Communications on Stochastic Analysis
Vol. 1, No. 2 (2007) 321-341

QUANTUM STOCHASTIC CALCULUS ON INTERACTING FOCK SPACES: SEMIMARTINGALE ESTIMATES AND STOCHASTIC INTEGRAL

VITONOFRIO CRISMALE

ABSTRACT. A quantum stochastic integration theory on interacting Fock spaces (IFS) is developed. We present the semi-martingale inequalities either in standard general IFS or in 1-mode type IFS, which allow us to introduce the definitions of integrable processes and construct stochastic integrals satisfying some useful properties which will be presented in [13].

1. Introduction

It is our scope to develop a quantum stochastic calculus on (standard) interacting Fock space. In this first part we establish the *semimartingale inequalities* for simple adapted processes and use them to define stochastic integral for a large class of operators. In the second part [13] we will use these inequalities to establish existence and uniqueness of the solution of quantum stochastic differential equations, a Ito formula and a unitarity condition.

Quantum stochastic calculus was initiated by Hudson and Parthasarathy in [16] and Barnett, Streater and Wilde in [11]. After these pioneering works, a great number of papers was devoted to develop a theory in non Boson cases (see e.g. [10] for the Fermion case, [15] for universal invariant case, [24] for free, [18] for general quasi-free, [12] for Boolean, [23] for full Fock module). Accardi, Fagnola and Quaegebeur in [3] reached a double result: on the one hand developing a theory independent of the particular representation chosen (as in the classical case) and on the other hand including all the quantum stochastic calculi already appeared (boson and fermion) into a unifying picture. Successively Fagnola in [14] showed that a suitable extension of this theory allowed to construct a quantum stochastic integral for the "free" noise case introduced by Kümmerer and Speicher in [17] and Speicher in [24].

In the 90's a new structure, the interacting Fock spaces (IFS), appeared in quantum probability. The interacting Fock spaces emerged from the *stochastic limit* in quantum electrodynamics (see [8] for details) and were systematically studied in several papers (see, for instance, [6], [19], [20], [21]). In 1998 Accardi and Bożejko in [1] showed that for 1-mode interacting Fock spaces the interacting factors can be expressed by means of the Jacobi coefficients of any probability

2000 *Mathematics Subject Classification.* Primary 81S25, 46L51; Secondary 60H05.
Key words and phrases. Interacting Fock spaces; Quantum stochastic calculus.

measure on the real line with all finite moments. This allows to construct a unitary isomorphism between such IFS and the $\mathbf{L}^2$-space associated with the probability distribution chosen. Such an approach was later generalized to finite dimensions bigger than one in [9] and to infinite dimensions in [4].

The aim of the present paper and its second part [13] is developing a quantum stochastic calculus for a class of standard interacting Fock spaces containing the 1-mode IFS. The free Fock space is a special interacting Fock space; it seems therefore natural to extend the results contained in [14], [17], [24] to the more general setting. We shall follow the approach in [14]; it is worth noticing that, if one takes as IFS the full (free) Fock space, the results here presented reduce to those shown in [14]. The paper is organized as follows. In Section 2, after introducing some "constraints" to get creation and annihilation on IFS as bounded operators and give the proof of the main technical tool used (the semi-martingale estimates), we define stochastic processes as a family of operators whose domains contain a subset of the number vectors. Successively we introduce a family of $*$-subalgebras of the algebra of bounded operators on IFS which plays the role of filtration in classical stochastic calculus. Finally we define the simple adapted processes and stochastic integrals on them with respect to the basic processes of creation and annihilation. Section 3 is devoted to the proof of semimartingale inequalities for simple adapted processes: roughly speaking such inequalities allow to majorize stochastic integrals of simple adapted processes by ordinary ones. They are the main technical tool in order to define the class of integrable processes and construct a stochastic integral enjoying some "nice" properties as we will see in [13].

We present two different proofs, namely for the case of non constant interacting functions and for 1-mode type interacting Fock spaces, getting different estimates. Those obtained in the general case clearly hold for the 1-mode type IFS. On the contrary, the peculiarity of the structure of these spaces allows us to achieve more subtle results, whose proofs can not be extended to the general structure (e.g. the case $\int_0^t F(s)\, dA^+(s)\, \xi$). Moreover, these results become necessary whenever one wants to cover all the symmetric distributions on the real line which can be expressed in terms of creation-annihilation operators in 1-mode type IFS (see [5] for details).

In Section 4 we use the semimartingale inequalities to define quantum stochastic integrals, in both the cases above introduced, for a class of processes wider than simple adapted ones. As in [3], the elements of this class are obtained as limits with respect to the locally convex topology generated by a family of seminorms on the domain of simple adapted processes and the topology of strong $*$-convergence on a proper domain. It is this choice of the topologies that will allow us in [13] to prove existence and uniqueness of the solution of quantum stochastic differential equations.

2. Simple adapted processes

Throughout these notes we will fix on setting stochastic calculus theory over standard interacting Fock space. Let be given

- $(X, \mathcal{X}, \mu)$ a measure space,

- $\{\lambda_n\}_n$ a sequence of measurable positive functions, where $\lambda_n : X^n \to \mathbb{R}_+$ for any $n \geq 1$ and $\lambda_0 := 1$.

We suppose there exists a sequence $(M_n)_n$ of non negative numbers such that for almost all $x_1, ..., x_n \in X$

$$\lambda_n (x_n, ..., x_1) \leq M_n \lambda_{n-1} (x_{n-1}, ..., x_1), \quad \forall n = 1, 2, \dots \tag{2.1}$$

Under our assumptions it follows that for $\mu-$almost all $(x_n, ..., x_1) \in X^n$

$$\lambda_n (x_n, ..., x_1) \leq M_n M_{n-1} \cdots M_1, \quad \forall n \geq 1$$

$$\lambda_{n+1} \ (x_0, x_n, ..., x_1) = 0, \quad \mu - a.a. \ x_0 \in X \ \text{ if } \lambda_n (x_n, ..., x_1) = 0 \tag{2.2}$$

Denote by $\mathcal{H}$ the Hilbert space $\mathbf{L}^2 (X, \mu)$. For any $n \geq 1$, over the $n-$th algebraic tensor product $\mathcal{H}^{\odot n} := \overbrace{\mathcal{H} \odot \mathcal{H} \odot \cdots \odot \mathcal{H}}^{n}$, thanks to (2.1) the following pre-scalar product is well defined

$$\langle F_n, G_n \rangle_n := \int_{M^n} \lambda_n (x_n, ..., x_1) \left(\overline{F}_n G_n\right) (x_n, ..., x_1) \, \mu (dx_n) \cdots \mu (dx_1) \tag{2.3}$$

where $F, G \in \mathcal{H}^{\odot n}$. By taking quotient and completing, one gets a Hilbert space which will be denoted as $\mathcal{H}_n$. Any vector in $\mathcal{H}_n$ is called a $n-$particle vector.

For any $n \geq 1$ and for any $f \in \mathcal{H}$, we define the creation operator as a linear operator such that

$$A^+ (f) F := f \otimes F \in \mathcal{H}_{n+1}, \ \text{ for any } F \in \mathcal{H}_n \tag{2.4}$$

whereas, if Φ denote the vacuum vector, i.e. $\Phi := 1 \oplus 0 \oplus 0 \oplus \cdots$

$$A^+ (f) \Phi = f$$

By (2.1) this operator belongs to $\mathbf{B} (\mathcal{H}_n, \mathcal{H}_{n+1})$, in fact for each $f \in \mathcal{H}$ and $F \in \mathcal{H}_n$, $\|A^+ (f) F\| \leq \sqrt{M_{n+1}} \, \|f\| \, \|F\|$. Its adjoint operator, called the annihilation operator, is well defined and also bounded from $\mathcal{H}_{n+1}$ into $\mathcal{H}_n$. It is easy to see that for any $n \geq 1$, for any $G \in \mathcal{H}_n$ and $\mu-$a.a. $x_0, x_1, ..., x_{n-1} \in X$

$$A (g) G (x_{n-1}, ..., x_1) = \int_X \frac{\lambda_n (x_0, x_{n-1}, ..., x_1)}{\lambda_{n-1} (x_{n-1}, ..., x_1)} \overline{g} (x_0) \, G (x_0, ..., x_1) \, \mu (dx_0) \tag{2.5}$$

and for any $f \in \mathcal{H}$

$$A (f) \Phi = 0$$

With the conventions $\mathcal{H}_0 := \mathbb{C}$ and $\mathcal{H}_1 := \mathcal{H}$ (i.e. $\lambda_1 = 1$) we call standard interacting Fock space (IFS) with interacting functions $\{\lambda_n\}_{n \in \mathbb{N}}$, the following space

$$\Gamma_I (\mathcal{H}) := \bigoplus_{n=0}^{\infty} \left(\mathcal{H}_n, \{\lambda_n\}_{n \in \mathbb{N}}\right) \tag{2.6}$$

Whenever all the interacting functions are constants, the space defined in (2.6) is called 1-mode type interacting Fock space (see [2] for more details). In particular, if for any $n \in \mathbb{N}$ $\lambda_n = 1$, we find the full (free) Fock space.

For the balance of these notes we restrict to the measure space $\mathbb{R}_+$ with the Lebesgue measure. We will treat two cases. First, the case with constant functions λ_n fulfilling

$$\frac{\lambda_n}{\lambda_{n+1}} \le M_n; \ \frac{\lambda_n}{\lambda_{n-1}} \le M_n, \quad \forall n \ge 1 \tag{2.7}$$

where $(M_n)_{n\ge 1}$ is a sequence of positive numbers.

Secondly, the case

$$\frac{\lambda_n(x_n,...,x_1)}{\lambda_{n+1}(x_0,x_n,...,x_1)} \le M; \frac{\lambda_{n+1}(x_0,x_n,...,x_1)}{\lambda_n(x_0,...,\widehat{x_k},...,x_1)} \le M, \quad \forall n \ge 1, \ \forall \ k = 0,1,...,n \tag{2.8}$$

for almost all $(x_0, x_n, ..., x_1) \in X^{n+1}$, where $M > 0$. Here the interacting functions are not necessarily constant (we refer it as "general" standard IFS), but M_n does not depend on n. The reason of introducing stronger constraints in this case consists in helping us to establish a semimartingale estimate, as presented in Proposition 3.11.

