

WHITE NOISE APPROACH TO THE ITÔ FORMULA FOR THE STOCHASTIC HEAT EQUATION

ALBERTO LANCONELLI

ABSTRACT. We derive an Itô's-type formula for the one dimensional stochastic heat equation driven by a space-time white noise. The proof is based on elementary properties of the \mathcal{S} -transform and on the explicit representation of the solution process. We also discuss the relationship with other versions of this Itô formula existing in literature.

1. Introduction

Consider the following stochastic partial differential equation (SPDE):

$$\begin{cases} \partial_t u_t(x) = \frac{1}{2} \partial_{xx} u_t(x) + \dot{W}_{t,x}, \\ u_t(0) = u_t(1) = 0, \quad u_0 = 0, \end{cases} \quad (1.1)$$

where $\partial_t := \frac{\partial}{\partial t}$, $\partial_{xx} := \frac{\partial^2}{\partial x^2}$, $\dot{W}_{t,x} := \frac{\partial^2 W_{t,x}}{\partial t \partial x}$ and $\{W_{t,x}, t \in [0, T], x \in [0, 1]\}$ is a Brownian sheet. By solution to this equation we mean an adapted two parameter stochastic process $\{u_t(x), t \in [0, T], x \in [0, 1]\}$ such that for $t \in [0, T]$ and $x \in [0, 1]$,

$$u_t(0) = u_t(1) = 0, \quad u_0(x) = 0,$$

and such that for all $l \in C_0^2([0, 1])$ the following equality

$$\langle u_t, l \rangle = \frac{1}{2} \int_0^t \langle u_s, l'' \rangle ds + \int_0^t \langle l, dW_s \rangle$$

holds for $t \in [0, T]$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{L}^2([0, 1])$ and

$$\int_0^t \langle l, dW_s \rangle := \int_0^t \int_0^1 l(x) dW_{s,x}.$$

It is well known (see e.g. [9]) that equation (1.1) has a unique solution $\{u_t(x), t \in [0, T], x \in [0, 1]\}$ which is continuous in the variables (t, x) and that it can be represented as

$$u_t(x) = \int_0^t \int_0^1 g_{t-s}(x, y) dW_{s,y},$$

where $\{g_t(x, y), t \in [0, T], x, y \in [0, 1]\}$ is the fundamental solution of the heat equation with homogenous Dirichlet boundary conditions, i.e.

$$\begin{cases} \partial_t g_t(x, y) = \frac{1}{2} \partial_{xx} g_t(x, y) \\ g_t(0, y) = g_t(1, y) = 0, \quad g_0(x, y) = \delta(x - y). \end{cases} \quad (1.2)$$

2000 *Mathematics Subject Classification.* Primary 60H15, 60H40; Secondary 60H07.

Key words and phrases. stochastic heat equation, Itô formula, \mathcal{S} -transform, Wick product.

Since for fixed $x \in [0, 1]$ the process $t \mapsto u_t(x)$ is not a semimartingale, the classical stochastic calculus can not be applied to it. It is therefore natural to ask whether an Itô-type formula can be found for this kind of process.

Two recent papers, [3] and [10], are devoted to the investigation of this problem. In [3] the authors develop a Malliavin calculus for the solution process $u_t(x)$ in order to obtain an Itô-type formula whose proof makes also use of projections on Wiener chaoses of different orders. In [10] the author applies the ordinary Itô formula to a regularized version of the solution process $u_t(x)$; then he studies the limit when that regularized process converges to the real one. The main feature of this procedure is the appearance of a renormalization of the square of an infinite dimensional stochastic distribution.

