Communications on Stochastic Analysis
Vol. 1, No. 3 (2007) 343-355

A RANDOM CHANGE OF VARIABLES AND APPLICATIONS TO THE STOCHASTIC POROUS MEDIUM EQUATION WITH MULTIPLICATIVE TIME NOISE

S. V. LOTOTSKY

ABSTRACT. A change of variables is introduced to reduce certain nonlinear stochastic evolution equations with multiplicative noise to the corresponding deterministic equation. The result is then used to investigate a stochastic porous medium equation.

1. Introduction

Let $U = U(t, x)$ be a solution of the porous medium equation

$$U_t = \mathbf{\Delta} U^m, \quad t > 0,\, x \in \mathbb{R}^d,\, m > 1, \tag{1.1}$$

and let $X = X(t)$ be a semi-martingale. Consider a stochastic version of (1.1),

$$du = \mathbf{\Delta} u^m dt + u\, dX(t); \tag{1.2}$$

the use of multiplicative noise preserves the positivity of the solution. The first main result of the paper is as follows:

Theorem 1.1. *There is a one-to-one correspondence between the solutions of the deterministic and stochastic porous medium equations, given by*

$$u(t, x) = h(t)U(H(t), x), \tag{1.3}$$

where

$$h(t) = 1 + \int_0^t h(s)dX(s),\ H(t) = \int_0^t h^{m-1}(s)ds.$$

Thus, many of the results known for the deterministic equation (1.1), as summarized, for example, in Aronson [1], have a clear counterpart in the stochastic case, and the objective of the paper is to the study this correspondence. To keep the presentation from becoming unnecessarily abstract, a particular semi-martingale X is considered: $X(t) = \int_0^t g(s)ds + \int_0^t f(s)dw(s)$, where w is a standard Brownian motion. The results derived in this manner can serve as a benchmark for further investigation of the stochastic porous medium equation, in particular, driven by space-time noise.

Here is a typical consequence of combining (1.3) with the known facts about the deterministic porous medium equation.

2000 *Mathematics Subject Classification.* Primary 60H15; Secondary 35R60, 76S05.
Key words and phrases. Barenblatt's solution, closed-form solutions, long-time asymptotic, random time change.
The author acknowledges support from the NSF CAREER award DMS-0237724.

Theorem 1.2. *Let $u_0(x)$ be a non-negative continuous function with compact support in $\mathbb{R}^d$. Then there exists a unique continuous non-negative random field $u = u(t,x)$ such that, for every smooth compactly supported function φ, and every $t > 0$, the equality*

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\,dx + \int_0^t \int_{\mathbb{R}^d} u^m(t,x)\boldsymbol{\Delta}\varphi(x)\,dx + \int_0^t \int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dw(t)$$

holds with probability one (in other words, u is the solution of $du = \boldsymbol{\Delta} u^m dt + u dw(t)$.) In addition,

(1) *The function u is Hölder continuous in t,x on every set of the form $[T,+\infty) \times \mathbb{R}^d$, $T > 0$;*
(2) *The mean mass is preserved:*
$$\mathbb{E}\int_{\mathbb{R}^d} u(t,x)dx = \int_{\mathbb{R}^d} u(0,x)dx;$$
(3) *The support of u is uniformly bounded: there exists a random variable η such that, with probability one, $0 < \eta < \infty$ and $u(t,x) = 0$ for $|x| > \eta$ and all $t > 0$.*
(4) *For every $x \in \mathbb{R}$, $\lim_{t\to\infty} u(t,x) = 0$ with probability one.*

Section 2 presents the general result about the change of variables (1.3) for a large class of nonlinear equations. Section 3 discusses the basic questions about existence, uniqueness, and regularity of the solution for the stochastic porous medium equation, and Section 4 investigates the long-time behavior of the solution. Because of the random time change, the long-time behaviors in the deterministic and stochastic cases can be different.

The following are some comments about the physical origin and relevance of equation (1.2), as well as connections with recent work on the subject. For a variety of distributed system, there are two functions that describe the state of the system at time t and point $x \in \mathbb{R}^d$; by the analogy with the classical problem of the gas flow, it is natural to call these functions the `density` $\rho(t,x)$ and the `velocity` $\boldsymbol{v}(t,x)$. A nearly universal `equation of continuity` states that

$$\kappa\rho_t + \mathrm{div}(\rho\,\boldsymbol{v}) = \sigma, \tag{1.4}$$

where $\sigma = \sigma(t,x,\rho)$ is the density of sources and sinks, and κ is the fraction of the space available to the system. Moreover, the underlying physical principles dictate the equation of motion:

$$\boldsymbol{v} = \boldsymbol{F}(t,x,\rho,\mathrm{grad}\rho) \tag{1.5}$$

for the known vector-valued function $\boldsymbol{F}$. An example is `Darcy's Law` for the ideal gas flow,

$$\boldsymbol{v} = -\frac{\nu}{\mu}\,\mathrm{grad}\,p, \tag{1.6}$$

where $p = p_0\rho^\alpha$, $\alpha > 0$, is the pressure, μ is the permeability of the medium, and ν is the viscosity of the gas.