Denote $\mathfrak{F}_I := \Gamma_I\left(\mathbf{L}^2(\mathbb{R}_+)\right)$ and

$$\mathfrak{D} := \left\{\Phi, u_1 \otimes ... \otimes u_k : k \in \mathbb{N}, u_j \in \mathbf{L}^2(\mathbb{R}_+, dx) \cap \mathbf{L}^1(\mathbb{R}_+, dx), \right.$$
$$\left. \|u_j\|_{\mathbf{L}^2} \le 1 \text{ and } \|u_j\|_{\mathbf{L}^1} \le 1\right\}$$

then $Lispan\mathfrak{D} \subseteq$Dom$(A^+(f))\cap$Dom$(A(f))$. Furthermore by the symbol $\mathfrak{L}(\mathfrak{D}, \mathfrak{F}_I)$ we denote the vector space of all linear operators with domain containing $\mathfrak{D}$, taking values in $\mathfrak{F}_I$ and such that their adjoint operator also contains $\mathfrak{D}$ in its domain. From the above discussion, it follows that this set is not empty.

The following definition introduces the notion of stochastic process.

Definition 2.1. A family $(X(t))_{t\ge 0}$ of elements of $\mathfrak{L}(\mathfrak{D}, \mathfrak{F}_I)$ is called a **stochastic process** in $\mathfrak{F}_I$ if for any $\xi \in \mathcal{D}$, the map $t \mapsto X(t)\xi$ is strongly measurable.

Throughout the paper we adopt the following notation:

- for any s, $t \in \mathbb{R}_+$ such that $s \le t$

  $$A(t) := A\left(\chi_{[0,t)}\right), \qquad A^+(t) := A^+\left(\chi_{[0,t)}\right)$$
  $$A(s,t) ;= A\left(\chi_{[s,t)}\right), \qquad A^+(s,t) := A^+\left(\chi_{[s,t)}\right)$$

  where for any $E \subseteq \mathbb{R}_+$, χ_E is the indicator function of the set E and

  $$M^{01} := (A(t))_{t\ge 0}; \quad M^{10} := \left(A^+(t)\right)_{t\ge 0}; \quad M^{00} := (t\mathbf{1})_{t\ge 0}$$

  are called respectively the annihilation, creation and deterministic processes. They are stochastic processes in the sense of Definition 2.1.
- for any $t \in [0, +\infty]$ one denotes by $A_{t]}$ the linear span of

  $$\mathcal{B}_t := \left\{A^{\varepsilon(n)}(g_n) A^{\varepsilon(n-1)}(g_{n-1}) \cdots A^{\varepsilon(1)}(g_1) : n \in \mathbb{N}\cup\{0\}, \right.$$
  $$\left. \varepsilon(k) \in \{-1, 1\}, \ g_k \in \mathbf{L}^2(0,t) \text{ for any } k = 1, 2, \cdots, n\right\} \tag{2.9}$$

where $A^{\varepsilon(n)}(g_n)\, A^{\varepsilon(n-1)}(g_{n-1}) \cdots A^{\varepsilon(1)}(g_1)$ is understood as the identity if $n = 0$ and

$$A^{\varepsilon}(g) := \begin{cases} A(f) & \text{if } \varepsilon = -1 \\ A^{+}(f) & \text{if } \varepsilon = 1 \end{cases} \tag{2.10}$$

It is clear that $A_{t]}$ is a $*$-subalgebra of $\mathbf{B}(\mathfrak{F}_I)$ for any t.

Remark 2.2. Recall that in 1-mode type IFS the annihilation operator is such that for any $n \in \mathbb{N}$, for any $f, f_n, \ldots, f_1 \in \mathcal{H}$

$$A(f)\, f_n \otimes f_{n-1} \otimes \cdots \otimes f_1 = \frac{\lambda_n}{\lambda_{n-1}} \langle f, f_n \rangle\, f_{n-1} \otimes \cdots \otimes f_1$$

Therefore each element of $A_{t]}$ can be written as a sum of operators of the form

$$c(t, f, g)\, A^{+}(f_h) \cdots A^{+}(f_1)\, A(g_l) \cdots A(g_1) \tag{2.11}$$

where $f \in \mathbf{L}^2\left((0,t), \mathbb{C}^h\right)$, $g \in \mathbf{L}^2\left((0,t), \mathbb{C}^l\right)$, $h, l \in \mathbb{N}$, $c(t, f, g) \in \mathbb{C}$.

A stochastic process is called **simple adapted** if it can be written as

$$\sum_{k=1}^{n} F(t_k)\, \chi_{[t_k, t_{k+1})} \tag{2.12}$$

where

$$n \in \mathbb{N}, 0 \le t_1 < t_2 < \ldots < t_{n+1} < +\infty,\ F(t_k) \in A_{t_k]},\ \forall\, k = 1, \ldots, n.$$

The vector space of simple adapted processes will be denoted by S. We notice that for any $t \geq 0$, any element $F(t)$ of the $*-$subalgebra $A_{t]}$ can be written as

$$F(t) = \sum_{h=1}^{m} \alpha_h F_h(t)$$

where $\alpha_h \in \mathbb{C}$, $F_h(t) \in \mathcal{B}_t$, $h = 1, \ldots, m$. In conclusion, any simple adapted process F is such that

$$F = \sum_{k=1}^{n} F(t_k)\, \chi_{[t_k, t_{k+1})},\ F(t_k) = \sum_{h=1}^{m} \alpha_h F_h(t_k),\ F_h(t_k) \in \mathcal{B}_{t_k}$$

Right and left stochastic integrals of simple adapted processes can be defined as usual in the following way:

$$\int_0^t F(s)\, dA(s) := \sum_{k=1}^{n} F(t_k)\, A(t_k \wedge t, t_{k+1} \wedge t)$$

$$\int_0^t dA(s)\, F(s) := \sum_{k=1}^{n} A(t_k \wedge t, t_{k+1} \wedge t)\, F(t_k)$$

$$\int_0^t F(s)\, dA^{+}(s) := \sum_{k=1}^{n} F(t_k)\, A^{+}(t_k \wedge t, t_{k+1} \wedge t)$$

$$\int_0^t dA^{+}(s)\, F(s) := \sum_{k=1}^{n} A^{+}(t_k \wedge t, t_{k+1} \wedge t)\, F(t_k)$$

where $t_k \wedge t := \min\{t_k, t\}$.

3. Semimartingale inequalities

This section is devoted to the proof of the 'semimartingale estimates for left and right stochastic integral with respect to simple adapted processes. They were firstly introduced in [3] and successively used in [14] as the main tool in order to extend the stochastic integral to a wider class of processes, as we do in Section 4, and to prove a quantum Ito formula, as we will do in [13].

According to our two cases, the section splits into two parts. We start with the second, i.e. the case with non constant interacting functions.

Lemma 3.1. *For any interacting Fock space $\mathfrak{F}_I$, with interacting functions $(\lambda_n)_n$ satisfying (2.8), the following inequalities hold:*

i) for any $d, k \in \mathbb{N}$, for any $x_1, \ldots, x_d, y_1, \ldots, y_{k+1} \in \mathbb{R}_+$

$$\frac{\lambda_{d+k+1}\left(y_{k+1}, \ldots, y_1, x_d, \ldots, x_1\right)}{\lambda_d\left(x_d, \ldots, x_1\right)} \leq M \frac{\lambda_{d+k}\left(y_{k+1}, \ldots, \widehat{y_j}, \ldots, y_1, x_d, \ldots, x_1\right)}{\lambda_d\left(x_d, \ldots, x_1\right)}$$

for all $j = 1, \ldots, k+1$, where, as usual, we indicate by $\widehat{y_j}$ that the argument y_j is omitted;

ii) for any $d, k \in \mathbb{N}$, for any $x_1, \ldots, x_d, y_1, \ldots, y_k, z \in \mathbb{R}_+$

$$\frac{\lambda_{d+k}\left(y_k, \ldots, y_1, x_d, \ldots, x_1\right)}{\lambda_d\left(x_d, \ldots, x_1\right)} \leq M \frac{\lambda_{d+k+1}\left(z, y_k, \ldots, y_1, x_d, \ldots, x_1\right)}{\lambda_d\left(x_d, \ldots, x_1\right)}$$

iii) for any $d, k, m \in \mathbb{N}$, for any $x_1, \ldots, x_d, y_1, \ldots, y_k, t_1, \ldots, t_m, z \in \mathbb{R}_+$

$$\frac{\lambda_{d+k+1}\left(z, y_k, \ldots, y_1, x_d, \ldots, x_1\right)}{\lambda_{d+m+1}\left(z, t_m, \ldots, t_1, x_d, \ldots, x_1\right)} \leq M^2 \frac{\lambda_{d+k}\left(y_k, \ldots, y_1, x_d, \ldots, x_1\right)}{\lambda_{d+m}\left(t_m, \ldots, t_1, x_d, \ldots, x_1\right)}$$

Proof. The inequalities above clearly follow from (2.8). □

The following proposition is the first part of semimartingale estimates for standard IFS with non constant interacting functions.

