (2.27).

In order to develop Gnedenko's approach in our context, we combine (2.4) with (2.5) to represent the total mass of the original BPS $\Xi_t^{(\eta)}$ as follows:

$$M_t^{(\eta)} = \sum_{i=1}^{\eta} \frac{1}{\eta} \ \cdot Y_{1-Q_t^{(\eta)},1-Q_t^{(\eta)}/(1-Q_t^{(\eta)})}(i). \quad (2.28)$$

The main idea of his method consists in approximating each term of the latter sum of i.i.d. scaled zero-modified geometric r.v.'s by an infinitely divisible r.v. whose Lévy measure coincides with the probability function of the original r.v. everywhere except the origin. But in view of (2.1)–(2.4), the *probability measure* of each term of the sum that emerges on the right-hand side of (2.28) is concentrated on the lattice $\frac{1}{\eta} \cdot \mathbf{Z}_+$ such that the value n/η is attained with probability

$$\frac{4}{(\eta \cdot t + 2) \cdot \eta \cdot t} \cdot \left(\frac{\eta \cdot t}{\eta \cdot t + 2} \right)^n. \tag{2.29}$$

It follows from (2.22) or Vinogradov [31, formula (3.4)] that the collection of quantities (2.29) on set $\mathbf{N}$ constitutes the *Lévy measure* of the next scaled Pólya-Aeppli r.v.:

$$\frac{1}{\eta} \cdot W_{2/(\eta \cdot t), Q_t^{(\eta)}}(i). \tag{2.30}$$

Here, $1 \le i \le \eta$, and Pólya-Aeppli r.v. $W_{\cdot,\cdot}$ is characterized in Definition 2.7. Next, it follows from Ref. 31, formula (3.2) that the sum of i.i.d.r.v.'s described by (2.30) is also a scaled Pólya-Aeppli r.v.:

$$\sum_{i=1}^{\eta} \frac{1}{\eta} \cdot W_{2/(\eta \cdot t), Q_t^{(\eta)}}(i) \stackrel{\mathrm{d}}{=} \frac{1}{\eta} \cdot W_{2/t, Q_t^{(\eta)}} .$$

It remains to note that by (2.25), the expression that emerges on the right-hand side of the above formula constitutes the total mass $\widetilde{M}_t^{(\eta)}$ of the '*quasi-stationary*' BPS $\Upsilon_t^{(\eta)}$.

These arguments can be used to obtain an alternative derivation of (2.21), although it would not be the easiest proof of this approximation. However, the idea of employing these particular accompanying infinitely divisible laws turns out to be important for the derivation of the *local* 'high-density' approximation given by Theorem 2.9.ii.

Remark 2.8. **(i)** In contrast to Gnedenko, we have not centered the original r.v.'s

$$\frac{1}{\eta} \cdot Y_{1-Q_t^{(\eta)}, 1-Q_t^{(\eta)}/(1-Q_t^{(\eta)})}(i).$$

We will demonstrate below that this leads to the violation of the property of *equality of variances* of the original and the approximating r.v.'s observed by Gnedenko (cf., e.g., Ref. 13, p. 279, the footnote).

(ii) The straightforward differentiation of c.g.f. of a zero-modified geometric law (that is easily obtained from (2.8)) stipulates that

$$\mathbf{E}\left[\frac{1}{\eta} \cdot Y_{1-Q_t^{(\eta)}, 1-Q_t^{(\eta)}/(1-Q_t^{(\eta)})}(i) \right] = \frac{1}{\eta} ,$$

and

$$\mathbf{Var}\left[\frac{1}{\eta} \cdot Y_{1-Q_t^{(\eta)}, 1-Q_t^{(\eta)}/(1-Q_t^{(\eta)})}(i) \right] = \frac{t}{\eta} .$$

Similarly, an application of Ref. 31, formula (3.2) yields that the numerical characteristics of the scaled Pólya-Aeppli r.v. (2.30) are as follows:

$$\mathbf{E}\left[\frac{1}{\eta} \cdot W_{2/(\eta \cdot t), Q_t^{(\eta)}}(i) \right] = \frac{1}{\eta} ,$$

and

$$\mathbf{Var}\left[\frac{1}{\eta} \cdot W_{2/(\eta \cdot t), Q_t^{(\eta)}}(i) \right] = \frac{\eta \cdot t + 1}{\eta^2} .$$

Recall that function $f_{3/2,1,1/t}(\cdot)$ represents the density of the absolutely continuous component of the distribution of the total mass of the limiting Dawson-Watanabe process given by (2.15) and (2.19), and let $\chi(\mathcal{A})$ denote the indicator of Borel set $\mathcal{A}$.