Assume that $\boldsymbol{F}(t,x,\rho,\operatorname{grad}\rho) = -q(\rho)\operatorname{grad}\rho - \boldsymbol{a}\psi(\rho)$ for some known functions q and ψ and a constant vector $\boldsymbol{a}$. Substituting (1.5) into (1.4), we then get

$$\kappa\,\rho_t = \boldsymbol{\Delta}\Phi(\rho) + \boldsymbol{a}\cdot \operatorname{grad}\Psi(\rho) + \sigma(t,x,\rho), \tag{1.7}$$

where $\Phi(x) = \int_0^x yq(y)dy$ and $\Psi(x) = x\psi(x)$. Equation (1.7) appears in a variety of problems, including population dynamics and gas and water propagation. The underlying physical setting usually requires the solution ρ of (1.7) to be non-negative for all $t > 0$ and $x \in \mathbb{R}$.

The following two particular cases of (1.7) are worth mentioning:

(1) Darcy's Law for the ideal gas flow (1.6) with constant μ, ν, κ and with $\sigma = 0$ results in equation (1.1) with $m = 1+\alpha$ for the normalized (dimensionless) and suitably scaled density u of the gas. The main difference between (1.1) and the heat equation ($m = 1$) is that initial disturbance is propagated by (1.1) with finite speed.

(2) The same relation (1.6) with constant μ, ν and with $\kappa = const$, $\sigma(t,x,\rho) = \sigma_0\,\rho$, $\sigma_0 = const.$ leads to a basic model for the spread of a crowd-avoiding population:

$$u_t = \boldsymbol{\Delta}u^m + \sigma_0\,u, \tag{1.8}$$

where u is a suitably normalized and scaled density of the population; see Gurtin and MacCamy [9, Section 2]. The number σ_0 is the reproduction intensity of the population; the population is birth-dominant if $\sigma_0 > 0$.

Comparing (1.2) *and* (1.8) *we conclude that* (1.2) *can describe the dynamics of the crowd-avoiding population with random time-dependent reproduction intensity;* if $X(t) = \int_0^t g(s)ds + \int_0^t f(s)dw(s)$, then

$$\sigma_0 = g(t) + f(t)\dot{w}(t), \tag{1.9}$$

where $\dot{w}$ is Gaussian white noise.

In general, allowing the function σ in (1.7) to be random is a natural way to introduce randomness in the porous medium equation. If the positivity of the solution is not guaranteed with these values of σ, the term $\boldsymbol{\Delta}u^m$ in the equation is replaced by $\boldsymbol{\Delta}(|u|^{m-1}u)$. Similar to (1.2), the randomness can be in time only, such as $\sigma(t,x,u) = u^\beta + u\dot{w}$ in Mel′nik [13, 12]. The randomness can also be in time and space, such as $\sigma(t,x,u) = \sum_k f_k(t,x)\dot{w}_k(t)$ in Kim [10], $\sigma(t,x,u) = F(u) + \dot{W}_Q(t,x)$ in Barbu et al. [2] and in Da Prato [5], or $\sigma(t,x,u) = F(u)+G(u)\dot{W}_Q(t,x)$ in Barbu et al. [3], with $\dot{W}_Q$ representing Gaussian noise that is white in time but is sufficiently regular in space. Note that, unlike (1.2), none of the above models can benefit from Theorem 1.1.

The derivation of (1.7) shows that another way to introduce randomness in the equation is to allow velocity $\boldsymbol{v}$ to be random, for example, by considering a stochastic differential equation satisfied by $\boldsymbol{v}$. Existence of a weak (in the probabilistic sense) solution of the resulting porous medium equation were recently obtained by Sango [16].

2. Nonlinear Equations with Multiplicative Noise

If $v = v(t,x)$, $t > 0$, $x \in \mathbb{R}^d$ satisfies the heat equation $v_t = \boldsymbol{\Delta}v$, and c is a real number, then the function $u(t,x) = v(t,x)e^{ct}$ satisfies $u_t = \boldsymbol{\Delta}u + cu$. Similarly, if

$w = w(t)$ is a standard Brownian motion, then

$$u(t,x) = v(t,x)e^{w(t)-(t/2)}$$

satisfies the stochastic Itô equation $du = \mathbf{\Delta} u dt + u dw(t)$. The objective of this section is to extend these results to some nonlinear equations.

Consider the equation

$$v_t = F(v, Dv, D^2 v, \ldots), \quad t > 0,\ x \in \mathbb{R}^d, \tag{2.1}$$

with some initial condition. In (2.1), $v = v(t,x)$ is the unknown function, F is a given function, $v_t = \partial v/\partial t$ and $D^k v$ denotes a generic k-th order partial derivative of v with respect to x.

We also consider the stochastic counterpart of (2.1) for the unknown random field $u = u(t,x)$, $t > 0$, $x \in \mathbb{R}^d$:

$$du = F(u, Du, D^2 u, \ldots)dt + u\,(f(t)dw(t) + g(t)dt), \tag{2.2}$$

with the same initial condition as in (2.1). In (2.2), $f = f(t)$ is a locally square-integrable deterministic function, g is a locally integrable deterministic function, w is a standard Brownian motion on a stochastic basis $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and the equation is in the Itô sense. We assume that $\mathbb{F}$ satisfies the usual conditions of completeness of $\mathcal{F}$ and right continuity of $\mathcal{F}_t$.