The aim of the present paper is to propose an alternative approach (and a corresponding version of the Itô formula) to the above mentioned problem which is somehow in between the techniques of the articles [3] and [10]; in fact we utilize notions of white noise analysis, an infinite dimensional stochastic distribution theory, and prove the resulting formula via scalar products with test functions, more precisely stochastic exponentials. The key point is the gaussianity of the solution process and its explicit representation as a stochastic convolution. Our formula is in the spirit of the Itô's-type formula for gaussian processes derived in the paper [8]. Moreover the idea of using the properties of the semigroup associated to the one dimensional Brownian motion (see the proof of Theorem 2.2) is analogous to the one used in [6]. As a result we obtain a short and direct way of proving the Itô formula for the stochastic heat equation based on the main tools of the white noise theory.

The paper is structured as follows: in Section 2.1 we recall basic notions and facts from the white noise theory; then in Section 2.2 we state and prove our main result; finally in the concluding section we compare our formula with those already existing in the literature.

2. Main result

2.1. Preliminaries. In this section we fix the notation and recall some basic results from the white noise theory. For additional information about this topic we refer the reader to the books [4] and [5] and to the paper [2].

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space and $\{W_{t,x}, t \in [0, T], x \in [0, 1]\}$ a Brownian sheet defined on it. Assume that

$$\mathcal{F} = \sigma(W_{t,x}, t \in [0, T], x \in [0, 1]),$$

so that each element $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ can be decomposed as a sum of multiple Itô integrals w.r.t. the Brownian sheet W , i.e.

$$X = \sum_{n \geq 0} I_n(h_n) \text{ (convergence in } \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})\text{),}$$

where for $n \geq 0$, h_n is a deterministic function belonging to $\mathcal{L}^2([0, T] \times [0, 1]^n)$ and $I_n(h_n)$ is the n -th order multiple Itô integral of h_n .

For example, the solution of the SPDE (1.1) has the decomposition:

$$u_t(x) = \int_0^t \int_0^1 g_{t-s}(x, y) dW_{s,y} = I_1(1_{[0,t]}(\cdot)g_{t-\cdot}(x, \cdot)).$$

If $A : D(A) \subset \mathcal{L}^2([0, T] \times [0, 1]) \rightarrow \mathcal{L}^2([0, T] \times [0, 1])$ is the unbounded operator

$$Ah(t, x) := A_t A_x h(t, x) := \sqrt{-\frac{\partial^2}{\partial t^2}} \sqrt{-\frac{\partial^2}{\partial x^2}} h(t, x)$$

with periodic boundary conditions, we define its *second quantization* as the operator,

$$\begin{aligned} \Gamma(A) : D(\Gamma(A)) \subset \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) &\rightarrow \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \\ \sum_{n \geq 0} I_n(h_n) &\mapsto \Gamma(A) \left(\sum_{n \geq 0} I_n(h_n) \right) := \sum_{n \geq 0} I_n(A^{\otimes n} h_n). \end{aligned}$$

For $p \geq 1$ the space

$$(S)_p := \{X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \text{ s.t. } E[|\Gamma(A^p)X|^2] < +\infty\}$$

is a subset of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ and if $q > p$ then $(S)_q \subset (S)_p$. The *Hida test function space* is defined as

$$(S) := \bigcap_{p \geq 1} (S)_p.$$

Its dual w.r.t. the inner product of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ is called *Hida distribution space* and is denoted by $(S)^*$. It can be shown that

$$(S)^* = \bigcup_{p \geq 1} (S)_{-p}.$$

Moreover by construction

$$(S) \subset \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}) \subset (S)^*.$$

For $f \in C_0^\infty([0, T] \times]0, 1[)$, the random variable

$$\mathcal{E}_T(f) := \exp \left\{ \int_0^T \int_0^1 f(s, y) dW_{s,y} - \frac{1}{2} \int_0^T \int_0^1 f^2(s, y) dy ds \right\},$$

belongs to (S) ; therefore if $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the dual pairing between $(S)^*$ and (S) then the application

$$X \in (S)^* \mapsto \mathcal{S}(X)(f) := \langle \langle X, \mathcal{E}_T(f) \rangle \rangle$$

is well defined and it is called *S-transform*. The function $\mathcal{S}(X)(\cdot)$ identifies uniquely the Hida distribution X . In particular, given $X, Y \in (S)^*$, we denote by $X \diamond Y$ the unique element of $(S)^*$ such that

$$\mathcal{S}(X \diamond Y)(f) = \mathcal{S}(X)(f) \mathcal{S}(Y)(f), \text{ for all } f \in C_0^\infty([0, T] \times]0, 1[);$$

the stochastic distribution $X \diamond Y$ is named *Wick product* of X and Y .

