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LOCAL THEOREMS RELATED TO LÉVY-TYPE BRANCHING MECHANISM

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ABSTRACT. We prove local limit theorems for the total mass processes of two branching-fluctuating particle systems which converge to discontinuous $(2, d, \beta)$-superprocess. To this end, we establish new subtle properties of the total mass for this class of superprocesses. Thus, the density of its absolutely continuous component exhibits a polynomial blow-up at the origin and has a regularly varying upper tail. Both particle systems considered are characterized by the same heavy-tailed branching mechanism that belongs to the domain of normal attraction of an extreme stable law with index $1+\beta \in (1,2)$. One of them starts from a Poisson field, whereas the initial number of particles for the other system is non-random. We demonstrate that the poissonization of the initial field of particles is related to Gnedenko's method of accompanying infinitely divisible laws. The comparison of our results with their 'continuous' counterparts (which pertain to convergence to the super-Brownian motion) reveals a worse discrepancy between the extinction probabilities. This is explained through the intrinsic difference between structures of individual surviving clusters.

1. Introduction

This work pertains to certain branching-fluctuating particle systems (or BPS's) and their limits in the case when the mechanism of local branching is *heavy-tailed*. Namely, it is assumed that this mechanism is governed by the particle production generating function

$$\psi_\beta(s) := (1+\beta)^{-1} \cdot (1-s)^{1+\beta}. \tag{1.1}$$

Here, argument $s \in [0,1]$, and parameter $\beta \in (0,1)$. Let $\mathbf{Z}_+$ and $\mathbf{N}$ denote the sets of all non-negative and all positive integers, respectively. Set $\mathbf{R}^1_+ := (0,\infty)$.

It is easily seen that (1.1) implies that if a particle splits, then a random number $\mathcal{K}$ of particles are produced such that $\forall n \in \mathbf{Z}_+$,

$$\mathbf{P}\{\mathcal{K}=n\} = \begin{cases} \frac{(-1)^n}{1+\beta} \cdot \begin{pmatrix} 1+\beta \\ n \end{pmatrix} & \text{if } n \neq 1; \\ 0 & \text{if } n = 1\,. \end{cases} \tag{1.2}$$

By (1.2), $\mathbf{E}\mathcal{K} = 1$. Hence, this branching mechanism is *critical*. We refer to the fulfillment of (1.1)–(1.2) with $\beta \in (0,1)$ as a '*Lévy-type branching mechanism*', although the term '*infinite-variance branching*' is sometimes employed (see Dawson et al. [6, p. 744]).

The results of this work are parallel to those of Vinogradov [31], where the critical binary branching case, which pertains to $\beta = 1$ in (1.1), is considered.

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The probability law (1.2) with $\beta \in (0,1)$ belongs to the domain of normal attraction of an extreme stable distribution with the stability index $1+\beta$ and skewness 1.

It is known that under natural regularity conditions on the motion and branching, the corresponding BPS's converge weakly to *discontinuous* $(2,d,\beta)$-superprocesses. See Dawson et al. [7]–[8], Dynkin [10, Ch. 3] and Le Gall [23, Ch. 1]. The fulfillment of (1.1)–(1.2) is the simplest example of such assumptions on local branching.

The term a '*discontinuous superprocess*' is justified by the fact that the sample paths of such measure-valued process are the *discontinuous* functions of the time variable. Alternatively, a similar BPS characterized by the *critical binary branching* mechanism (for which $\beta = 1$ in (1.1)) converges to the limit whose trajectories are continuous. This limit is frequently called the *super-Brownian motion*. See (2.13)–(2.18), the comments to these formulas, and Dawson and Vinogradov [9, p. 230] for more detail.

In this paper, we derive new local limit theorems. They serve as local counterparts to the well-known integral limit theorems on weak convergence of BPS's to discontinuous $(2,d,\beta)$-superprocesses in the case of the very specific mechanism of local branching given by (1.1)–(1.2).