Definition 2.1. Given a stopping time τ, a `classical solution` of equation (2.2) on the set $(0,\tau]\!] = \{(t,\omega) : t \leq \tau\}$ is a random field $u = u(t,x)$ with the following properties:

1. u is continuous in (t,x) for all $(t,\omega) \in (0,\tau]\!]$ and $x \in \mathbb{R}^d$;
2. all the necessary partial derivatives of u with respect to x exist and are continuous in (t,x) for all $(t,\omega) \in (0,\tau]\!]$ and $x \in \mathbb{R}^d$;
3. The equality

$$u(t,x) = u(0,x) + \int_0^t F(u(s,x), Du(s,x), \ldots)ds + \int_0^t u(s,x)(f(s)dw(s) + g(s)ds)$$

for all $(t,\omega) \in (0,\tau]\!]$ and $x \in \mathbb{R}^d$.

Taking in the above definition $f = g = 0$, we get the definition of the classical solution of equation (2.1). It turns out that if the function F is homogeneous, then there is a one-to-one correspondence between the classical solutions of (2.1) and (2.2). The key component in this correspondence is a random time change.

Definition 2.2. (a) We say that the function F is `homogeneous of degree` $\gamma \geq 1$ if, for every $\lambda > 0$,

$$F(\lambda x, \lambda y, \lambda z, \ldots) = \lambda^\gamma F(x, y, z, \ldots). \tag{2.3}$$

(b) We say that equation (2.1) is `homogeneous of degree` $\gamma \geq 1$ if the function F is homogeneous of degree γ.

Proposition 2.3. *Assume that the function F is homogeneous of degree γ. Define the functions*

$$h(t) = \exp\left(\int_0^t g(s)ds + \int_0^t f(s)dw(s) - \frac{1}{2}\int_0^t f^2(s)ds\right),\ H_\gamma(t) = \int_0^t h^{\gamma-1}(s)ds. \tag{2.4}$$

Then a function $v = v(t,x)$ is a classical solution of (2.1) *if and only if*

$$u(t,x) = v(H_\gamma(t),x)h(t) \tag{2.5}$$

is a classical solution of (2.2).

Proof. Assume that v is a classical solution of (2.1). Note that $H_\gamma'(t) = h^{\gamma-1}(t)$ and

$$dh(t) = h(t)(f(t)dw(t) + g(t)dt). \tag{2.6}$$

By the Itô formula,

$$du(t,x) = v_t(H_\gamma(t),x)h^\gamma(t)dt + u(t,x)(f(t)dw(t) + g(t)dt).$$

Using (2.1) and homogeneity of F,

$$\begin{aligned} v_t(H_\gamma(t),x)h^\gamma(t) &= F\Big(h(t)v(H_\gamma(t),x), h(t)Dv(H_\gamma(t),x), h(t)D^2v(H_\gamma(t),x),\dots\Big) \\ &= F(u, Du, D^2u, \dots), \end{aligned}$$

and therefore u is a classical solution of (2.2).

Conversely, assume that u is a classical solution of (2.2). Since $h(t) > 0$ for all t, ω, the function H_γ is strictly increasing and has an inverse function R_γ. Define $v(t,x) = z(R_\gamma(t))u(R_\gamma(t),x)$, where $z(t) = 1/h(t)$. Note that

$$dz(t) = z(t)\Big(-g(t)dt - f(t)dw(t) + f^2(t)dt\Big).$$

Then, by the Itô formula, we conclude that v is a classical solution of (2.1). □

As a simple illustration of Proposition 2.3 consider Burger's equation $v_t = vv_x,\ t > 0$ with initial condition $v(0,x) = -x$. This equation is homogeneous of degree 2 and has a classical solution $v(t,x) = -x/(1+t)$. Then the stochastic equation $du = uu_x dt + udw(t)$ with the same initial condition has a solution

$$u(t,x) = -x\,\frac{e^{w(t)-(t/2)}}{1 + \int_0^t e^{w(s)-(s/2)}ds}.$$

Proposition 2.3 can be generalized in several directions:

(1) The functions f, g can be random and adapted. In fact, the process $\int_0^t g(s)ds + \int_0^t f(s)dw(s)$ can be replaced with a semi-martingale $X(t)$, possibly with jumps. Then $h(t)$ becomes the stochastic (Dolean) exponential of X; see Liptser and Shiryaev [11, Section 2.4].

(2) Other types of stochastic integral and other types of random perturbation can be considered, as long as the corresponding analog of equation (2.6) can be solved and an analogue of the Itô formula applied. For example, consider the fractional Brownian motion B^H and take $f = 1$, $g = 0$. With a suitable interpretation of the integral udB^H we get $h(t) = e^{B^H(t)-(1/2)t^{2H}}$; see, for example, Nualart [14, Chapter 5].