Notice also that if $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ then

$$\mathcal{S}(X)(f) = \langle \langle X, \mathcal{E}_T(f) \rangle \rangle = E[X \mathcal{E}_T(f)],$$

and if ξ is an Itô integrable stochastic process then

$$\begin{aligned} \mathcal{S}\left(\int_0^T \int_0^1 \xi_{t,x} dW_{t,x}\right)(f) &= E\left[\int_0^T \int_0^1 \xi_{t,x} dW_{t,x} \mathcal{E}_T(f)\right] \\ &= \int_0^T \int_0^1 E[\xi_{t,x} \mathcal{E}_T(f)] dx dt. \end{aligned}$$

2.2. Itô-type formula. Before the main theorem of this paper, we state and prove the following auxiliary result.

Lemma 2.1. *Let $u_t(x)$ be the solution of the SPDE (1.1); then $\partial_{xx}u_t(x) \in (S)^*$.*

Proof. It is known that the fundamental solution $g_t(x, y)$ of the heat equation with homogenous Dirichlet boundary conditions (1.2) can be represented as

$$g_t(x, y) = \sum_{n \geq 1} e^{-\lambda_n t} e_n(x) e_n(y),$$

where $e_n(x) = \sqrt{2} \sin(\pi n x)$ and $\lambda_n = \pi^2 n^2$. This gives

$$\begin{aligned} A_y^{-2}(\partial_{xx}g_t(x, y)) &= A_y^{-2}\left(\sum_{n \geq 1} e^{-\lambda_n t} (-\lambda_n) e_n(x) e_n(y)\right) \\ &= A_y^{-2}\left(\sum_{n \geq 1} e^{-\lambda_n t} e_n(x) A_y^2 e_n(y)\right) \\ &= \sum_{n \geq 1} e^{-\lambda_n t} e_n(x) e_n(y) \\ &= g_t(x, y). \end{aligned}$$

Therefore,

$$\begin{aligned} E[\Gamma(A^{-2})\partial_{xx}u_t(x)^2] &= \int_0^t \int_0^1 |A_s^{-2} A_y^{-2} \partial_{xx}g_{t-s}(x, y)|^2 dy ds \\ &= \int_0^t \int_0^1 |A_s^{-2} g_{t-s}(x, y)|^2 dy ds \\ &\leq C_t \int_0^t \int_0^1 |g_{t-s}(x, y)|^2 dy ds < +\infty, \end{aligned}$$

where C_t is a constant; this proves that $\partial_{xx}u_t(x) \in (S)_{-2} \subset (S)^*$. \square

Theorem 2.2. *For any $\varphi \in C_b^2(\mathbb{R})$ and $l \in C_0^2(]0, 1[)$ one has*

$$\begin{aligned} \langle \varphi(u_t), l \rangle &= \langle \varphi(0), l \rangle + \int_0^t \langle \varphi'(u_s) l, dW_s \rangle + \frac{1}{2} \langle \int_0^t (\varphi'(u_s) \diamond \partial_{xx}u_s) ds, l \rangle \\ &\quad + \frac{1}{2} \int_0^t \langle \varphi''(u_s) g_{2s}, l \rangle ds. \end{aligned} \tag{2.1}$$

Proof. Without loss of generality we can assume that $\varphi(0) = 0$. We aim at proving that

$$E[\langle \varphi(u_t), l \rangle \mathcal{E}_T(f)] = E[\mathcal{R} \mathcal{E}_T(f)],$$

for all $f \in C_0^\infty([0, T[\times]0, 1])$ where \mathcal{R} denotes the R.H.S. of (2.1). This fact together with the property

$$\text{span}\{\mathcal{E}_T(f), f \in C_0^\infty([0, T[\times]0, 1])\} \text{ is dense in } \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$$

will imply that

$$\langle \varphi(u_t), l \rangle = \mathcal{R}, \quad \mathcal{P} - \text{almost surely,}$$

which is the statement of the theorem.