The next assertion reveals a subtle but important difference between the accuracy of local approximations for the classes of Pólya-Aeppli and convolutions of zero-modified geometric laws. The result of part (ii) follows from a combination of part (i) with the estimates in the local Poisson theorem and conditioning arguments. It is plausible that this method can be extended to cover the case(s) of other initial distributions of BPS's. Also, part (i) of the following theorem constitutes an essential refinement of Ref. 31, Corollary 3.1. Namely, it involves the *accuracy* of approximation and makes it possible to tackle together both the singular and absolutely continuous components of the limit.

Let $C_1(t,x) \le C_2(t,x)$ denote certain positive real constants which depend on both $t \in \mathbf{R}^1_+$ and $x \in [0,\infty)$, and $C(t) \in \mathbf{R}^1_+$ be a constant that may depend on $t \in \mathbf{R}^1_+$.

Theorem 2.9. *Assume that the integer-valued parameter $\eta \to \infty$. Fix real $t > 0$, suppose that $x \ge 0$ is a fixed real, and consider those values of η for which the product $x \cdot \eta$ takes on integer values only. Then x belongs to the range of r.v.'s $M_t^{(\eta)}$ and $\widetilde{M}_t^{(\eta)}$. For such x,*
(i) The probability function of the total mass $\widetilde{M}_t^{(\eta)}$ of the 'quasi-stationary' BPS $\Upsilon_t^{(\eta)}$ has the next asymptotics:

$$\left| \mathbf{P}\{\widetilde{M}_t^{(\eta)} = x\} - \chi(\{x=0\}) \cdot e^{-2/t} - \frac{1}{\eta} \cdot f_{3/2,1,1/t}(x) \right| \le C_1(t,x)/\eta^2. \tag{2.31}$$

(ii) For each fixed real $x > 0$, the probability function of the total mass $M_t^{(\eta)}$ of the original BPS $\Xi_t^{(\eta)}$ can be approximated as follows:

$$\left| \mathbf{P}\{M_t^{(\eta)} = x\} - \frac{1}{\eta} \cdot f_{3/2,1,1/t}(x) \right| \le C_2(t,x)/\eta^2. \tag{2.32}$$

In addition,

$$\left| \mathbf{P}\{M_t^{(\eta)} = 0\} - e^{-2/t} - \frac{\mathbf{1}}{\mathbf{2}} \cdot \frac{1}{\eta} \cdot f_{3/2,1,1/t}(0) \right| \le C(t)/\eta^2. \tag{2.33}$$

Proof of Theorem 2.9 is deferred to Section 3.

It is of interest that the combination of (2.19), (2.31) and (2.33) implies that the order of decay given by (2.27) is *sharp*. Namely, one ascertains that $\forall$ fixed $t \in \mathbf{R}^1_+$ and as $\eta \to \infty$,

$$\mathbf{P}\{\widetilde{M}_t^{(\eta)} = 0\} - \mathbf{P}\{M_t^{(\eta)} = 0\} \sim \frac{1}{\eta} \cdot \frac{2}{t^2} \cdot e^{-2/t} + \mathcal{O}(1/\eta^2). \tag{2.34}$$

The next remark is aimed at comparing parts (i) and (ii) of the above theorem.

Remark 2.10. **(i)** Let us stress the presence of factor $\mathbf{1/2}$ on the left-hand side of formula (2.33), which provides the second-order approximation for the probability of extinction of the original BPS $\Xi_t^{(\eta)}$. This is in a sharp contrast to formula (2.31). Therefore, it seems plausible that a subtle difference between the second-order '*high-density*' approximations for BPS's $\Xi_t^{(\eta)}$ and $\Upsilon_t^{(\eta)}$ becomes noticeable mostly due to the presence of the point mass