(3) Since transformation (2.5) does not involve the space variable x, Proposition 2.3 also works for initial-boundary value problems.
(4) Transformation (2.5) can establish a one-to-one correspondence between generalized (and even viscosity) solutions of (2.1) and (2.2), but the precise definition of the solution and the corresponding arguments in the proof require more information about the function F.

Let us emphasize that transformation (2.5) does not lead to a closed-form solution of the stochastic equation (2.2) unless there is a closed-form solution of the deterministic equation (2.1). Some methods of finding closed-form solutions of nonlinear equations of the type (2.1) are described in [6].

3. Stochastic Porous Medium Equation

Recall that the classical porous medium equation is

$$U_t(t,x) = \mathbf{\Delta}(U^m(t,x)), \quad t > 0, \tag{3.1}$$

where $U_t = \partial U/\partial t$ and $\mathbf{\Delta}$ is the Laplace operator. This equation can model various physical phenomena for every $m > 0$; in what follows, we will consider only $m > 1$ ($m = 1$ is the heat equation). We also assume that $x \in \mathbb{R}^d$ so that there are no boundary conditions. Note that, without special restrictions on m, the definition of the solution of (3.1) must include a certain non-negativity condition on U; this condition is also consistent with the physical interpretation of the solution as a density of some matter.

The `scaled pressure` $V = V(t,x)$ corresponding to the porous medium equation (3.1) is defined by

$$V(t,x) = \frac{m}{m-1} U^{m-1}(t,x) \tag{3.2}$$

and satisfies

$$V_t = (m-1)V\mathbf{\Delta}V + |\nabla V|^2, \tag{3.3}$$

where ∇ is the gradient. The function V is extensively used in the study of the analytic properties of (3.1).

Let $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis with the usual assumptions, $w = w(t)$, a standard Brownian motion on $\mathbb{F}$, and $\tau > 0$, a stopping time. Let $f = f(t)$ and $g = g(t)$ be non-random functions such that f is locally square-integrable and g is locally integrable.

Consider the following equation:

$$du(t,x) = \mathbf{\Delta}(u^m(t,x))dt + u(t,x)(f(t)dw(t) + g(t)dt), \quad t > 0,\ x \in \mathbb{R}^d. \tag{3.4}$$

Definition 3.1. A non-negative, continuous random field $u = u(t,x)$ is called a `solution` of equation (3.4) on the set $(0,\tau]\!] = \{(t\,\omega) : t \leq \tau\}$ if, for every smooth compactly supported function $\varphi = \varphi(x)$ the following equality holds for all $(t,\omega) \in (0,\tau]\!]$:

$$(u,\varphi)(t) = (u,\varphi)(0) + \int_0^t (u^m, \mathbf{\Delta}\varphi)(s)ds + \int_0^t (u,\varphi)(s)(f(s)dw(s) + g(s)ds), \tag{3.5}$$

where

$$(u, \varphi)(t) = \int_{\mathbb{R}^d} u(t,x)\varphi(x)dx.$$

Note that, with $f(t) = g(t) = 0$, this definition also applies to the deterministic equation (3.1).

Define the functions

$$\begin{aligned} h(t) &= \exp\left(\int_0^t g(s)ds + \int_0^t f(s)dw(s) - \frac{1}{2}\int_0^t f^2(s)ds\right), \\ H(t) &= \int_0^t h^{m-1}(s)ds. \end{aligned} \tag{3.6}$$

Proposition 3.2. *There is a one-to-one correspondence between the solutions of* (3.1) *and* (3.4) *given by*

$$u(t,x) = U(H(t),x)h(t). \tag{3.7}$$

Proof. Note that the porous medium equation (3.1) is homogeneous of degree m. Then (3.7) is strongly suggested by Proposition 2.3. Since the solutions in question are not necessarily classical, the formal argument involves application of the Itô formula in the integral relation (3.5), but is completely analogous to the proof of Proposition 2.3. □

Two immediate consequences of (3.7) are the comparison principle and maximum principle for equation (3.4); both follow from the corresponding results for the deterministic equation (3.1), see the book by Vázquez [17, Theorem 9.2].

Corollary 3.3. *(a)* `Comparison principle`: *If u, $\widetilde{u}$ are two solutions of* (3.4) *and $u(0,x) \le \widetilde{u}(0,x)$ for all $x \in \mathbb{R}^d$, then $u(t,x) \le \widetilde{u}(t,x)$ for all $(t,\omega) \in (0,\tau]\!]$, $x \in \mathbb{R}^d$.*

(b) `Maximum principle`: *If u is a solution of* (3.4) *and $0 \le u(0,x) \le M$ for all $x \in \mathbb{R}^d$, then $0 \le u(t,x) \le Mh(t)$ for all $(t,\omega) \in (0,\tau]\!]$, $x \in \mathbb{R}^d$.*

Remark 3.4. 1. In the particular case $f = 0$, $g = const.$ the relation (3.7) was discovered by Gurtin and MacCamy [9].

2. While the current paper deals only with non-negative solutions of the porous medium equation, relation (3.7) also holds for all the solutions of $U_t = \boldsymbol{\Delta}(|U|^{m-1}U)$ and $du = \boldsymbol{\Delta}(|u|^{m-1}u)dt + udw(t)$.