Let us fix an arbitrary $f \in C_0^\infty([0, T[\times]0, 1])$; a simple application of the Girsanov theorem yields:

$$\begin{aligned} E[\varphi(u_t(x))\mathcal{E}_T(f)] &= E\left[\varphi\left(\int_0^t \int_0^1 g_{t-s}(x, y)dW_{s,y}\right)\mathcal{E}_T(f)\right] \\ &= E\left[\varphi\left(\int_0^t \int_0^1 g_{t-s}(x, y)dW_{s,y} + \int_0^t \int_0^1 g_{t-s}(x, y)f(s, y)dyds\right)\right]. \end{aligned}$$

Now observe that

$$\int_0^t \int_0^1 g_{t-s}(x, y)dW_{s,y} + \int_0^t \int_0^1 g_{t-s}(x, y)f(s, y)dyds,$$

is a Gaussian random variable with mean given by

$$m(t, x) := \int_0^t \int_0^1 g_{t-s}(x, y)f(s, y)dyds,$$

and variance equal to

$$\sigma^2(t, x) := \int_0^t \int_0^1 g_{t-s}^2(x, y)dyds = \int_0^t g_{2s}(x, x)ds,$$

where the second equality is obtained from the semigroup property of g_t together with the change of variable $s \mapsto t - s$.

Therefore we can write

$$\begin{aligned} E[\varphi(u_t(x))\mathcal{E}_T(f)] &= E\left[\varphi\left(\int_0^t \int_0^1 g_{t-s}(x, y)dW_{s,y} + \int_0^t \int_0^1 g_{t-s}(x, y)f(s, y)dyds\right)\right] \\ &= (P_{\sigma^2(t, x)}\varphi)(m(t, x)), \end{aligned} \tag{2.2}$$

where $\{P_t\}_{t \geq 0}$ denotes the one dimensional heat semigroup,

$$(P_t\varphi)(x) := \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \varphi(y)e^{-\frac{(x-y)^2}{2t}} dy.$$

Identity (2.2) turns out to be very useful; in fact we can apply the ordinary chain rule and get

$$\begin{aligned}
E[\varphi(u_t(x))\mathcal{E}_T(f)] &= (P_{\sigma^2(t,x)}\varphi)(m(t,x)) \\
&= \int_0^t \frac{d}{dv}(P_{\sigma^2(v,x)}\varphi)(m(v,x))dv \\
&= \int_0^t (P_{\sigma^2(v,x)}\varphi')(m(v,x))\frac{dm(v,x)}{dv}dv \\
&\quad + \frac{1}{2} \int_0^t (P_{\sigma^2(v,x)}\varphi'')(m(v,x))g_{2v}(x,x)dv \\
&= \mathcal{A}(t,x) + \mathcal{B}(t,x),
\end{aligned}$$

where

$$\mathcal{A}(t,x) := \int_0^t (P_{\sigma^2(v,x)}\varphi')(m(v,x))\frac{dm(v,x)}{dv}dv,$$

and

$$\mathcal{B}(t,x) := \frac{1}{2} \int_0^t (P_{\sigma^2(v,x)}\varphi'')(m(v,x))g_{2v}(x,x)dv.$$