Proposition 3.2. *Let $F \in S$, $d \in \mathbb{N}$. For standard interacting Fock spaces, with non constant interacting functions, for all $\xi = g_d \otimes \cdots \otimes g_1 \in \mathfrak{D}$ and $h = 1, \ldots, d$, let us denote $\eta_h := g_d \otimes \cdots \otimes g_{h+1} \otimes g_{h-1} \otimes \cdots \otimes g_1 \in \mathfrak{D}$ (with the convention that $\eta_h := 0$ if $\xi = \Phi$, $\eta_h := \Phi$ if $d = 1$). Then, for all $t \in \mathbb{R}_+$ we have*

$$\left\|\int_0^t F(s)\, dA(s)\, \xi\right\|^2 \leq M^2 t \int_0^t \left\|F(s)\, \eta_d\right\|^2 ds \tag{3.1}$$

$$\left\|\int_0^t dA^+(s)\, F(s)\, \xi\right\|^2 \leq M \int_0^t \left\|F(s)\, \xi\right\|^2 ds \tag{3.2}$$

$$\left\|\int_0^t dA(s)\, F(s)\, \xi\right\|^2 \leq M^2 \sum_{h=0}^{d-1} \int_0^t \left\|F(s)\, \eta_{d-h}\right\|^2 ds \tag{3.3}$$

$$\left\|\int_0^t F(s)\, ds \xi\right\|^2 \leq t \int_0^t \left\|F(s)\, \xi\right\|^2 ds \tag{3.4}$$

Proof. We begin by showing (3.1).

$$\left\| \int_0^t F(s)\, dA(s)\, \xi \right\|^2 = \left\| \sum_{k=1}^n F(t_k)\, A(t_k \wedge t, t_{k+1} \wedge t)\, \xi \right\|^2$$

$$= \left\| \sum_{k=1}^n \int_{t_k \wedge t}^{t_{k+1} \wedge t} dx_d F(t_k) \frac{\lambda_d(x_d, \dots, x_1)}{\lambda_{d-1}(x_{d-1}, \dots, x_1)} g_d(x_d)\, \eta_d \right\|^2$$

$$\le \left[\sum_{k=1}^n \int_{t_k \wedge t}^{t_{k+1} \wedge t} dx_d \frac{\lambda_d(x_d, \dots, x_1)}{\lambda_{d-1}(x_{d-1}, \dots, x_1)} \left\| g_d(x_d)\, F(t_k)\, \eta_d \right\| \right]^2$$

By Cauchy-Schwarz inequality and (2.8), the right hand side above is less than or equal to

$$M^2 t \int_0^t \left\| g_d(s) \right\|^2 \left\| F(s)\, \eta_d \right\|^2 ds \le M^2 t \int_0^t \left\| F(s)\, \eta_d \right\|^2 ds$$

Now we turn to prove (3.2).

$$\begin{aligned} &\left\| \int_0^t dA^+(s)\, F(s)\, \xi \right\|^2 \\ &= \sum_{k,h=1}^n \left\langle A^+(t_k \wedge t, t_{k+1} \wedge t)\, F(t_k)\, \xi, A^+(t_h \wedge t, t_{h+1} \wedge t)\, F(t_h)\, \xi \right\rangle \\ &= \sum_{k,h=1}^n \left\langle F(t_k)\, \xi, A(t_k \wedge t, t_{k+1} \wedge t)\, A^+(t_h \wedge t, t_{h+1} \wedge t)\, F(t_h)\, \xi \right\rangle \end{aligned} \tag{3.5}$$

By the adaptness of the process, the quantity above vanishes when $h \neq k$: this implies that only the diagonal elements of the sum above survive, i.e. the quantity in the right hand side of (3.5) is equal to

$$\begin{aligned} &\sum_{k=1}^n \left\langle F(t_k)\, \xi, A(t_k \wedge t, t_{k+1} \wedge t)\, A^+(t_k \wedge t, t_{k+1} \wedge t)\, F(t_k)\, \xi \right\rangle \\ &= \sum_{k=1}^n \sum_{l,r=1}^m \overline{\alpha}_l \alpha_r \left\langle F_l(t_k)\, \xi, A(t_k \wedge t, t_{k+1} \wedge t)\, A^+(t_k \wedge t, t_{k+1} \wedge t)\, F_r(t_k)\, \xi \right\rangle \end{aligned} \tag{3.6}$$

where, for any $r = 1, \dots, m$, $F_r(t_k) = A^{\varepsilon(p)}\left(g_{r_p}\right) \cdots A^{\varepsilon(1)}\left(g_{r_1}\right)$, $p \in \mathbb{N}$, $g_{r_j} \in \mathbf{L}^2(0, t_k)$, $\varepsilon(j) \in \{-1, 1\}$ for any $j = 1, \dots, p$. Let us consider

$$c := \left| \{ j = 1, \dots, p : \varepsilon(j) = 1 \} \right| - \left| \{ j = 1, \dots, p : \varepsilon(j) = -1 \} \right|$$

where, as usual, $|\cdot|$ denotes the cardinality. Notice that $c \ge -d$, otherwise the scalar product vanishes. Therefore, by Cauchy-Schwarz inequality, (3.6) is less than or equal to

$$\sum_{k=1}^{n}\sum_{l,r=1}^{m}\left|\overline{\alpha}_l\alpha_r\right|\left\|F_l\left(t_k\right)\xi\right\|\int_{t_k\wedge t}^{t_{k+1}\wedge t}dx\frac{\lambda_{d+c+1}\left(x,x_c,\ldots,x_1,y_d,\ldots y_1\right)}{\lambda_{d+c}\left(x_c,\ldots,x_1,y_d,\ldots y_1\right)}\left\|F_r\left(t_k\right)\xi\right\|$$

$$\leq M\sum_{k=1}^{n}\int_{t_k\wedge t}^{t_{k+1}\wedge t}\left\|F\left(t_k\right)\xi\right\|^2dx=M\int_0^t\left\|F\left(s\right)\xi\right\|^2ds$$

where the variables $x_1,\ldots,x_c$ appear only when $c>0$. For (3.3) we consider

$$\left\|\int_0^t dA\left(s\right)F\left(s\right)\xi\right\|^2=\left\|\sum_{k=1}^{n}\sum_{r=1}^{m}\alpha_rA\left(t_k\wedge t,t_{k+1}\wedge t\right)F_r\left(t_k\right)\xi\right\|^2 \tag{3.7}$$

For any $r=1,\ldots,m$, let us denote by p the number of operators in $F_r\left(t_k\right)$, hence $F_r\left(t_k\right)=A^{\varepsilon(p)}\left(g_{r_p}\right)\cdots A^{\varepsilon(1)}\left(g_{r_1}\right)$, $p\in\mathbb{N}$, $g_{r_j}\in\mathbf{L}^2\left(0,t_k\right)$, $\varepsilon\left(j\right)\in\{-1,1\}$ for any $j=1,\ldots,p$ and, in the same notations as the previous case, we call $c'=-c$. By the adaptness of the process and the non crossing principle (see [7]), the quantity above can be different from zero only if $0\leq c'<d$, $\varepsilon\left(p\right)=-1$. As a consequence there exist exactly c' annihilators in $F_r\left(t_k\right)$ acting on the vector ξ, whereas the remaining annihilators are coupled with creators belonging to the same $F_r\left(t_k\right)$. Therefore $p=2q+c'$. Let $\{z_1',\ldots,z_{c'}'\}$, $z_1'<\cdots<z_{c'}'$ be the index set in $\{1,\ldots,p\}$ relative to the annihilators acting on ξ and $\{z_1,\ldots,z_{2q}\}$ its complementary. Again by the non crossing principle, for any $j=1,\ldots,c'$, $\varepsilon\left(z_j'\right)=-\varepsilon\left(d-j+1\right)$. Let $\{l_h,r_h\}_{h=1}^q$ be the left-right index set for the non crossing pair partition determined by $\varepsilon\in\{-1,1\}^{2q}$ on the set $\{z_1,\ldots,z_{2q}\}$ (see [7] for details). For any $j=1,\ldots,c'$, $g^{\left(r,z_j'\right)}$ denotes the test function of an arbitrary annihilator in $F_r\left(t_k\right)$ not coupled with any creator therein and for any $h=1,\ldots,q$, $g^{\left(r,l_h\right)},g^{\left(r,r_h\right)}$ are the test functions for the remaining operators in $F_r\left(t_k\right)$. Then

$$\begin{aligned}
&A\left(t_k\wedge t,t_{k+1}\wedge t\right)F_r\left(t_k\right)g_d\otimes\cdots\otimes g_1\left(x_{d-c'-1},\ldots,x_1\right)\\
&=\left[\int_0^t\cdots\int_0^t\int_{t_k\wedge t}^{t_{k+1}\wedge t}dx_{d-c'}\frac{\lambda_{d-c'}\left(x_{d-c'},\ldots,x_1\right)}{\lambda_{d-c'-1}\left(x_{d-c'-1},\ldots,x_1\right)}g_{d-c'}\left(x_{d-c'}\right)\times\right.\\
&\times\prod_{j=1}^{c'}\left(\frac{\lambda_{d-j+1}\left(x_{d-j+1},\ldots,x_1\right)}{\lambda_{d-j}\left(x_{d-j},\ldots,x_1\right)}\left(g^{\left(r,z_j'\right)}g_{d-j+1}\right)\left(x_{d-j+1}\right)dx_{d-j+1}\right)\times\\
&\times\left.\prod_{h=1}^{q}\Lambda\cdot\left(g^{\left(r,l_h\right)}g^{\left(r,r_h\right)}\right)\left(x_{r_h}\right)dx_{r_h}\right]\cdot\left(g_{d-c'-1}\otimes\cdots\otimes g_1\right)\left(x_{d-c'-1},\ldots,x_1\right)
\end{aligned}$$

where Λ is a product of a certain number of fractions of λ_n's, whose explicit form is not necessary for our purposes. Hence

$$\left\|A\left(t_k\wedge t,t_{k+1}\wedge t\right)F_r\left(t_k\right)\xi\right\|\leq M\int_{t_k\wedge t}^{t_{k+1}\wedge t}dx_{d-c'}\left|g_{d-c'}\right|\left(x_{d-c'}\right)\left\|F_r\left(t_k\right)\eta_{d-c'}\right\|$$

Since c' depends on the choice of $F_r(t_k)$, it can takes all values among 0 and $d-1$; consequently

$$\|A(t_k \wedge t, t_{k+1} \wedge t) F(t_k) \xi\| \le M \sum_{h=0}^{d-1} \int_{t_k \wedge t}^{t_{k+1} \wedge t} dx_{d-h} |g_{d-h}| (x_{d-h}) \|F(t_k) \eta_{d-h}\|$$

and

$$\begin{aligned} \left\| \int_0^t dA(s) F(s) \xi \right\|^2 &\le M^2 \sum_{h=0}^{d-1} \left[\int_0^t |g_{d-h}|(s) \|F(s) \eta_{d-h}\| ds \right]^2 \\ &\le M^2 \sum_{h=0}^{d-1} \int_0^t \|F(s) \eta_{d-h}\|^2 ds \end{aligned}$$

where we used the Cauchy-Schwarz inequality, which also directly gives (3.4). □

Remark 3.3. We notice the above result can be obtained also if one weakens (2.8) by the following conditions

$$\frac{\lambda_n(x_n, ..., x_1)}{\lambda_{n+1}(x_0, x_n, ..., x_1)} \le M_n; \frac{\lambda_{n+1}(x_0, x_n, ..., x_1)}{\lambda_n(x_0, ..., \widehat{x_k}, ..., x_1)} \le M_n, \quad \forall n \ge 1, \ \forall\, k = 0, 1, ..., n$$

where (M_n) is a sequence of positive numbers. On the contrary, the proof of a semimartingale estimate for $\int_0^t F(s) dA^+(s) \xi$ requires conditions (2.8).