It is important to have the definition of the solution of (3.4) on a random time interval, because even a classical solution of (3.1) can blow up in finite time, and then the corresponding solution of (3.4) will blow up in random time. The `quadratic pressure` solution provides an example. By direct computation, the function $U^{[\mathrm{qp}]}$ defined by

$$U^{[\mathrm{qp}]}(t,x) = \left(\frac{t_1|x|^2}{t_q - t}\right)^{1/(m-1)}, \; t_q = \frac{m-1}{2mq(2+d(m-1))}, \quad q > 0;\; t_1 = t_q|_{q=1}, \tag{3.8}$$

is a classical solution of (3.1) for $(t,x) \in (0,t_q) \times \mathbb{R}^d$; $U^{[\mathrm{qp}]}$ is known as the `quadratic pressure solution` because the corresponding pressure is

$$V^{[\mathrm{qp}]}(t,x) = \frac{mt_1|x|^2}{(m-1)(t-t_q)};$$

for details, see Aronson [1, pages 3–5] or Vázquez [17, Section 4.5]. By Proposition 2.3,

$$u^{[\mathrm{qp}]}(t,x) = U^{[\mathrm{qp}]}(H(t),x)h(t)$$

is a classical solution of (3.4) on $(0,\tau_q]\!]$, where

$$\tau_q = \inf(t>0 : H(t) = t_q)$$

is the blow-up time; it is certainly possible to have $\tau_q < \infty$ with positive probability. On the other hand, if $f=0$, then, with a suitable choice of g, the life of every non-global solution of (3.1) can be extended indefinitely: note that $g(t) = -\alpha < 0$ corresponds to $H(t) = (1-e^{-\alpha t})/\alpha < 1/\alpha$, and it remains to take $\alpha > 0$ sufficiently large.

The following is the main result about global existence, uniqueness, and regularity of the solution of (3.4).

Theorem 3.5. *Assume that the initial condition $u(0,x)$ is non-random and has the following properties:*

(1) *non-negative and bounded:* $0 \le u(0,x) \le C$, $x \in \mathbb{R}^d$;
(2) *continuous;*
(3) *integrable:* $\int_{\mathbb{R}^d} u(0,x)dx = M$, $0 < M < \infty$;
(4) *square-integrable:* $\int_{\mathbb{R}^d} u^2(0,x)dx < \infty$.

Then there exists a unique non-negative solution $u = u(t,x)$ of (3.4). This solution is defined for all $t>0$, $x \in \mathbb{R}^d$ and has the following properties:

(1) *the mean total mass satisfies* $\mathbb{E}\int_{\mathbb{R}^d} u(t,x)dx = Me^{\int_0^t g(s)ds}$, $t>0$;
(2) $u(t,\cdot)$ *is Hölder continuous (as a function of x) for every $t>0$;*
(3) *if* $\left(\int_{\mathbb{R}^d} u^p(0,x)dx\right)^{1/p} = M_p < \infty$, *then*

$$\left(\mathbb{E}\int_{\mathbb{R}^d} u^p(t,x)dx\right)^{1/p} \le M_p \exp\left(\int_0^t g(s)ds + \frac{(p-1)}{2}\int_0^t f^2(s)ds\right). \tag{3.9}$$

In addition, if the functions f,g are locally bounded, then the function u is Hölder continuous on $[T,+\infty) \times \mathbb{R}^d$ for every $T>0$.

Proof. For the deterministic equation, Sabinina [15] proved existence, uniqueness, and conservation of mass (see also the book by Vázquez [17, Chapter 9]; Caffarelli and Friedman [4] proved Hölder continuity. It is also known [17, Theorem 9.3] that

$$\int_{\mathbb{R}^d} U^p(t,x)dx \le \int_{\mathbb{R}^d} U^p(0,x)dx, \quad t>0,\ p>1,$$

where U is the solution of (3.1). It remains to use Proposition 3.2 and note that

$$\mathbb{E}h^p(t) = \exp\left(p\int_0^t g(s)ds + \frac{p(p-1)}{2}\int_0^t f^2(s)ds\right), \quad t>0.$$

Also, if the functions f, g are locally bounded, then the function h is Hölder continuous of any order less than 1/2. □

Note that the initial condition of the quadratic pressure solution, $U^{[\mathrm{qp}]}(0,x) = q^{1/(m-1)}|x|^{2/(m-1)}$, is not bounded. Theorem 3.5 shows that a blow-up of the solution can be avoided with suitable growth restrictions on the initial condition. These restrictions are sufficient but not necessary: consider, for example the `linear pressure solution`

$$u^{[\mathrm{lp}]}(t,x) = \left(\frac{m-1}{m}\max\big(H(t)+x,\,0\big)\right)^{1/(m-1)} h(t), \quad t>0,\ x\in\mathbb{R}.$$