Recalling the definition of $m(v,x)$ we have

$$\begin{aligned}
\frac{dm(v,x)}{dv} &= f(v,x) + \frac{1}{2} \int_0^v \int_0^1 \partial_{xx}g_{v-s}(x,y)f(s,y)dyds \\
&= f(v,x) + \frac{1}{2} \partial_{xx} \left(\int_0^v \int_0^1 g_{v-s}(x,y)f(s,y)dyds \right),
\end{aligned}$$

and hence

$$\begin{aligned}
\mathcal{A}(t,x) &= \int_0^t (P_{\sigma^2(v,x)}\varphi')(m(v,x)) \left(f(v,x) \right. \\
&\quad \left. + \frac{1}{2} \partial_{xx} \left(\int_0^v \int_0^1 g_{v-s}(x,y)f(s,y)dyds \right) \right) dv \\
&= \int_0^t (P_{\sigma^2(v,x)}\varphi')(m(v,x))f(v,x)dv \\
&\quad + \frac{1}{2} \int_0^t (P_{\sigma^2(v,x)}\varphi')(m(v,x))\partial_{xx} \left(\int_0^v \int_0^1 g_{v-s}(x,y)f(s,y)dyds \right) dv \\
&= \int_0^t E[\varphi'(u_v(x))\mathcal{E}_T(f)]f(v,x)dv \\
&\quad + \frac{1}{2} \int_0^t E[\varphi'(u_v(x))\mathcal{E}_T(f)]\partial_{xx}E[u_v(x)\mathcal{E}_T(f)]dv,
\end{aligned}$$

where the last equality is due to identities (2.2). Moreover since from Lemma 2.1 $\partial_{xx}u_v(x) \in (S)^*$ we have

$$\partial_{xx}E[u_v(x)\mathcal{E}_T(f)] = \partial_{xx}\mathcal{S}(u_v(x))(f) = \mathcal{S}(\partial_{xx}u_v(x))(f),$$

where $\mathcal{S}(\partial_{xx}u_v(x))(f)$ denotes the \mathcal{S} -transform of $\partial_{xx}u_v(x)$.

If now $l \in C_0^2([0, 1])$ is a test function in the space variable x , by the properties of the Wick product we conclude that

$$\begin{aligned}
\int_0^1 \mathcal{A}(t, x)l(x)dx &= \int_0^1 \int_0^t E[\varphi'(u_v(x))\mathcal{E}_T(f)]l(x)f(v, x)dvdx \\
&\quad + \frac{1}{2} \int_0^1 \int_0^t E[\varphi'(u_v)\mathcal{E}_T(f)]\mathcal{S}(\partial_{xx}u_v(x))(f)l(x)dvdx \\
&= \int_0^1 \int_0^t E[\varphi'(u_v(x))\mathcal{E}_T(f)]l(x)f(v, x)dvdx \\
&\quad + \frac{1}{2} \int_0^1 \int_0^t \mathcal{S}(\varphi'(u_v(x)) \diamond \partial_{xx}u_v(x))(f)l(x)dvdx \\
&= E\left[\left(\int_0^t \int_0^1 \varphi'(u_v(x))l(x)dW_{v,x}\right)\mathcal{E}_T(f)\right] \\
&\quad + \frac{1}{2}E\left[\left(\int_0^1 \left(\int_0^t \varphi'(u_v(x)) \diamond \partial_{xx}u_v(x)dv\right)l(x)dx\right)\mathcal{E}_T(f)\right].
\end{aligned}$$

Looking again at identities (2.2) we also discover that

$$\mathcal{B}(t, x) = \frac{1}{2} \int_0^t E[\varphi''(u_v(x))\mathcal{E}_T(f)]g_{2v}(x, x)dv;$$

combining the expressions for \mathcal{A} and \mathcal{B} we can now conclude that

$$\begin{aligned}
\langle E[\varphi(u_t)\mathcal{E}_T(f)], l \rangle &= E\left[\left(\int_0^t \int_0^1 \varphi'(u_v(x))l(x)dW_{v,x}\right.\right. \\
&\quad \left.\left.+ \frac{1}{2} \langle \int_0^t \varphi'(u_v) \diamond \partial_{xx}u_v dv, l \rangle\right)\mathcal{E}_T(f)\right] \\
&\quad + \frac{1}{2} \langle \int_0^t E[\varphi''(u_v)\mathcal{E}_T(f)]g_{2v}dv, l \rangle \\
&= E\left[\left(\int_0^t \langle \varphi'(u_v)l, dW_v \rangle + \frac{1}{2} \langle \int_0^t (\varphi'(u_v) \diamond \partial_{xx}u_v)dv, l \rangle\right)\mathcal{E}_T(f)\right] \\
&\quad + \frac{1}{2}E\left[\langle \int_0^t \varphi''(u_v)g_{2v}dv, l \rangle\mathcal{E}_T(f)\right].
\end{aligned}$$