In order to have a semimartingale estimate for $\int_0^t F(s) dA^+(s) \xi$ one needs some preliminary results.

Let us consider $F \in S$, then there exist $n \in \mathbb{N}$, $0 \le t_1 < t_2 < \ldots < t_{n+1} < +\infty$ such that $F(s) = \sum_{k=1}^n F(t_k) \chi_{[t_k, t_{k+1})}(s)$, $F(t_k) \in A_{t_k]}$. Using the same notations and arguments developed in the proof of (3.2) in Proposition 3.2, we have:

$$\begin{aligned} &\left\| \int_0^t F(s) dA^+(s) \xi \right\|^2 \\ &= \sum_{k=1}^n \left\langle \xi, A(t_k \wedge t, t_{k+1} \wedge t) (F(t_k))^* F(t_k) A^+(t_k \wedge t, t_{k+1} \wedge t) \xi \right\rangle \end{aligned}$$

By definition the right hand side above can be written as

$$\sum_{k=1}^n \sum_{l,r=1}^m \overline{\alpha}_l \alpha_r \left\langle \xi, A(t_k \wedge t, t_{k+1} \wedge t) (F_l(t_k))^* F_r(t_k) A^+(t_k \wedge t, t_{k+1} \wedge t) \xi \right\rangle$$

The adaptness of $F_l(t_k)$ and $F_r(t_k)$ implies that the non zero contributions to the scalar products above can be obtained only if in any element of the sums above, $A(t_k \wedge t, t_{k+1} \wedge t)$ is coupled with $A^+(t_k \wedge t, t_{k+1} \wedge t)$. This, together with the non crossing principle for interacting Fock spaces, gives us some conditions on $(F_l(t_k))^* F_r(t_k)$, where $l, r = 1, \ldots, m$ shown in the following lemmata.

Firstly, for any $l, r = 1, \ldots, m$, we denote

$$A^{\varepsilon_{(l,r)}(p)} \left(g_{p,t_k}^{(l,r)} \right) \cdots A^{\varepsilon_{(l,r)}(1)} \left(g_{1,t_k}^{(l,r)} \right) := (F_l(t_k))^* F_r(t_k) \tag{3.8}$$

where $p \in \mathbb{N}$, $\varepsilon_{(l,r)}(p), \ldots, \varepsilon_{(l,r)}(1) \in \{-1, 1\}$, $g_{j,t_k}^{(l,r)} \in \mathbf{L}^2(0, t_k)$, $j = 1, \ldots, p$.

Lemma 3.4. *In the same notations introduced above*

$$A\left(t_k\wedge t,t_{k+1}\wedge t\right)A^{\varepsilon_{(l,r)}(p)}\left(g^{(l,r)}_{p,t_k}\right)\cdots A^{\varepsilon_{(l,r)}(1)}\left(g^{(l,r)}_{1,t_k}\right)A^{+}\left(t_k\wedge t,t_{k+1}\wedge t\right)\xi$$

is different from zero only if the following conditions are satisfied:

i) $\left|\left\{\varepsilon_{(l,r)}(j)=1:j=1,\dots,p\right\}\right|=\left|\left\{\varepsilon_{(l,r)}(j)=-1:j=1,\dots,p\right\}\right|$;

ii) for any $j=2,\dots,p$

$$\left|\left\{\varepsilon_{(l,r)}(k)=1:1\le k<j\right\}\right|\ge\left|\left\{\varepsilon_{(l,r)}(k)=-1:1\le k<j\right\}\right|$$

hence $\varepsilon_{(l,r)}(1)=1$, $\varepsilon_{(l,r)}(p)=-1$.

Proof. Indeed let us firstly suppose that i) does not hold; for instance we suppose $\left|\left\{\varepsilon_{(l,r)}(j)=1:j=1,\dots,p\right\}\right|<\left|\left\{\varepsilon_{(l,r)}(j)=-1:j=1,\dots,p\right\}\right|$. By the non crossing principle, there exists an annihilator in the sequence $\left(F_l\left(t_k\right)\right)^{*}F_r\left(t_k\right)$ coupled with $A^{+}\left(t_k\wedge t,t_{k+1}\wedge t\right)$, thus giving zero.

The case $\left|\left\{\varepsilon_{(l,r)}(j)=1:j=1,\dots,p\right\}\right|>\left|\left\{\varepsilon_{(l,r)}(j)=-1:j=1,\dots,p\right\}\right|$ is similar.

If instead there exists $j=2,\dots,p$ such that ii) is not verified, then, by the non crossing principle, on the right hand side of $A^{\varepsilon_{(l,r)}(j)}\left(g^{(l,r)}_{j,t_k}\right)$ there exists an annihilator coupled with $A^{+}\left(t_k\wedge t,t_{k+1}\wedge t\right)$ thus giving zero. The last part is trivial. □

The result above ensures that p must be even, i.e. $p=2n$. From now on we introduce the notation $\varepsilon_{(l,r)}=\left(\varepsilon_{(l,r)}(2n),\dots,\varepsilon_{(l,r)}(1)\right)\in\{-1,1\}^{2n}_{+}$ to express that the partition $\varepsilon_{(l,r)}$ realizes conditions i), ii) of the Lemma above. Moreover it is well known (see [7] for details) that $\varepsilon_{(l,r)}\in\{-1,1\}^{2n}_{+}$ induces a unique non crossing pair partition on the set $\{1,\dots,2n\}$, denoted by $\left\{\left(l_{n_{(l,r)}},r_{n_{(l,r)}}\right),\dots,\left(l_1,r_1\right)\right\}$, which can be assumed increasingly ordered with respect to the left indices l_j's. As in [2] we introduce the depth function for a given partition.

Definition 3.5. For any $n\in\mathbb{N}$ and $\varepsilon\in\{-1,1\}^{n}$ the map

$$d_\varepsilon:\{1,\dots,n\}\to\{0,\pm1,\dots,\pm n\}$$

defined as

$$d_\varepsilon(j):=\sum_{k=1}^{j}\varepsilon(k)=\left|\{\varepsilon(k):\varepsilon(k)=1,\ k<j\}\right|-\left|\{\varepsilon(k):\varepsilon(k)=-1,\ k<j\}\right|$$

such that for any $j=1,\dots,n$, is called the depth function of ε.

For any sequence of operators of the type considered above, $d_\varepsilon(j)$ is the number of creators (annihilators if negative) which are on the right hand side of $A^{\varepsilon(j)}$ or, equivalently, the number of pairs containing j in their "interior".

The following definition is given in order to prove a useful result for the last semi-martingale estimate.

Definition 3.6. A non crossing pair partition $\{l_j,r_j\}_{j=1}^{n}$ of $\{1,2,\dots,2n\}$ such that $l_1<l_2<\dots<l_n$ is called connected if for any $k=1,\dots,n-1$ one has $\{l_k,r_k\}\subset\{l_n,r_n\}$. A subset $\{l_i,r_i\}_{i\in I}$, $I\subseteq\{1,\dots,n\}$, of $\{l_j,r_j\}_{j=1}^{n}$ is called a connected component of $\{l_j,r_j\}_{j=1}^{n}$ if it is a connected non crossing pair partition.

A non crossing pair partition $\{l_j, r_j\}_{j=1}^n$ is called interval partition if, for any $j = 1, \dots, n$, $l_j = r_j + 1$.

Lemma 3.7. *Let us suppose that the pair partition $\{l_j, r_j\}_{j=1}^n$ induced by $\varepsilon \in \{-1, 1\}_+^{2n}$ is such that $1 = r_n < \dots < r_1 < l_1 < \dots < l_n = 2n$. Then, for any $g_{l_1}, \dots, g_{l_n}, g_{r_1}, \dots, g_{r_n} \in \mathcal{H}$, any $d \in \mathbb{N}$, $\xi \in \mathcal{H}_d$, $x_d, \dots, x_1 \in \mathbb{R}_+$ one has:*

$$\left[A\left(g_{l_n}\right) \cdots A\left(g_{l_1}\right) A^+\left(g_{r_1}\right) \cdots A^+\left(g_{r_n}\right) \xi\right]\left(x_d, \dots, x_1\right)$$

$$= \left[\int_{\mathbb{R}_+^n} \frac{\lambda_{d+d_\varepsilon(l_1)}\left(y_{l_n}, \dots, y_{l_1}, x_d, \dots, x_1\right)}{\lambda_d\left(x_d, x_{d-1}, \dots, x_1\right)} \prod_{j=1}^n \left(\overline{g}_{l_j} g_{r_j}\right)\left(y_{l_j}\right) dy_{l_j} \xi\right]\left(x_d, \dots, x_1\right)$$

where $y_{l_n}, \dots, y_{l_1} \in \mathbb{R}_+$ and $d_\varepsilon\left(l_1\right) = n$.