at zero. This is the only singular point of the *mixed* limiting distribution given by (2.14)–(2.15).
(ii) The estimates of remainders given by (2.31)–(2.32) are not uniform in x. However, it is plausible that the expression that emerges on the left-hand side of inequality (2.31) admits a uniform in x upper bound having the same order of η^{-2}.
(iii) In view of the exact formulas for the expectations and variances of the r.v.'s given in Remark 2.8.ii, the numerical characteristics of the total mass $M_t^{(\eta)}$ of the original BPS $\Xi_t^{(\eta)}$ are equal to 1 and t. Hence, they perfectly match those of the total mass $Tw_{3/2}(1, 1/t)$ of the limiting Dawson-Watanabe process. In contrast, the variance of r.v. $\widetilde{M}_t^{(\eta)}$ equals $(\eta t + 1)/\eta$. Therefore, it is only *asymptotically equivalent* to the variance t of the limit $Tw_{3/2}(1, 1/t)$ (compare to (2.16)). The comprehension of the intrinsic reason(s) why a system with a '*less accurate*' variance exhibits more natural convergence properties remains an open problem.

3. Technical Proofs

Proof of Theorem 2.3. For simplicity, assume that $r \neq 0$. Otherwise the validity of (2.7) follows from the comment given below the formulation of the theorem.

Our method of proof involves reducing the calculation of specific integrals which emerge in the general case of $r \neq 0$ to known integrals in the special case of $r = 0$. Recall that the latter case corresponds to a negative binomial subclass of the family being considered. In order to identify the probability function, we utilize the inversion formula (cf., e.g., Feller [12, v. 2, formulas (XV.3.13)-(XV.3.14)]): $\forall n \in \mathbf{Z}_+$,

$$\mathbf{p}_{\gamma,r,k}(n) = \frac{1}{2\cdot\pi} \cdot \int_{-\pi}^{\pi} e^{-i\cdot n\cdot t} \cdot f_{U_{\gamma,r}(k)}(t) \cdot dt.$$

The closed-form representation for the characteristic function $f_{U_{\gamma,r}(k)}(\cdot)$ of r.v. $U_{\gamma,r}(k)$ (that is introduced by (2.6)) is easily obtained from the binomial expansion of the middle expression in (2.8) for its p.g.f. $\psi_{\gamma,r,k}(\cdot)$. It follows immediately that

$$f_{U_{\gamma,r}(k)}(t) = \frac{\gamma^k}{(1-(1-\gamma+\gamma\cdot r)\cdot e^{i\cdot t})^k} \cdot \sum_{\ell=0}^{k} \binom{k}{\ell} \cdot (-r)^\ell \cdot e^{i\cdot t\cdot \ell}.$$

Hence,

$$\mathbf{p}_{\gamma,r,k}(n) = \frac{\gamma^k}{2\cdot\pi} \cdot \sum_{\ell=0}^{k} \binom{k}{\ell} \cdot (-r)^\ell \cdot \int_{-\pi}^{\pi} e^{-i\cdot(n-\ell)\cdot t} \cdot \frac{dt}{(1-(1-\gamma+\gamma\cdot r)\cdot e^{i\cdot t})^k}. \tag{3.1}$$

It remains to note that in the case when $n \geq \ell$, the integral that emerges on the right-hand side of (3.1) equals

$$2\cdot\pi\cdot(1-\gamma+\gamma\cdot r)^{n-\ell} \cdot \binom{n-\ell+k-1}{n-\ell}. \tag{3.2}$$

The validity of (3.2) can be derived analytically by using the methods of complex analysis. Alternatively, one can obtain (3.2) from the inversion formula for the standard negative binomial distribution. The same reasoning implies that for each integer $n < \ell$, this integral equals 0. The verification of these assertions is straightforward and can be carried out by employing combinatorial arguments given in Feller [12, v. 1, Sections VI.8 and XI.2]. The details are left to the reader.

Set $m := k - \ell$. A subsequent combination of (3.1)–(3.2) along with some algebra implies the validity of (2.7). □

Proof of Proposition 2.5. Consider the following generating function that corresponds to the set of probabilities (2.10), but is not a p.g.f.:

$$\mathbf{Q}_{\gamma,r,k}(z) := \sum_{k=0}^{\infty} \mathbf{q}_{\gamma,r,k} \cdot z^k.$$

By Ref. 12, v. 1, Theorem XI.1,

$$\mathbf{Q}_{\gamma,r,k}(z) = \frac{1 - \psi_{\gamma,r,k}(z)}{1 - z}\,, \tag{3.3}$$

where $|z| < 1$, and p.g.f. $\psi_{\gamma,r,k}(\cdot)$ is given by (2.8). Combining (2.8) and (3.3) with some algebra, one obtains that

$$\mathbf{Q}_{\gamma,r,k}(z) = \sum_{\ell=0}^{k-1} \frac{\mathbf{A}_{\gamma,r}(\ell) \cdot (r \cdot z - 1)^\ell}{(z - 1/(1 - \gamma + \gamma \cdot r))^{\ell+1}}\,. \tag{3.4}$$