By analogy with (3.2), define the `scaled pressure` corresponding to equation (3.4) by

$$v(t,x) = \frac{m}{m-1}u^{m-1}(t,x). \tag{3.10}$$

An application of the Itô formula shows that v satisfies

$$\begin{aligned} dv = &\left((m-1)v\boldsymbol{\Delta} v + |\nabla v|^2 + \frac{(m-1)(m-2)}{2}vf^2\right)dt \\ &+ (m-1)v\,(f dw + g dt). \end{aligned} \tag{3.11}$$

On the other hand, equation (3.3) is homogeneous of degree 2, and so Proposition 2.3 suggests an alternative definition:

$$v(t,x) = V(\widetilde{H}(t),x)\widetilde{h}(t), \tag{3.12}$$

where

$$\begin{aligned} \widetilde{h}(t) = \exp&\left((m-1)\int_0^t f(s)dw(s) - \frac{(m-1)^2}{2}\int_0^t f^2(s)ds\right. \\ &\left.+\ \frac{(m-1)(m-2)}{2}\int_0^t f^2(s)ds + (m-1)\int_0^t g(s)ds\right) \end{aligned}$$

and $\widetilde{H}(t) = \int_0^t \widetilde{h}(s)ds$. An observation that $\Big((m-1)(m-2)-(m-1)^2\Big)/2 = -(m-1)/2$ shows that, in fact, $\widetilde{H}(t) = H(t)$ and

$$V(\widetilde{H}(t),x)\widetilde{h}(t) = \frac{m}{m-1}U^{m-1}(H(t),x)h^{m-1}(t).$$

In other words, (3.10) and (3.12) define the same function. Another way to phrase this conclusion is to say that, for the porous medium equation (3.1), the change of variables defined by Proposition 2.3 commutes with the transformation $U \mapsto (m/(m-1))U^{m-1}$.

4. Barenblatt's solution and long-time behavior

Barenblatt's solution of $U_t = \boldsymbol{\Delta}(U^m)$ is

$$U^{[\mathrm{BT}]}(t,x;b) = \frac{1}{t^\alpha}\left(\max\left(b - \frac{m-1}{2m}\,\beta\,\frac{|x|^2}{t^{2\beta}},\ 0\right)\right)^{1/(m-1)}, \quad t>0,\ x\in\mathbb{R}^d, \tag{4.1}$$

where $b > 0$ and

$$\beta = \frac{1}{(m-1)d+2},\ \ \alpha = \beta d.$$

For the derivation of $U^{[\mathrm{BT}]}$ see Aronson [1, pages 3–4], Evans [7, Section 4.2.2], or Vázquez [17, Section 4.4.2]. The function has $U^{[\mathrm{BT}]}$ the following properties:

(1) the `total mass` of the solution, $\int_{\mathbb{R}^d} U^{[\mathrm{BT}]}(t,x;b)dx$, does not depend on t and is uniquely determined by b: by direct computation,

$$\int_{\mathbb{R}^d} U^{[\mathrm{BT}]}(t,x;b)dx = b^{1/(2\beta(m-1))}\left(\frac{m-1}{2\pi\, m}\,\beta\right)^{-d/2}\frac{\Gamma\left(\frac{m}{m-1}\right)}{\Gamma\left(\frac{m}{m-1}+\frac{d}{2}\right)}, \tag{4.2}$$

where Γ is the Gamma function (see also Aronson [1, Pages 3–4] and Vázquez [17, Section 17.5]);

(2) $U^{[\mathrm{BT}]}$ is a classical solution of $U_t = \mathbf{\Delta}(U^m)$ in the region

$$\left\{(t,x) : |x| \neq \sqrt{\frac{2mb}{(m-1)\beta}}\ t^{\beta}\right\}; \tag{4.3}$$

(3) For every $p, q, t_0 > 0$, $x_0 \in \mathbb{R}^d$, the function

$$\tilde{U}(t,x) = \left(\frac{p}{q^2}\right)^{1/(m-1)} U^{[\mathrm{BT}]}(pt+t_0, qx+x_0; b) \tag{4.4}$$

is also a solution of (3.1).

By Proposition 2.3, the function

$$u^{[\mathrm{BT}]}(t,x;b) = U^{[\mathrm{BT}]}(H(t),x;b)h(t), \tag{4.5}$$

with H, h given by (3.6), is a solution of the stochastic porous medium equation (3.4). In particular,

$$\mathbb{E}\int_{\mathbb{R}^d} u^{[\mathrm{BT}]}(t,x;b)dx = \left(\int_{\mathbb{R}^d} U^{[\mathrm{BT}]}(t,x;b)dx\right) e^{\int_0^t g(s)ds}.$$

Barenblatt's solution $U^{[\mathrm{BT}]}$ determines the long-time behavior of *every* global solution of the deterministic equation (3.1). Similarly, with obvious restrictions on H, $u^{[\mathrm{BT}]}$ determines the long-time behavior of the solutions of the stochastic porous medium equation from Theorem 3.5.