Proof. The proof is straightforward by noticing that $d_\varepsilon\left(l_j\right) - 1 = d_\varepsilon\left(l_{j+1}\right)$. $\square$

As a consequence of the lemma above, any sequence of operators indexed by a pair partition $\varepsilon \in \{-1, 1\}_+^{2n}$ such that $d_\varepsilon\left(l_1\right) = n$, once applied to a d-particle vector, give only one fraction of the λ_n's, as in the case of a single pair of operators. The difference consists in the fact here the interacting functions in each fraction are no longer index consecutive. Now we investigate the case in which a sequence of annihilators and creators acting on a certain vector induces a more general connected pair partition.

Lemma 3.8. *Let us given a non-crossing pair partition $\{l_j, r_j\}_{j=1}^n$ such that for any $j = 1, \dots, n-1$, $l_j = r_j + 1$, $l_n = 2n$, $r_n = 1$. Then, for any $g_{l_1}, \dots, g_{l_n}, g_{r_1}, \dots, g_{r_n} \in \mathcal{H}$, any $d \in \mathbb{N}$, $\xi \in \mathcal{H}_d$, $x_d, \dots, x_1 \in \mathbb{R}_+$ one has:*

$$\left[A\left(g_{l_n}\right) A\left(g_{l_{n-1}}\right) A^+\left(g_{r_{n-1}}\right) \cdots A\left(g_{l_1}\right) A^+\left(g_{r_1}\right) A^+\left(g_{r_n}\right) \xi\right]\left(x_d, \dots, x_1\right)$$

$$= \left[\int_{\mathbb{R}_+^n} dy_{l_1} dy_{l_n} \frac{\lambda_{d+2}\left(y_{l_n}, y_{l_1}, x_d, \dots, x_1\right)}{\lambda_d\left(x_d, \dots, x_1\right)} \left(\overline{g}_{l_n} g_{r_n}\right)\left(y_{l_n}\right) \left(\overline{g}_{l_1} g_{r_1}\right)\left(y_{l_1}\right) \times\right.$$

$$\left. \times \left(\prod_{j=2}^{n-1} \frac{\lambda_{d+2}\left(y_{l_n}, y_{l_j}, x_d, \dots, x_1\right)}{\lambda_{d+1}\left(y_{l_n}, x_d, \dots, x_1\right)} \left(\overline{g}_{l_j} g_{r_j}\right)\left(y_{l_j}\right) dy_{l_j}\right) \xi\right]\left(x_d, \dots, x_1\right)$$

where $y_{l_1}, \dots, y_{l_n} \in \mathbb{R}_+$.

Proof. In fact

$$\left[A\left(g_{l_n}\right) \cdots A\left(g_{l_1}\right) A^+\left(g_{r_1}\right) A^+\left(g_{r_n}\right) \xi\right]\left(x_d, \dots, x_1\right)$$

$$= \left[\int_{\mathbb{R}_+} dy_{l_1} \frac{\lambda_{d+2}\left(y_{l_n}, y_{l_1}, x_d, \dots, x_1\right)}{\lambda_{d+1}\left(y_{l_n}, x_d, \dots, x_1\right)} \left(\overline{g}_{l_1} g_{r_1}\right)\left(y_{l_1}\right) \times\right.$$

$$\times \int_{\mathbb{R}_+} dy_{l_2} \frac{\lambda_{d+2}\left(y_{l_n}, y_{l_2}, x_d, \dots, x_1\right)}{\lambda_{d+1}\left(y_{l_n}, x_d, \dots, x_1\right)} \left(\overline{g}_{l_2} g_{r_2}\right)\left(y_{l_2}\right) \times$$

$$\times \cdots \times \int_{\mathbb{R}_+} dy_{l_{n-1}} \frac{\lambda_{d+2}\left(y_{l_n}, y_{l_{n-1}}, x_d, \dots, x_1\right)}{\lambda_{d+1}\left(y_{l_n}, x_d, \dots, x_1\right)} \left(\overline{g}_{l_{n-1}} g_{r_{n-1}}\right)\left(y_{l_{n-1}}\right) \times$$

$$\left. \times \int_{\mathbb{R}_+} dy_{l_n} \frac{\lambda_{d+1}\left(y_{l_n}, x_d, \dots, x_1\right)}{\lambda_d\left(x_d, \dots, x_1\right)} \left(\overline{g}_{l_n} g_{r_n}\right)\left(y_{l_n}\right) \xi\right]\left(x_d, \dots, x_1\right)$$

and the thesis follows. □

From now on we will speak of "fractions of the λ_n's" referred to fractions which can not be further simplified (i.e. they are irreducible); we speak of "product of fractions of the λ_n's" referred to a product which can not be further simplified: as a consequence, we can enumerate how many factors there are in a certain product of fractions of the λ_n's.

Hence, given a connected pair partition of creation-annihilation operators acting on a vector ξ, satisfying the assumptions of the lemma above, the number of fractions of the λ_n's is exactly given by the number of the index consecutive pairs. This, together with Lemma 3.7, suggests us to generalize such a result for an arbitrary sequence of annihilators and creators inducing a connected pair partition.

Proposition 3.9. *Let us given a non crossing pair partition* $\{l_j, r_j\}_{j=1}^n$ *such that* $l_n = 2n$, $r_n = 1$ *and denote*

$$k := \left|\left\{\{l_h, r_h\} \subseteq \{l_j, r_j\}_{j=1}^n : r_h = l_h - 1,\ h = 1, \ldots, n\right\}\right|$$

Then, after computing

$$A^{\varepsilon(l_n)}\left(g_{l_n}\right) \cdots A\left(g_{l_j}\right) \cdots A^+\left(g_{r_j}\right) \cdots A^{\varepsilon(r_n)}\left(g_{r_n}\right) \xi$$

there appear exactly a product of k *fractions of the* λ_n*'s.*

Proof. The thesis can be obtained by iteration. Let us fix the first pair of consecutive left-right indices from the right in the sequence, say $\{l_{k_1}, r_{k_1}\}$. If $l_{k_1}+1$ is a right index or $r_{k_1}-1$ is a left index, we turn to the successive index consecutive pair. On the contrary, if $l_{k_1}+1$ is a left index and $r_{k_1}-1$ is a right index, by the non crossing principle, on the right hand side of $A^+\left(g_{r_{k_1}-1}\right)$ there appear only creation operators. By Lemma 3.7, the action of $A\left(g_{l_{k_1}+1}\right) A\left(g_{l_{k_1}}\right) A^+\left(g_{r_{k_1}}\right) A^+\left(g_{r_{k_1}-1}\right)$ on the $d+j$ particle vector on the right hand side ($j = 1, \ldots, n-1$), give rise to a unique fraction of the λ_n's. After we repeat the same arguments for all the pairs of consecutive left-right indices, finally obtaining the same type of partition described in Lemma 3.8. □

Let us take $F_l(t_k), F_r(t_k) \in \mathcal{B}_{t_k}$ and introduce the following notation:

$$\left(F_l(t_k)\right)^* F_r(t_k) = A^{\varepsilon(2n)}\left(g_{2n}\right) \cdots A^{\varepsilon(1)}\left(g_1\right), \varepsilon \in \{-1, 1\}_+^{2n} \tag{3.9}$$

and denote

$$\overline{N}_{t_k}^{(l,r)} := \left|\left\{\{l_j, r_j\},\ j = 1, \ldots, n : r_j = l_j - 1\right\}\right|$$

i.e. $\overline{N}_{t_k}^{(l,r)}$ is the number k introduced in Proposition 3.9.

Lemma 3.10. *For any* $d \in \mathbb{N}$ *and* $\xi = g_d \otimes \ldots \otimes g_1 \in \mathfrak{D}$

$$\begin{aligned} &\left|\left\langle \xi, A\left(t_k \wedge t, t_{k+1} \wedge t\right) \left(F_l(t_k)\right)^* F_r(t_k) A^+\left(t_k \wedge t, t_{k+1} \wedge t\right) \xi \right\rangle\right| \\ &\leq M^{2\overline{N}_{t_k}^{(l,r)}-1} \left\langle \xi, \int_{t_k \wedge t}^{t_{k+1} \wedge t} dy \left\| \left(F_l(t_k)\right)^* F_r(t_k) \right\| \xi \right\rangle \end{aligned} \tag{3.10}$$

Proof. Firstly we notice the sequence

$$A\left(t_k\wedge t,t_{k+1}\wedge t\right)\left(F_l\left(t_k\right)\right)^*F_r\left(t_k\right)A^+\left(t_k\wedge t,t_{k+1}\wedge t\right)$$

determines a connected pair partition with $\overline{N}_{t_k}^{(l,r)}$ index consecutive pairs. Then, by Proposition 3.9, its action on ξ gives exactly $\overline{N}_{t_k}^{(l,r)}$ fractions of the λ_n's. The proof of (3.10) is given by induction on $\overline{N}_{t_k}^{(l,r)}$. In fact, ε being defined in (3.9), take $\varepsilon'\in\{-1,1\}^{2(n+1)}$ such that

$$\begin{cases}\varepsilon'(1)=1\\ \varepsilon'(j)=\varepsilon(j-1)\,,\ j=2,\dots,n+1\\ \varepsilon'(2n+2)=-1\end{cases}\tag{3.11}$$