Here, constant

$$\mathbf{A}_{\gamma,r}(\ell) := -\frac{(1 - \gamma) \cdot \gamma^\ell}{(1 - \gamma + \gamma \cdot r)^{\ell+1}}\,. \tag{3.5}$$

Since each term of the sum that emerges on the right-hand side of formula (3.4) is a rational function, it constitutes a generating function for a certain numerical sequence in the sense of Ref. 12, v. 1, Section XI.2. This justifies our next step that consists in fixing an arbitrary ℓ^{th} term of this sum and investigating the asymptotic behavior of n^{th} term of the corresponding numerical sequence $\mathbf{q}^{(\ell)}_{\gamma,r,k}(n)$ as $n \to \infty$. Here, $0 \le \ell \le k - 1$.

Observe that the multiplicity of the root of the denominator of this term is $\ell+1$. Hence, a slight correction of Ref. 12, v. 1, Exercise XI.25 yields that

$$\mathbf{q}^{(\ell)}_{\gamma,r,k}(n) \sim \mathbf{A}_{\gamma,r}(\ell) \cdot \frac{[(1 - \gamma) \cdot (1 - r)]^\ell}{\ell!} \cdot n^\ell \cdot (1 - \gamma + \gamma \cdot r)^{n+1} \tag{3.6}$$

as $n \to \infty$. By (3.6) and the arguments given in Ref. 12, v. 1, p. 276, the asymptotics of $\mathbf{q}_{\gamma,r,k}(n)$ as $n \to \infty$ is determined by the last, $(k-1)^{st}$ term of the sum $\mathbf{q}^{(0)}_{\gamma,r,k}(n) + ... + \mathbf{q}^{(k-1)}_{\gamma,r,k}(n)$. The proof of (2.11) is completed by an application of (3.5)–(3.6). □

Proof of Theorem 2.9. (i) First, recall that less accurate results on the asymptotics up to equivalence were derived separately for $x = 0$ and $x > 0$ in Ref. 31, Corollary 3.1. The approach implemented in that work involved a relationship between the confluent hypergeometric function (2.24) and the modified Bessel function of the first kind (compare to (2.15)). Their relationship is presented as formula (3.17) therein. In contrast, the proof given below is more straightforward yielding a more accurate and elegant result.

It follows from formula (2.19) and Ref. 31, formula (3.5) that

$$\mathbf{P}\{\widetilde{M}_t^{(\eta)} = 0\} = \exp\{-2/t\} \cdot \exp\left\{ \frac{4}{(\eta \cdot t + 2) \cdot t} \right\} \\ = e^{-2/t} + \frac{1}{\eta} \cdot f_{3/2,1,1/t}(0) + \mathcal{O}(1/\eta^2) \tag{3.7}$$

as $\eta \to \infty$. Next, it is known that each Pólya-Aeppli distribution can be represented as a Poisson mixture with the mixing measure given by a specific Poisson-exponential law (cf., e.g., Ref. 19, p. 481). Namely, it is relatively easy to verify that the distribution of the total number $\eta \cdot \widetilde{M}_t^{(\eta)}$ of particles of quasi-stationary BPS $\Upsilon_t^{(\eta)}$ which are alive at time t admits the Poisson mixture representation with the unit value of the Poisson parameter and the mixing measure given by the Poisson-exponential law $Tw_{3/2}(\eta,\ (t \cdot \sqrt{\eta})^{-1})$. Therefore, one gets that for each fixed $x > 0$ and those η for which $x \cdot \eta$ takes on an integer value,

$$\mathbf{P}\{\eta \cdot \widetilde{M}_t^{(\eta)} = x \cdot \eta\} = \frac{1}{(x \cdot \eta)!} \cdot [\, \mathbf{P}\left\{Tw_{3/2}(\eta,\ (t \cdot \sqrt{\eta})^{-1}) = 0\right\} \\ + \int_{0+}^{\infty} e^{-z} \cdot z^{x \cdot \eta} \cdot f_{3/2,\eta,(t \cdot \sqrt{\eta})^{-1}}(z) \cdot dz \,]. \tag{3.8}$$

Here, function $f_{3/2,\cdot,\cdot}(\cdot)$ is given by (2.15).