This completes the proof. \square

3. Comparisons

3.1. Zambotti's formula. In [10] the author obtains the following Itô formula:

$$\begin{aligned}
\langle \varphi(u_t), l \rangle &= \langle \varphi(0), l \rangle + \frac{1}{2} \int_0^t \langle l'', \varphi(u_s) \rangle ds + \int_0^t \langle \varphi'(u_s)l, dW_s \rangle \\
&\quad - \frac{1}{2} \int_0^t \langle l, : |\partial_x u_s|^2 : \varphi''(u_s) \rangle ds,
\end{aligned} \tag{3.1}$$

where the last term is defined as the limit of renormalized diverging quantities. The procedure to derive this formula is to approximate the solution of the SPDE

via the smoother process

$$u_t^\epsilon(x) := \int_0^t \int_0^1 g_{t-s+\epsilon}(x, y) dW_{s,y},$$

and then to pass to the limit as $\epsilon \rightarrow 0$. We are now going to show that formula (2.1) is equivalent to formula (3.1).

It is easy to see that the line of reasoning of the proof of Theorem 2.2 can be also carried for the process $u_t^\epsilon(x)$; in this case the variance takes the form

$$\sigma_\epsilon^2(t, x) := E[|u_t^\epsilon(x)|^2] = \int_0^t \int_0^1 g_{t-s+\epsilon}^2(x, y) dy ds,$$

and hence the last term of formula (2.1) can be rewritten as

$$\begin{aligned} \int_0^t \varphi''(u_s^\epsilon(x)) \frac{d\sigma_\epsilon^2(s, x)}{ds} ds &= \int_0^t \varphi''(u_s^\epsilon(x)) \left(\int_0^1 g_\epsilon^2(x, y) dy \right. \\ &\quad \left. + \int_0^s \int_0^1 \partial_s(g_{s-v+\epsilon}^2(x, y)) dy dv \right) ds \\ &= \int_0^t \varphi''(u_s^\epsilon(x)) \left(\int_0^1 g_\epsilon^2(x, y) dy \right. \\ &\quad \left. + \int_0^s \langle g_{s-v+\epsilon}(x), \partial_{xx} g_{s-v+\epsilon}(x) \rangle dv \right) ds \\ &= \int_0^t \varphi''(u_s^\epsilon(x)) \int_0^1 g_\epsilon^2(x, y) dy ds \\ &\quad + \int_0^t \varphi''(u_s^\epsilon(x)) \int_0^s \langle g_{s-v+\epsilon}(x), \partial_{xx} g_{s-v+\epsilon}(x) \rangle dv ds \\ &= \int_0^t \varphi''(u_s^\epsilon(x)) \int_0^1 g_\epsilon^2(x, y) dy ds \\ &\quad + \int_0^t \int_0^s \int_0^1 D_{v,y} \varphi'(u_s^\epsilon(x)) D_{v,y} \partial_{xx} u_s^\epsilon(x) dy dv ds \\ &= \int_0^t \varphi''(u_s^\epsilon(x)) \int_0^1 g_\epsilon^2(x, y) dy ds \\ &\quad + \int_0^t (\varphi'(u_s^\epsilon(x)) \partial_{xx} u_s^\epsilon(x) - \varphi'(u_s^\epsilon(x)) \diamond \partial_{xx} u_s^\epsilon(x)) ds. \end{aligned}$$