Let us suppose $\overline{N}_{t_k}^{(l,r)}=1$. If $\left\{l'_j,r'_j\right\}_{j=1}^{n+1}$ is the non-crossing pair partition determined by ε', then $r'_{n+1}<r'_n<\ldots<r'_1<l'_1<\ldots<l'_n<l'_{n+1}$ with l'_{n+1},r'_{n+1} the indices relative respectively to $A\left(t_k\wedge t,t_{k+1}\wedge t\right)$ and $A^+\left(t_k\wedge t,t_{k+1}\wedge t\right)$. By Lemmata 3.7 and 3.1 one has

$$\begin{aligned}
&\left|\left\langle\xi,A\left(t_k\wedge t,t_{k+1}\wedge t\right)\left(F_l\left(t_k\right)\right)^*F_r\left(t_k\right)A^+\left(t_k\wedge t,t_{k+1}\wedge t\right)\xi\right\rangle\right|\\
&=\left|\left\langle\xi,\int_{t_k\wedge t}^{t_{k+1}\wedge t}dy_{l'_{n+1}}\int_{\mathbb{R}^n}\frac{\lambda_{d+d_{\varepsilon'(l'_1)}}\left(y_{l'_1},\dots,y_{l'_{n+1}},x_d,\dots,x_1\right)}{\lambda_d\left(x_d,\dots,x_1\right)}\right.\right.\\
&\left.\left.\times\left[\prod_{j=1}^n\left(\overline{g}_{l'_j}g_{r'_j}\right)\left(y_{l'_j}\right)dy_{l'_j}\right]\xi\right\rangle\right|\\
&\le M\left\langle\xi,\int_{t_k\wedge t}^{t_{k+1}\wedge t}dy_{l'_{n+1}}\int_{\mathbb{R}^n}\frac{\lambda_{d+d_{\varepsilon'(l'_1)}-1}\left(y_{l'_1},\dots,\widehat{y_{l'_{n+1}}},x_d,\dots,x_1\right)}{\lambda_d\left(x_d,\dots,x_1\right)}\times\right.\\
&\left.\times\left[\prod_{j=1}^n\left|\left(\overline{g}_{l'_j}g_{r'_j}\right)\right|\left(y_{l'_j}\right)dy_{l'_j}\right]\xi\right\rangle
\end{aligned}$$

Since $d_{\varepsilon'\left(l'_j\right)}=d_{\varepsilon(l_j)}+1$ and $l'_j=l_j$ for any $j=1,\dots,n$, the quantity above is equal to

$$\begin{aligned}
&M\left\langle\xi,\int_{t_k\wedge t}^{t_{k+1}\wedge t}dy_{l'_{n+1}}\int_{\mathbb{R}^n}\frac{\lambda_{d+d_{\varepsilon(l_1)}}\left(y_{l_1},\dots,y_{l_n},x_d,\dots,x_1\right)}{\lambda_d\left(x_d,\dots,x_1\right)}\times\right.\\
&\left.\times\left[\prod_{j=1}^n\left|\left(\overline{g}_{l_j}g_{r_j}\right)\right|\left(y_{l_j}\right)dy_{l_j}\right]\xi\right\rangle\\
&=M\left\langle\xi,\int_{t_k\wedge t}^{t_{k+1}\wedge t}dy\left\|\left(F_l\left(t_k\right)\right)^*F_r\left(t_k\right)\right\|\xi\right\rangle
\end{aligned}$$

where the last equality is achieved by Lemma 3.7 again. Let us suppose the result holds for any $\overline{N}_{t_k}^{(l,r)}\le N$ and prove it for $\overline{N}_{t_k}^{(l,r)}=N+1$. With ε' defined as in (3.11) and $\left\{l'_j,r'_j\right\}_{j=1}^{n+1}$ the left-right index set uniquely determined by ε', since

$l_1' < \cdots < l_{n+1}'$, we have l_1' is the index relative to the first annihilator moving from the right hand side. By the non-crossing arguments, it is easy to see that $\{l_1', r_1'\}$ is the first index consecutive pair from the right hand side. Let $\left\{\overline{l}_h', \overline{r}_h'\right\}_{h=1}^{d_{\varepsilon'}(l_1')}$ be the subset of $\left\{l_j', r_j'\right\}_{j=1}^{n+1}$ in which all the right indices are on the right hand side of l_1' and $\overline{r}'_{d_{\varepsilon'(l_1')}} = r_1'$. If y is the variable relative to the operator $A(t_k \wedge t, t_{k+1} \wedge t)$, then

$$\begin{aligned}
&\left|\left\langle \xi, A(t_k \wedge t, t_{k+1} \wedge t)(F_l(t_k))^* F_r(t_k) A^+(t_k \wedge t, t_{k+1} \wedge t)\xi\right\rangle\right| \\
&= \left|\left\langle \xi, A(t_k \wedge t, t_{k+1} \wedge t) \cdots A\left(g_{l_1'}\right) A^+\left(g_{r_1'}\right) \cdots A^+(t_k \wedge t, t_{k+1} \wedge t)\xi\right\rangle\right| \\
&= |\langle \xi, A(t_k \wedge t, t_{k+1} \wedge t) A(g_{2n}) \cdots \times \\
&\times \left[\int_0^t \frac{\lambda_{d+d_{\varepsilon'(l_1')}+1}(y_{\overline{l}_1'}, \ldots, y_{\overline{l}'_{d_{\varepsilon'(l_1')}-1}}, y_{l_1'}, y, x_d, \ldots, x_1)}{\lambda_{d+d_{\varepsilon'(l_1')}}(y_{\overline{l}_1'}, \ldots, y_{\overline{l}'_{d_{\varepsilon'(l_1')}-1}}, \widehat{y_{l_1'}}, y, x_d, \ldots, x_1)} \left(\overline{g}_{l_1'} g_{r_1'}\right)(y_{l_1'})\, dy_{l_1'}\right] \times \\
&\times \cdots A^{\varepsilon(1)}(g_1) A^+(t_k \wedge t, t_{k+1} \wedge t)\xi\right\rangle\Big|
\end{aligned}$$

Moreover, by Lemma 3.1, we obtain the quantity above is less than or equal to

$$\begin{gathered}
M^2 \left\langle \xi, A(t_k \wedge t, t_{k+1} \wedge t) A^{\varepsilon(2n)}(g_{2n}) \cdots \times \right. \\
\times \left[\int_0^t \frac{\lambda_{d+d_{\varepsilon'(l_1')}}(y_{\overline{l}_1'}, \ldots, y_{\overline{l}'_{d_{\varepsilon'(l_1')}-1}}, y_{l_1'}, \widehat{y}, x_d, \ldots, x_1)}{\lambda_{d+d_{\varepsilon'(l_1')}-1}(y_{\overline{l}_1'}, \ldots, y_{\overline{l}'_{d_{\varepsilon'(l_1')}-1}}, \widehat{y_{l_1'}}, \widehat{y}, x_d, \ldots, x_1)} \left|\left(\overline{g}_{l_1'} g_{r_1'}\right)\right|(y_{l_1'})\, dy_{l_1'}\right] \times \\
\left. \times \cdots A^{\varepsilon(1)}(g_1) A^+(t_k \wedge t, t_{k+1} \wedge t)\xi\right\rangle
\end{gathered}$$

Now in the sequence of operators on the right hand side of the scalar product above, we have exactly N index consecutive pairs. The induction hypothesis gives us the quantity is less than or equal to

$$M^2 M^{2N-1} \left\langle \xi, \int_{t_k \wedge t}^{t_{k+1} \wedge t} ds \left\|F_l(t_k)^* F_r(t_k)\right\| \xi\right\rangle$$

and the thesis follows. ☐

Before proving the last semi-martingale inequality, we introduce the following useful notation:

$$\overline{N}_{t_k} := \max_{1 \le l, r \le m} \overline{N}_{t_k}^{(l,r)}, \quad \overline{N} := \max_{k=1,\ldots,n} \overline{N}_{t_k}$$

Proposition 3.11. *Using the same notations as above, one has*

$$\left\|\int_0^t F(s)\, dA^+(s)\xi\right\|^2 \le M^{2\overline{N}-1} \int_0^t \|F(s)\xi\|^2\, ds \tag{3.12}$$

Proof. In fact, using the adaptness arguments, the left hand side of (3.12) is equal to

$$\sum_{k=1}^n \sum_{l,r=1}^m \left|\left\langle \xi, \overline{\alpha}_l \alpha_r A(t_k)(F_l(t_k))^* F_r(t_k) A^+(t_k)\xi\right\rangle\right|$$

Using Lemma 3.10, one finds this is less than or equal to

$$\sum_{k=1}^{n} M^{2\overline{N}_{t_k}-1} \left| \left\langle \xi, \int_{t_k \wedge t}^{t_{k+1}\wedge t} \|F(t_k)\|^2 \xi \right\rangle \right| \leq M^{2\overline{N}-1} \int_0^t \|F(s)\xi\|^2 ds$$

where we used the Cauchy-Schwarz inequality. □

Let us consider the 1-mode type IFS case, i.e. the case (2.7).

Fixed $t \in [0, +\infty)$, any $G \in \mathcal{B}_t$ is represented by a sequence of creation and annihilation operators we need to know the number of. For example, if $G = A^{\varepsilon(n)}(g_n) \cdots A^{\varepsilon(1)}(g_1)$, where $n \in \mathbb{N}$, $g_1, \ldots, g_n \in \mathbf{L}^2(0,t)$, $\varepsilon \in \{-1,1\}^n$, this number is equal to n. Since the representation of G is not unique, such a number runs over a set, whose minimum we call the *order* of G and denote by $ordG$. We recall that any $F \in \mathcal{S}$ can be written as

$$F = \sum_{k=1}^{n} F(t_k)\chi_{[t_k,t_{k+1})},\ F(t_k) = \sum_{h=1}^{m} \alpha_h F_h(t_k),\ F_h(t_k) \in \mathcal{B}_{t_k}$$

By definition $ordF_h(t_k) < +\infty$ for any t_k, then we define

$$N_{t_k}^{(F)} := \max_{1\leq h\leq m} ordF_h(t_k), \quad N^{(F)} := \max_{1\leq k\leq n} N_{t_k}^{(F)}$$

For any $d \in \mathbb{N}$ let us take

$$\omega_{N_{t_k}^{(F)}+d+1} := \max_{1\leq h\leq m} \omega_{ordF_h(t_k)+d+1}$$
$$\omega_{N^{(F)}+d+1} := \max_{1\leq k\leq n} \omega_{N_{t_k}^{(F)}+d+1}$$

where, as usual, for any $n \in \mathbb{N}$, $\omega_n := \frac{\lambda_n}{\lambda_{n-1}}$ and $\omega_0 := 1$. Moreover we put

$$F_+(t_k) := \sum_{h=0}^{H} c\left(t_k, f^{(h,k)}\right) A^+\left(f_h^{(h,k)}\right) \cdots A^+\left(f_1^{(h,k)}\right)$$
$$F_-(t_k) := \sum_{h=0}^{H} c\left(t_k, f^{(h,k)}\right) A\left(f_h^{(h,k)}\right) \cdots A\left(f_1^{(h,k)}\right)$$

$f^{(h,k)} := \left(f_h^{(h,k)}, \ldots f_1^{(h,k)}\right) \in \mathbf{L}^2\left((0,t), \mathbb{C}^h\right)$. If for any $d \geq 1$

$$M_{\overline{d}} := \max\{M_1, \ldots, M_d\}$$

we find the following semimartingale estimates for left and right stochastic integrals of simple adapted processes.