The asymptotics of the integral that emerges on the right-hand side of (3.8) (with the estimate of remainder) is obtained by Laplace's method (cf., e.g., Ref. 2, Chapter 4). To this end, one should make the change of variables $v := z/\eta$. The combination of this approach with (3.8) and Stirling's formula (with the estimate of remainder) yields that

$$\mathbf{P}(\eta \cdot \widetilde{M}_t^{(\eta)} = x \cdot \eta) = \frac{1}{(x \cdot \eta)!} \cdot e^{-2/t} \cdot [\, 1 + \frac{2}{t} \cdot \eta^{x\eta} \\ \cdot \int_{0+}^{\infty} e^{-\eta(v - x \log\ v)} \cdot \frac{1}{\sqrt{v}} \cdot \exp\left\{ -\frac{2}{t} \cdot v \right\} \cdot I_1\Big(\frac{4}{t} \cdot \sqrt{v} \Big) \,] \cdot dv \\ = \frac{1}{\eta} \cdot f_{3/2,1,1/t}(x) + \mathcal{O}(1/\eta^2) \tag{3.9}$$

as $\eta \to \infty$. To conclude the proof of (2.31), it remains to combine (3.7) with (3.9).
(ii) The proof of validity of (2.33) repeats the derivation of (3.7). It easily follows from the combination of (2.1) with (2.4)–(2.5).

The proof of (2.32) relies on approximating the probability of interest by expression (3.14) below. It is motivated by Gnedenko's method of accompanying infinitely divisible laws. To some extent, we utilize the ideas similar to those used in Dawson and Vinogradov [9, Proposition 1.6 and Proof of Proposition 1.10]. First, it is relatively easy to demonstrate that

$$\mathbf{P}\{\eta \cdot M_t^{(\eta)} = x \cdot \eta\} = \mathbf{P}\left\{ \sum_{i=1}^{\mathbf{B}(\eta, Q_t^{(\eta)})} \mathcal{H}_i = x \cdot \eta \right\} \tag{3.10}$$

(compare to Panjer and Willmot [27, pp. 263-264]). Here, $\mathbf{B}(\eta, Q_t^{(\eta)})$ is a binomial r.v. with η trials and the probability of success in a single trial equal to $Q_t^{(\eta)}$. This r.v.

is assumed to be independent of the sequence $\{\mathcal{H}_i,\ i \geq 1\}$ of i.i.d.r.v.'s with common *shifted geometric* distribution such that

$$\mathcal{H}_i \stackrel{\mathrm{d}}{=} Y_{Q_t^{(\eta)},0} + 1. \tag{3.11}$$

Here, the standard geometric r.v. $Y_{Q_t^{(\eta)},0}$ is defined by (2.1)–(2.2) with parameters $\gamma = Q_t^{(\eta)}$ and $r = 0$. The probabilistic interpretation of r.v.'s which emerge in formulas (3.10)–(3.11) can be found in Dawson and Vinogradov [9, p. 231]. In the sequel, we will approximate binomial r.v. $\mathbf{B}(\eta, Q_t^{(\eta)})$ with a Poisson r.v. $\Pi(2\eta/(\eta \cdot t + 2))$. The latter variable is also assumed to be independent of the above sequence of shifted geometric r.v.'s $\{\mathcal{H}_i,\ i \geq 1\}$. The remainder of this approximation will be estimated by employing sharp upper bounds in the local Poisson theorem given in Karymov [22].

Let us recall that $x > 0$ and employ the conditioning arguments to conclude that

$$\mathbf{P}\left\{ \sum_{i=1}^{\mathbf{B}(\eta, Q_t^{(\eta)})} \mathcal{H}_i = x \cdot \eta \right\} = \sum_{k=1}^{\eta} \mathbf{P}\left\{ \sum_{i=1}^{k} \mathcal{H}_i = x \cdot \eta \right\} \cdot \mathbf{P}\{\mathbf{B}(\eta, Q_t^{(\eta)}) = k\}\ . \tag{3.12}$$