Here we have used the property of the Wick product which shows its interplay with the Hida-Malliavin derivative, namely

$$\begin{aligned} X \diamond \int_0^T \int_0^1 h(s, y) dW_{s,y} \\ = X \int_0^T \int_0^1 h(s, y) dW_{s,y} - \int_0^T \int_0^1 (D_{s,y} X) h(s, y) dy ds. \end{aligned}$$

See [5] and [7] for details. Therefore formula (2.1) becomes

$$\begin{aligned} \langle \varphi(u_t^\epsilon), l \rangle &= \langle \varphi(0), l \rangle + \int_0^t \langle \varphi'(u_s^\epsilon) l, dW_s \rangle + \frac{1}{2} \langle \int_0^t \varphi'(u_s^\epsilon) \diamond \partial_{xx} u_s^\epsilon ds, l \rangle \\ &\quad + \frac{1}{2} \langle \int_0^t \varphi''(u_s^\epsilon) \int_0^1 g_\epsilon^2(\cdot, y) dy ds, l \rangle \\ &\quad + \frac{1}{2} \langle \int_0^t (\varphi'(u_s^\epsilon) \partial_{xx} u_s^\epsilon - \varphi'(u_s^\epsilon) \diamond \partial_{xx} u_s^\epsilon) ds, l \rangle \\ &= \langle \varphi(0), l \rangle + \int_0^t \langle \varphi'(u_s^\epsilon) l, dW_s \rangle \\ &\quad + \frac{1}{2} \langle \int_0^t \varphi''(u_s^\epsilon) \int_0^1 g_\epsilon^2(\cdot, y) dy ds, l \rangle + \frac{1}{2} \langle \int_0^t \varphi'(u_s^\epsilon) \partial_{xx} u_s^\epsilon ds, l \rangle. \end{aligned}$$

Moreover

$$\varphi'(u_s^\epsilon(x)) \partial_{xx} u_s^\epsilon(x) = \partial_{xx} \varphi(u_s^\epsilon(x)) - \varphi''(u_s^\epsilon(x)) (\partial_x u_s^\epsilon(x))^2;$$

a substitution in the previous equality gives

$$\begin{aligned} \langle \varphi(u_t^\epsilon), l \rangle &= \langle \varphi(0), l \rangle + \int_0^t \langle \varphi'(u_s^\epsilon) l, dW_s \rangle \\ &\quad + \frac{1}{2} \langle \int_0^t \varphi''(u_s^\epsilon) (\int_0^1 g_\epsilon^2(\cdot, y) dy) ds, l \rangle \\ &\quad + \frac{1}{2} \langle \int_0^t (\partial_{xx} \varphi(u_s^\epsilon) - \varphi''(u_s^\epsilon) (\partial_x u_s^\epsilon)^2) ds, l \rangle \\ &= \langle \varphi(0), l \rangle + \int_0^t \langle \varphi'(u_s^\epsilon) l, dW_s \rangle \\ &\quad - \frac{1}{2} \langle \int_0^t \varphi''(u_s^\epsilon) ((\partial_x u_s^\epsilon)^2 - \int_0^1 g_\epsilon^2(\cdot, y) dy) ds, l \rangle \\ &\quad + \frac{1}{2} \langle \int_0^t \varphi(u_s^\epsilon) ds, l'' \rangle. \end{aligned}$$

Since this identity coincides with the expression derived in [10] before taking the limit as $\epsilon \rightarrow 0$, the equivalence of the two formulas is proved.