Proposition 3.12. *Under the same notations of Proposition 3.2, for 1-mode type IFS one has:*

$$\left\| \int_0^t F(s)\, dA(s)\, \xi \right\|^2 \leq M_d^2 \int_0^t \|F(s)\eta_d\|^2 ds \tag{3.13}$$

$$\left\| \int_0^t dA(s)\, F(s)\, \xi \right\|^2 \leq M_{\overline{d}}^2 \sum_{h=0}^{d-1} \int_0^t \|F_-(s)\eta_{d-h}\|^2 ds \tag{3.14}$$

$$\left\|\int_0^t dA^+(s)\,F(s)\,\xi\right\|^2 \le M_{N(F)+d+1}\int_0^t \|F(s)\,\xi\|^2\,ds \tag{3.15}$$

$$\left\|\int_0^t F(s)\,dA^+(s)\,\xi\right\|^2 \le M_{N(F)+d+1}\int_0^t \|F_+(s)\,\xi\|^2\,ds \tag{3.16}$$

Proof. In fact, from (2.7)

$$\begin{aligned}\left\|\int_0^t F(s)\,dA(s)\,\xi\right\|^2 &= \left\|\sum_{k=1}^n F(t_k)\,A(t_k\wedge t, t_{k+1}\wedge t)\,\xi\right\|^2\\ &= \left\|\sum_{k=1}^n \omega_d\int_{t_k\wedge t}^{t_{k+1}\wedge t} dx_d g_d(x_d)\,F(t_k)\,\eta_d\right\|^2\\ &\le M_d^2\left\|\int_0^t ds g_d(s)\,F(s)\,\eta_d\right\|^2\end{aligned}$$

and (3.13) follows from the Cauchy-Schwarz inequality. Let us prove (3.14).

$$\left\|\int_0^t dA(s)\,F(s)\,\xi\right\|^2 = \left\|\sum_{k=1}^n A(t_k\wedge t, t_{k+1}\wedge t)\,F_-(t_k)\,\xi\right\|^2 \tag{3.17}$$

where the equality above follows from (2.11) and the adaptness of $F(t_k)$. If

$$F_-^{(h)}(r) := \sum_{k=1}^n c\left(t_k, f^{(h,k)}\right)A\left(f_h^{(h,k)}\right)\cdots A\left(f_1^{(h,k)}\right)\chi_{[t_k,t_{k+1})}(r)$$

for any $k = 1,\dots,n$

$$\begin{aligned}&A(t_k\wedge t, t_{k+1}\wedge t)\,F_-(t_k)\,g_d\otimes\cdots\otimes g_1\\ &= \sum_{h=0}^{H\wedge(d-1)} c\left(t_k, f^{(h,k)}\right)\left(\prod_{j=1}^h \omega_{d-j+1}\left\langle f_j^{(h,k)}, g_{d-j+1}\right\rangle\right)\times\\ &\times\left(\int_{t_k\wedge t}^{t_{k+1}\wedge t}\omega_{d-h}g_{d-h}(r)\,dr\right)g_{d-h-1}\otimes\ldots\otimes g_1\end{aligned}$$

Then

$$\begin{aligned}&\sum_{k=1}^n A(t_k\wedge t, t_{k+1}\wedge t)\,F_-(t_k)\,\xi\\ &= \sum_{h=0}^{H\wedge(d-1)}\omega_{d-h}\int_0^t g_{d-h}(r)\,F_-^{(h)}(r)\,\eta_{d-h}dr\end{aligned}$$

As a consequence of orthogonality of $F_{-}^{(h)}(r)\,\eta_{d-h}$ and $F_{-}^{(h')}(r)\,\eta_{d-h'}$ when $h \neq h'$, the right hand side of (3.17) is equal to:

$$\sum_{h=0}^{H\wedge(d-1)} \left\| \omega_{d-h} \int_0^t g_{d-h}(r)\, F_{-}^{(h)}(r)\,\eta_{d-h} dr \right\|^2$$
$$\leq M_d^2 \sum_{h=0}^{d-1} \int_0^t \left\| F_{-}(r)\,\eta_{d-h} \right\|^2 dr$$

where in the last estimate we used the Cauchy-Schwarz inequality. For (3.15), by means of the usual adaptness arguments

$$\left\| \int_0^t dA^{+}(s)\, F(s)\, \xi \right\|^2$$
$$\leq \sum_{k=1}^{n} \left| \left\langle F(t_k)\,\xi, A(t_k\wedge t, t_{k+1}\wedge t)\, A^{+}(t_k\wedge t, t_{k+1}\wedge t)\, F(t_k)\,\xi \right\rangle \right|$$
$$= \sum_{k=1}^{n} \sum_{l,r=1}^{m} \left|\overline{\alpha}_l \alpha_r\right| \cdot \left| \left\langle F_l(t_k)\,\xi, A(t_k\wedge t, t_{k+1}\wedge t)\, A^{+}(t_k\wedge t, t_{k+1}\wedge t)\, F_r(t_k)\,\xi \right\rangle \right|$$
$$= \sum_{k=1}^{n} \sum_{l,r=1}^{m} \omega_{ordF_r(t_k)+d+1} \left|\overline{\alpha}_l \alpha_r\right| \cdot \left| \left\langle F_l(t_k)\,\xi, \int_{t_k\wedge t}^{t_{k+1}\wedge t} dr F_r(t_k)\,\xi \right\rangle \right|$$
$$\leq \sum_{k=1}^{n} \omega_{N_{t_k}^{(F)}+d+1} \int_{t_k\wedge t}^{t_{k+1}\wedge t} dr \left\| F(t_k)\,\xi \right\|^2 \leq \omega_{N^{(F)}+d+1} \int_0^t \left\| F(s)\,\xi \right\|^2 ds$$

Hence (3.15) follows. Finally, for (3.16), as a consequence of the adaptness of $F(t_k)$, we have

$$\left\| \int_0^t F(s)\, dA^{+}(s)\,\xi \right\|^2 = \left\| \sum_{k=1}^{n} F_{+}(t_k)\, A^{+}(t_k\wedge t, t_{k+1}\wedge t)\,\xi \right\|^2$$

which is equal to

$$\sum_{k=1}^{n} \left\langle \xi, A(t_k\wedge t, t_{k+1}\wedge t)\, F_{-}(t_k)\, F_{+}(t_k)\, A^{+}(t_k\wedge t, t_{k+1}\wedge t)\,\xi \right\rangle$$

Again by adaptness we have that for any $k = 1, \ldots, n$ the non zero contributions to the scalar product above can be obtained when $A(t_k\wedge t, t_{k+1}\wedge t)$ acts on $A^{+}(t_k\wedge t, t_{k+1}\wedge t)$. It can be checked the quantity above is less than or equal to

$$\sum_{k=1}^{n} \frac{\lambda_{N_{t_k}^{(F)}+d+1}}{\lambda_d} \cdot \frac{\lambda_d}{\lambda_{N_{t_k}^{(F)}+d}} \left| \left\langle \xi, \int_{t_k\wedge t}^{t_{k+1}\wedge t} dr F_{-}(t_k)\, F_{+}(t_k)\,\xi \right\rangle \right|$$
$$\leq \sum_{k=1}^{n} M_{N_{t_k}^{(F)}+d+1} \left| \left\langle \xi, \int_{t_k\wedge t}^{t_{k+1}\wedge t} dr F_{-}(t_k)\, F_{+}(t_k)\,\xi \right\rangle \right|$$
$$\leq M_{N^{(F)}+d+1} \int_0^t \left\| F_{+}(s)\,\xi \right\|^2 ds$$

where we used the Cauchy-Scwharz inequality. □

4. Stochastic Integral

In this section we extend the definition of stochastic integral to the vector space of processes that can be approximated by sequences of elements of S. We will follow the methods of [3] and [14] in order to set a definition of a stochastic integral satisfying our semimartingale inequalities.

Let us take $\xi = u_d \otimes \ldots \otimes u_1 \in \mathfrak{D}$ and the set $J(\xi) \subset \mathfrak{D}$ whose elements are Φ and $u_{\sigma(h)} \otimes \ldots \otimes u_{\sigma(1)}$, $h \in \{1, ..., d\}$, $\sigma : \{1, ..., h\} \to \{1, ..., d\}$ increasing. As in [3] we want to establish a $\tau-$ semimartingale inequality with respect to a topology τ induced by a family of semi–norms. We recall that in [3] the topology τ is induced by the seminorms

$$\|F\|^2_{\xi,t,\mu} := \int_0^t \|F(s)\xi\|^2 \, d\mu(s),$$

where F is a simple adapted process, $\xi \in \mathfrak{D}$, $t \in \mathbb{R}_+$ are arbitrarily chosen.