Theorem 4.1. *Assume that $\lim_{t\to\infty} H(t) = +\infty$ with probability one. Let $u = u(t,x)$ be the solution of the stochastic porous medium equation constructed under the assumptions of Theorem 3.5. Then, for every $x \in \mathbb{R}^d$,*

$$\lim_{t\to\infty} (H(t))^{\beta d}|u(t,x) - u^{[\mathrm{BT}]}(t,x;b)| = 0 \tag{4.6}$$

with probability one, where b is such that

$$\int_{\mathbb{R}^d} u(0,x)dx = b^{1/(2\beta(m-1))}\left(\frac{m-1}{2\pi\, m}\,\beta\right)^{-d/2}\frac{\Gamma\left(\frac{m}{m-1}\right)}{\Gamma\left(\frac{m}{m-1}+\frac{d}{2}\right)}. \tag{4.7}$$

Proof. This follows from Proposition 2.3 and the corresponding result for the deterministic equation,

$$\lim_{t\to\infty} t^{\beta d}|U(t,x) - U^{[\mathrm{BT}]}(t,x;b)| = 0$$

(see Friedman and Kamin [8]). □

Remark 4.2. Clearly, if $\lim_{t\to\infty} H(t)$ is finite, then the long-time behavior of the solutions of (3.4) will be quite different and, in particular, not as universal. By definition, the function H is non-decreasing and therefore has a limit (finite or infinite) with probability one. If $\int_0^\infty f^2(s)ds < \infty$, then $H(t) \uparrow +\infty$ as long as $\liminf_{t\to\infty} \int_0^t g(s)ds > -\infty$. If $\int_0^\infty f^2(t)dt = +\infty$, then, by the law of iterated logarithm, $\lim_{t\to\infty} H(t)$ can be finite with probability one, and more information about $f(t)$ and $g(t)$ is necessary to proceed (see the example below).

Further information about the asymptotic behavior of the solution can be obtained under additional assumptions about the functions f, g.

Theorem 4.3. *Assume that* $\int_0^\infty f^2(t)dt = 2\sigma^2$ *and* $\int_0^t g(t)dt = \mu$ *for some* $\sigma, \mu \in \mathbb{R}$. *Then, for every* $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} |u(t,x) - e^{\xi}\, U^{[\mathrm{BT}]}(e^{(m-1)\xi}t, x; b)| = 0 \tag{4.8}$$

with probability one, where b *satisfies* (4.7) *and* ξ *is a Gaussian random variable with mean* $\mu - \sigma^2$ *and variance* σ^2.

Proof. Under the assumptions of the theorem, we have, with probability one,

$$\lim_{t\to\infty}\left(\int_0^t g(s)ds + \int_0^t f(s)dw(s) - \frac{1}{2}\int_0^t f^2(s)ds\right) = \xi. \tag{4.9}$$

Then $\lim_{t\to\infty} h^{m-1}(t) = e^{(m-1)\xi}$ and therefore

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t h^{(m-1)}(s)ds = e^{(m-1)\xi}.$$

The result follows by continuity of the function $U^{[\mathrm{BT}]}$. Note that no rate can be specified in (4.8) without assumptions about the rate of convergence in (4.9). □

Remark 4.4. It follows that equation (3.4) DOES NOT have a nontrivial invariant measure in any traditional function space. Indeed, by (4.8), if $\lim_{t\to\infty} h(t)$ exists, then, for every $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} u(t,x) = \lim_{t\to\infty} U^{[\mathrm{BT}]}(t,x;b) = 0,$$

that is, either the solution decays uniformly or the mass spreads out to infinity. On the other hand, if $\lim_{t\to\infty} h(t)$ does not exist, then, by (3.7), no non-trivial limit of $u(t,x)$ can exist either. The same remark applies to the equation in a bounded domain.

Other information about certain global solutions of the stochastic porous medium equation can be obtained by comparison with Barenblatt's solution.

Theorem 4.5. *Assume that the initial condition $u(0,x)$ of* (3.4) *is continuous, non-negative, and compactly supported in $\mathbb{R}^d$. Then, with probability one,*

(1) *the solution $u(t,x)$ is non-negative and has compact support in $\mathbb{R}^d$ for all $t>0$;*
(2) *the* `interface`*, that is, the boundary of the set $\{x\in\mathbb{R}^d : u(t,x)>0\}$, is moving with finite speed.*
(3) *if $\lim_{t\to\infty} H(t)<\infty$, then the support of the solution remains bounded for all $t>0$.*

Proof. By the maximum principle (Corollary 3.3(b)), if $u(0,x)\geq 0$, then so is $u(t,x)$. Furthermore, if $u(0,x)=0$ for $|x|>R$ and $u(0,x)\leq C$, then, for sufficiently large C_1,

$$u(0,x)\leq \max(C_1-|x|^2,0).$$

By comparison principle (Corollary 3.3(a)) we then get $u(t,x)\leq \tilde{U}(H(t),x)h(t)$, where $\tilde{U}(t,x)$ is a function of the type (4.4) with $p=q=t_0=1$, $x_0=0$, and b sufficiently large. Therefore, $u(t,x)=0$ for $|x|>C_2H^{\beta}(t)$ for a suitable (non-random) number C_2. □