3.2. Gradinaru-Nourdin-Tindel's formula. In [3] the authors interpret the solution $\{u_t(x), t \in [0, T], x \in [0, 1]\}$ of the SPDE (1.1) as a random process $t \mapsto u_t$ taking values on the Hilbert space $\mathcal{L}^2([0, 1])$. With this approach they prove that for any smooth $F : \mathcal{L}^2([0, 1]) \rightarrow \mathbb{R}$ the following formula holds:

$$F(u_t) = F(0) + \int_0^t \langle F'(u_s), \delta u_s \rangle + \frac{1}{2} \int_0^t \text{Tr}(e^{s\partial_{xx}} F''(u_s)) ds. \quad (3.2)$$

Here the term $\int_0^t \langle F'(u_s), \delta u_s \rangle$ denotes a Skorohod type integral w.r.t. the process $\{u_t\}_{t \in [0, T]}$ while $\text{Tr}(e^{s\partial_{xx}} F''(u_s))$ stands for the trace of the operator $e^{s\partial_{xx}} F''(u_s)$. To prove this fact the authors show that the projections on Wiener chaoses of different orders of the RHS of (3.2) coincides with the corresponding projection of the LHS of (3.2). This procedure makes use of the Kolmogorov equation associated

to the SPDE (1.1) in its abstract formulation (see e.g. [1]) and of the duality between the Skorohod integral and the Malliavin derivative (see e.g. [7]).

If for instance one chooses $F : \mathcal{L}^2([0, 1]) \rightarrow \mathbb{R}$ to be

$$F(h) := \int_0^1 \varphi(h(x))l(x)dx, \quad \varphi \in C_b^2(\mathbb{R}), l \in C_0^2([0, 1]),$$

then formula (3.2) reads:

$$\langle \varphi(u_t), l \rangle = \langle \varphi(0), l \rangle + \int_0^t \langle \varphi'(u_s)l, \delta u_s \rangle + \frac{1}{2} \int_0^t \langle \varphi''(u_s)g_{2s}, l \rangle ds. \quad (3.3)$$

A straightforward comparison between formula (2.1) and (3.3) reveals that

$$\int_0^t \langle \varphi'(u_s)l, \delta u_s \rangle = \int_0^t \langle \varphi'(u_s)l, dW_s \rangle + \frac{1}{2} \left\langle \int_0^t (\varphi'(u_s) \diamond \partial_{xx} u_s) ds, l \right\rangle;$$

in other words, the Skorohod type integral defined in [3] as the adjoint of the Malliavin derivative w.r.t. the process u_s can be seen as a Wick-multiplication by $\frac{\partial u_s}{\partial s}$.

References

1. Da Prato, G. and Zabczyk, J.: *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge, 1992.
2. Deck, T., Potthoff, J., and Våge, G.: A review of white noise analysis from a probabilistic standpoint, *Acta Appl. Math.* **48** (1997) 91–112.
3. Gradinaru, M., Nourdin, I., and Tindel, S.: Ito's- and Tanaka's-type formulae for the stochastic heat equation: the linear case *J. Funct. Anal.* **228** (2005) 114–143.
4. Holden, H., Øksendal, B., Ubøe, J. and Zhang, T.-S.: *Stochastic Partial Differential Equations- A Modeling, White Noise Functional Approach*, Birkhäuser, Boston, 1996.
5. Kuo, H.-H.: *White noise distribution theory*, CRC Press, Boca Raton, 1996.
6. Lanconelli, A.: Wick product and backward heat equation *Mediterranean J. of Math.* **2** (2005) 367–379.
7. Nualart, D.: *The Malliavin calculus and related topics*, Springer, Berlin, 1995.
8. Nualart, D. and Taqqu, M. S.: Wick-Itô formula for Gaussian processes *Stoch. Anal. Appl.* **24** (2006) 599–614
9. Walsh, J. B.: An introduction to stochastic partial differential equations, in: *Ecole d'ete de probabilités de Saint-Flour XIV*, (1984) 236-439, Springer, Berlin.
10. Zambotti, L.: Itô-Tanaka's formula for stochastic partial differential equations driven by additive space-time white noise, in: *SPDEs and applications VII*, Lect. Notes Pure Appl. Math. **245**, (2006) 337–347, Boca Raton.

ALBERTO LANCONELLI: DIPARTIMENTO DI MATEMATICA, UNIVERSITA' DEGLI STUDI DI BARI,
VIA E.ORABONA 4, 70125 BARI, ITALIA
E-mail address: lanconelli@dm.uniba.it