Subsequently, the sum that emerges on the right-hand side of (3.12) can be represented in the following form:

$$\begin{aligned} &\sum_{k=1}^{\infty} \mathbf{P}\left\{ \sum_{i=1}^{k} \mathcal{H}_i = x \cdot \eta \ \Big|\ \Pi\Big(\frac{2 \cdot \eta}{\eta \cdot t + 2}\Big) = k \right\} \cdot \mathbf{P}\left\{ \Pi\Big(\frac{2 \cdot \eta}{\eta \cdot t + 2}\Big) = k \right\} \\ &- \sum_{k=\eta+1}^{\infty} \mathbf{P}\left\{ \sum_{i=1}^{k} \mathcal{H}_i = x \cdot \eta \right\} \cdot \mathbf{P}\left\{ \Pi\Big(\frac{2 \cdot \eta}{\eta \cdot t + 2}\Big) = k \right\} \\ &+ \sum_{k=1}^{\eta} \mathbf{P}\left\{ \sum_{i=1}^{k} \mathcal{H}_i = x\eta \right\} \cdot \Big[\mathbf{P}\{\mathbf{B}(\eta, Q_t^{(\eta)}) = k\} - \mathbf{P}\Big\{ \Pi\Big(\frac{2\eta}{\eta t + 2}\Big) = k \Big\} \Big]. \end{aligned} \tag{3.13}$$

We will show below that the middle and rightmost sums which emerge in formula (3.13) are asymptotically negligible. Also, the leftmost sum that emerges in (3.13) pertains to a Poisson random sum of shifted geometric r.v.'s. By (2.22), this r.v. follows a specific Pólya-Aeppli law. Moreover, it turns out that this sum equals

$$\mathbf{P}\{\eta \cdot \widetilde{M}_t^{(\eta)} = x \cdot \eta\}. \tag{3.14}$$

Hence, the further evaluation of this quantity is reduced to part (i) of the theorem that pertains to '*accompanying infinitely divisible laws*', which are those of r.v.'s $\widetilde{M}_t^{(\eta)}$.

Next, it is obvious that the absolute value of the middle sum that emerges in formula (3.13) does not exceed $\mathbf{P}\{\ \Pi(2\eta/(\eta \cdot t + 2) > \eta\}$. The latter probability is easily estimated by virtue of the exponential Chebyshev inequality. One concludes that the upper bound decays towards zero faster than any negative power of η. The details are straightforward and left to the reader.

Finally, it remains to estimate the rightmost sum that emerges in (3.13). To this end, we employ (3.11) along with the well-known formula for the probability function of a negative binomial r.v. to obtain that for each integer $1 \le k \le x \cdot \eta$,

$$\mathbf{P}\left\{ \sum_{i=1}^{k} \mathcal{H}_i = x \cdot \eta \right\} = \binom{x \cdot \eta - 1}{x \cdot \eta - k} \cdot \left(\frac{2}{\eta \cdot t} \right)^k \cdot \left(\frac{\eta \cdot t}{\eta \cdot t + 2} \right)^{x \cdot \eta}. \tag{3.15}$$

Also, it is evident in view of (3.11) that for each integer $k > x \cdot \eta$, the probability that emerges on the left-hand side of (3.15) equals zero.

At this stage, we decompose the rightmost sum over k that emerges in (3.13) into two parts. The first sum $\sum_1$ includes the values of $k \le Const$, whereas the second sum $\sum_2$ pertains to the values of index k which tend to infinity with η. In order to estimate the absolute value of $\sum_1$, we combine (3.15) with the *uniform* upper bound in the local Poisson theorem (cf., e.g., Ref. 22, Corollary 1). Subsequently, one easily derives that

$$| \Sigma_1 | = \mathcal{O}(1/\eta^2) \tag{3.16}$$

as $\eta \to \infty$. In contrast, the sum over $\rho(\eta) \le k \le \min(x \cdot \eta, \eta)$ is estimated by the use of the *nonuniform* upper bound in the local Poisson theorem that can be found in Ref. 22, Theorem 4. Here, $\rho(\eta)$ is a certain (non-random) numerical sequence that tends to infinity as $\eta \to \infty$. A combination of this bound with (3.15) ascertains that

$$| \Sigma_2 | \le \frac{D_1}{\eta^2} \cdot \sum_{\ell=\rho(\eta)}^{\infty} \frac{D_2^{\ell}}{(\ell - 2)^{\ell-2}} \,. \tag{3.17}$$

Here, D_1 and D_2 are certain positive constants which depend on t and x but do not depend on η. The rest is trivial, since the sum that emerges in (3.17) constitutes the tail of a convergent series. To conclude, combine (3.16)–(3.17). □

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