Example. Consider the equation

$$du=\mathbf{\Delta}u^2\,dt+u\,dw(t) \tag{4.10}$$

and assume that $u(0,x)$ is continuous, non-negative, compactly supported, and $\int_{\mathbb{R}^d}u(0,x)dx>0$. Then there exists a random variable η such that $0<\eta<\infty$ with probability one and $u(t,x)=0$ for all $|x|>\eta$ and all $t>0$. Indeed, in this case

$$h(t)=e^{w(t)-(t/2)},\ H(t)=\int_0^t e^{w(s)-(s/2)}ds,$$

and, by the previous theorem, it is enough to show that $H(t)$ is bounded with probability one. Let T_0 be the last time $w(t)$ exceeds $t/4$: $T_0=\sup\{t>0 : w(t)>t/4\}$. Then

$$H(t)<\int_0^{T_0} e^{w(t)-(t/2)}dt+4e^{T_0/4},\quad t>0.$$

By the law of iterated logarithm, $T_0<\infty$ with probability one, and therefore $\lim_{t\to\infty}H(t)<\infty$ with probability one.

Notice also that $\lim_{t\to\infty}h(t)=0$ with probability one, and consequently, for every $x\in\mathbb{R}^d$,

$$\lim_{t\to\infty}u(t,x)=\lim_{t\to\infty}h(t)U(H(t),x)=0,$$

because the solution $U=U(t,x)$ of the deterministic equation (3.1) with initial condition $U(0,x)=u(0,x)$ is a uniformly bounded function. On the other hand, we know that $\int_{\mathbb{R}^d}U(t,x)dx=\int_{\mathbb{R}^d}u(0,x)dx$, and, since $\mathbb{E}h(t)=1$ for all $t>0$, we conclude that $\mathbb{E}\int_{\mathbb{R}^d}u(t,x)dx=\int_{\mathbb{R}^d}u(0,x)dx$. In other words, the solution of the stochastic porous medium equation (4.10) is supported in the same (random) compact set for all $t>0$ and decays to zero as $t\to\infty$, while preserving the mean total mass.

References

1. Aronson, D. G.: The porous medium equation, in: *Nonlinear diffusion problems*, (1986), 1–46, vol. 1224 of *Lecture Notes in Math.* Springer, Berlin,
2. Barbu, V., Bogachev, V. I., Da Prato, G., and Röckner, M.: Weak solutions to the stochastic porous media equation via Kolmogorov equations: the degenerate case. *J. Funct. Anal.* **237**:1 (2006), 54–75.
3. Barbu, V., Da Prato, G., and Röckner, M.: Existence and uniqueness of nonnegative solutions to the stochastic porous media equation. *Indiana Univ. Math. J.* (To appear).
4. Caffarelli, L. A. and Friedman, A.: Regularity of the free boundary of a gas flow in an n-dimensional porous medium. *Indiana Univ. Math. J.* **29**:3 (1980), 361–391.
5. Da Prato, G., Röckner, M., Rozovskii, B. L., and Wang, F.-Y.: Strong solutions of stochastic generalized porous media equations: existence, uniqueness, and ergodicity. *Comm. Partial Differential Equations* **31**:1-3 (2006), 277–291.
6. Estévez, P. G., Qu, C., and Zhang, S.: Separation of variables of a generalized porous medium equation with nonlinear source. *J. Math. Anal. Appl.* **275**:1 (2002), 44–59.
7. Evans, L. C.: *Partial differential equations*, vol. 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
8. Friedman, A. and Kamin, S.: The asymptotic behavior of gas in an n-dimensional porous medium. *Trans. Amer. Math. Soc.* **262**:2 (1980), 551–563.
9. Gurtin, M. E. and MacCamy, R. C.: On the diffusion of biological populations. *Math. Biosci.* **33**:1-2 (1977), 35–49.
10. Kim, J. U.: On the stochastic porous medium equation. *J. Differential Equations* **220**:1 (2006), 163–194.
11. Liptser, R. S. and Shiryayev, A. N.: *Theory of martingales*, vol. 49 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1989.
12. Mel′nik, S. A.: The dynamics of solutions of the Cauchy problem for a stochastic equation of parabolic type with power non-linearities (the stochastic term is linear). *Teor. Ĭmovīr. Mat. Stat.*, **64** (2001), 110–117.
13. Mel′nik, S. A.: Estimates for the solution of the Cauchy problem for a stochastic differential equation of parabolic type with power nonlinearities (a strong source). *Differ. Uravn.* **38**:6 (2002), 802–808.
14. Nualart, D.: *The Malliavin calculus and related topics*, second ed. Probability and its Applications. Springer-Verlag, Berlin, 2006.
15. Sabinina, E. S.: On the Cauchy problem for the equation of nonstationary gas filtration in several space variables. *Soviet Math. Dokl.* **2** (1961), 166–169.
16. Sango, M.: Weak solutions for a doubly degenerate quasilinear parabolic equation with random forcing. *Discrete Contin. Dyn. Syst. Ser. B* **7**:4 (2007), 885–905 (electronic).
17. Vázquez, J. L.: *The porous medium equation*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007.

Current address, S. V. Lototsky: Department of Mathematics, USC, Los Angeles, CA 90089
E-mail address, S. V. Lototsky: `lototsky@math.usc.edu`
URL: `http://math.usc.edu/∼lototsky`