In our case, for any $F \in S$, $\xi \in \mathfrak{D}$, $t \in \mathbb{R}_+$ and for any $N \geq 1$, the topology τ is determined by the seminorms

$$q_{\xi,t,\overline{N}}(F) := \max\left\{M^2 t, M^{2\overline{N}-1}, t\right\} \times \\ \times \sum_{\eta \in J(\xi)} \left\{ \left(\int_0^t \|F(s)\eta\|^2 \, ds \right)^{\frac{1}{2}} + \left(\int_0^t \|F^*(s)\eta\|^2 \, ds \right)^{\frac{1}{2}} \right\} \tag{4.1}$$

or

$$q_{\xi,t,N^{(F)}}(F) := \max\left\{M_{\frac{2}{d}}, M_{N^{(F)}+d+1}, t\right\} \times \\ \times \sum_{\eta \in J(\xi)} \sum_{\varepsilon \in \{-1,0,1\}} \left\{ \left(\int_0^t \|F_\varepsilon(s)\eta\|^2 \, ds \right)^{\frac{1}{2}} + \left(\int_0^t \|F^*_\varepsilon(s)\eta\|^2 \, ds \right)^{\frac{1}{2}} \right\} \tag{4.2}$$

according to whether we consider the general case, with non constant interacting functions, or the 1-mode type IFS, and $F_{-1} = F_-$, $F_0 = F$, $F_1 = F_+$. From now on we will consider only the general case, as the 1-mode type IFS can be obtained just replacing (4.1) by (4.2). Denote by $\alpha\beta$ an arbitrary element of the set $\{01, 10\}$. As a consequence of Proposition 3.2 and Proposition 3.11, the maps

$$F \in S \mapsto \int_0^t F(s) \, dM^{\alpha\beta}(s) \in \mathfrak{L}(\mathfrak{D}, \mathfrak{F}_I)$$

$$F \in S \mapsto \int_0^t dM^{\alpha\beta}(s) \, F(s) \in \mathfrak{L}(\mathfrak{D}, \mathfrak{F}_I)$$

are continuous with respect to the topology on S induced by semi-norms (4.1) and the topology of strong $*$-convergence on $\mathfrak{D}$. Hence, denoting these topologies by τ and τ' respectively, we say that the basic processes are $(\tau - \tau')$ -semimartingales, according to [3], Definition 2.1.

Let $\overline{S}$ be the vector space of processes F such that there exists a sequence $\left(F^{(n)}\right)_{n \geq 0}$ in S for which the following property holds:

$(*)$ for any $t \in \mathbb{R}_+$ $\left(F^{(n)}\right)_{n\geq 0}$ converges to $F(t)$ in the topology τ' of strong $*$-convergence on $\mathfrak{D}$.

The following definition gives us the class of integrable processes.

Definition 4.1. A process $F \in \overline{S}$ is said to be integrable if there exists a sequence $\left(F^{(n)}\right)_{n\geq 0}$ in S satisfying the condition $(*)$ and such that for any $\xi \in \mathfrak{D}$, $t \in \mathbb{R}_+$

$$\lim_{n\to+\infty} q_{\xi,t,\overline{N}}\left(F^{(n)} - F\right) = 0. \tag{4.3}$$

We denote by $\mathfrak{I}$ the class of all integrable processes and notice that for any $t \in \mathbb{R}_+$, any $F \in \mathfrak{I}$, and any $(F^n)_{n\geq 0}$ satisfying Definition 4.1, the sequences of stochastic integrals:

$$\left(\int_0^t F^{(n)}(s)\, dM^{\alpha\beta}(s)\right)_{n\geq 0}$$

$$\left(\int_0^t dM^{\alpha\beta}(s)\, F^{(n)}(s)\right)_{n\geq 0}$$

are convergent in the topology of strong $*$-convergence on $\mathfrak{D}$ as a consequence of Propositions 3.2 and 3.11. Hence we can define the left and right stochastic integrals of elements of $\mathfrak{I}$ with respect to the basic processes in the following way

$$\int_0^t F(s)\, dM^{\alpha\beta}(s)\,\xi := \lim_{n\to+\infty} \int_0^t F^{(n)}(s)\, dM^{\alpha\beta}(s)\,\xi$$

$$\int_0^t dM^{\alpha\beta}(s)\, F(s)\,\xi := \lim_{n\to+\infty} \int_0^t dM^{\alpha\beta}(s)\, F^{(n)}(s)\,\xi,$$

for any $\xi \in \mathfrak{D}$.

Remark 4.2. In the construction of quantum stochastic integrals in both the cases (general and 1-mode type), we used the locally convex topology induced by the family of seminorms (4.1) or (4.2) and the topology of strong $*$-convergence on $\mathfrak{L}(\mathfrak{D}, \mathfrak{F}_I)$. Nevertheless one can notice that it is possible to define a class of integrable processes and their quantum stochastic integrals by using only the strong $*$-convergence and the limit in the induced topology. The emergence of the other topology, together with condition (4.3), reveals in order that such integrals have some nice properties, allowing, for example, to give existence and uniqueness of the solution for a wide class of quantum stochastic differential equations, as we will se in [13].

The following result contains the semimartingale inequalities for the stochastic integrals of elements of $\mathfrak{I}$ and will be useful in [13].

Proposition 4.3. *For any $F \in \mathfrak{I}$, any $\xi \in \mathfrak{D}$, the following estimates hold:*

$$\left\|\int_0^t F(s)\, dM^{\alpha\beta}(s)\,\xi\right\| \leq q_{\xi,t,\overline{N}}(F), \quad \left\|\int_0^t dM^{\alpha\beta}(s)\, F(s)\,\xi\right\| \leq q_{\xi,t,\overline{N}}(F) \tag{4.4}$$

Proof. The thesis follows from Definition 4.1, Proposition 3.2 and Proposition 3.11. □

Acknowledgments. The author expresses his deep gratitude towards profs. Luigi Accardi, Franco Fagnola, Yun Gang Lu and Michael Skeide for useful comments and discussions. Moreover he thanks an anonymous referee for pointing out some unclear passages in the original manuscript.

References

1. Accardi, L. and Bożejko, M.: Interacting Fock Space and Gaussianization of probability measures*; Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1** no. 4 (1998) 663-670.
2. Accardi, L., Crismale, and V., Lu, Y. G.: Constructive universal central limit theorems based on interacting Fock spaces; *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **8** no. 4 (2005) 631-650.
3. Accardi, L., Fagnola, F., and Quaegebeur, J.: A representation free quantum stochastic calculus; *J. Funct. Anal.* **104** (1992) 149-197.
4. Accardi, L., Kuo, H. H., and Stan, A.: Characterization of probability measures through the canonically associated interacting Fock spaces*; Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **7** no. 3 (2004) 485-505.
5. Accardi, L., Kuo, H. H., and Stan, A.: Probability measures in terms of creation, annihilation and neutral operators*;* in *Quantum Probability and Infinite Dimensional Analysis*, *QP-PQ* **XVIII** (2005), World Scientific, Singapore.
6. Accardi, L. and Lu, Y. G.: The Wigner semi-cirle law in quantum electrodynamics; *Comm. Math. Phys.* **180** no. 3 (1996) 605-632.
7. Accardi, L., Lu, Y. G., and Volovich, I.: *The QED Hilbert module and interacting Fock spaces.* International Institute for Advances Studies Publications, Kyoto, 1997.
8. Accardi, L., Lu, Y. G., and Volovich, I.: *Quantum theory and its stochastic limit.* Springer, Berlin, 2002.
9. Accardi, L. and Nahni, M.: Interacting Fock Space and orthogonal Polynomials in several variables*; Preprint* (2002).
10. Applebaum, D. B. and Hudson, R. L.: Fermion Ito's formula and stochastic evolutions*; Comm. Math. Phys.* **96** (1984) 473-496.
11. Barnett, C., Streater, R. F., and Wilde, I. F.: The Ito-Clifford integral; *J. Funct. Anal.* **48** (1982) 172-212.
12. Ben Ghorbal, A. and Schürmann, M.: Quantum stochastic calculus on Boolean Fock space; *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **7** no. 4 (2004) 631–650.
13. Crismale, V.: Quantum stochastic calculus on interacting Fock spaces: stochastic differential equations and Ito formula; Preprint.
14. Fagnola, F.: On quantum stochastic integration with respect to "free" noises; in *Quantum Probability and Rel. Top., QP-PQ* **VI** (1991) 285-304, Worl Scientific, Singapore.
15. Hudson, R. L. and Lindsay, J. M.: A noncommutative martingale representation theorem for non-Fock Brownian Motion*; J. Funct. Anal.* **61** (1985) 202-221.
16. Hudson, R. L. and Parthasarathy, K. R.: Quantum Ito's formula and stochastic evolutions*; Comm. Math. Phys.* **93** (1984) 301-323.
17. Kümmerer, B. and Speicher, R.: Stochastic integration on the Cuntz algebra O_∞; *J. Funct. Anal.* **103** (1992) 372-408.
18. Lindsay, J. M. and Wilde, I. F.: On non-Fock Boson Stochastic integrals; *J. Funct. Anal.* **65** (1986) 76-82.
19. Lu, Y. G.: On the interacting free Fock space and the deformed Wigner law*; Nagoya Math. J.* **145** (1997) 1-28.
20. Lu, Y. G.: An interacting Free Fock space and the arcsine law; *Probab. Math. Statist.* **17** (1997) 149-166.
21. Lu, Y. G.: Interacting Fock spaces related to the Anderson model; *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1** no. 2 (1998) 247-283.
22. Parthasarathy, K. R.: *An introduction to Quantum Stochastic Calculus.* Birkhäuser, Basel, 1992.

23. Skeide, M.: Quantum stochastic calculus on full Fock modules; *J. Funct. Anal.* **173** (2000) 401-452.
24. Speicher, R.: Stochastic integration on the full Fock space with the help of a kernel calculus; *Publ. Res. Inst. Math. Sci.* **27** no. 1 (1991) 149-184.

Dipartimento di Matematica, Università degli studi di Bari, Via E. Orabona, 4 - I-70125 Bari, Italy
E-mail address: crismalev@dm.uniba.it