Recall that these constraints can be weakened in the case when one is interested in the integral theorems only (see the references given above). Such discrepancy is not unusual for the theory of limit theorems of the Probability Theory, where in contrast to integral theorems, the derivation of their local counterparts is often more delicate and involves the imposition of additional constraints. This is especially relevant when one approximates local behavior of discrete structures by continuous functions.

In the case when one deals with *measure-valued* processes, the problem of the derivation of local approximations can be considered from two different angles. First, one can concentrate on the local fluctuations due to a spatial motion of individual particles. Alternatively, one can disregard fluctuations caused by the motion mechanism and concentrate on the derivation of local approximations for the total mass of a BPS being considered.

In the case when $\beta = 1$ for which the limit is the continuous super-Brownian motion, the latter approach was implemented in Vinogradov [29], [31]. The distribution theory background pertaining to the continuous case is given in Vinogradov [30]. It is of interest that both the development of methods and the derivation of results presented in [29] and [31] were facilitated by the discovery of their connection to the previously known general integral limit theorems on weak convergence to representatives of the *power-variance family* of probability laws. Hereinafter, it is denoted by *PVF*.

In particular, the univariate distributions of total mass $M_1(t)$ of the super-Brownian motion are the members of PVF with the value of the power parameter equal to 3/2. Such relations are no longer applicable in a more difficult *discontinuous* setting. Thus, even the Lebesgue decomposition of the univariate laws of total mass $M_\beta(t)$ of the limiting discontinuous $(2,d,\beta)$-superprocess has been previously unknown.

Theorem 2.1.i of the next section demonstrates that similar to the continuous case, the Lebesgue decomposition of r.v. $M_\beta(t)$ is comprised of a point mass at the origin and an absolutely continuous component over $\mathbf{R}^1_+$. However, the asymptotic properties of the density of this component are qualitatively different from those of its continuous counterpart.

First, it blows up at the origin. Its blow-up rate at 0 is specified by (2.20). Secondly, the density of the absolutely continuous component of r.v. $M_\beta(t)$ exhibits a polynomial decay at $+\infty$ (see (2.19)). The proof involves an application of known and the derivation of new results on regular variation and large deviations. Further studies of the properties of $M_\beta(t)$ are deferred to Section 3.

The other main results of this work are Theorems 2.2 and 2.3 of the next section. They provide *local approximations* along with the estimates of remainders for the univariate distributions of the total mass processes of two representative BPS's. These discrete structures were considered in numerous publications, which include author's joint works [9], [7] and [8]. The consideration of these approximations is justified by local properties of r.v. $M_\beta(t)$ established in Theorem 2.1.

There is an important relationship between the discrete structures considered in Theorems 2.2 and 2.3. Namely, there is a deep analogy between the imposition of the *Poisson initial condition* (or *poissonization*) in Theorem 2.2 and *Gnedenko's method of accompanying infinitely divisible laws*. It is made rigorous in Remark 2.5. Also, Gnedenko's method is related to properties of the *individual surviving clusters* considered in Proposition 2.4. Gnedenko's method is reviewed in Sec. 24 of [14]. Although his method of the derivation of *integral* theorems is not directly employed here, but this similarity is reflected in the proof of Theorem 2.3.i. See also the relevant Remarks 2.6–2.7.

To facilitate the reading, we deferred the technical proofs to the concluding Section 4. They involve the use of Poisson mixtures, Laplace's method, large deviations, conditioning arguments and subtle estimates in the local Poisson theorem. The latter results make it possible to derive Theorem 2.3.i from Theorem 2.2.i.

2. Main results and discussion

Let us start from the description of the models. First, consider BPS $\Xi^{(\eta)}_{\beta,m,t}$, where $t \geq 0$ is interpreted as the *time variable*. For an arbitrary fixed time instant $t > 0$, BPS $\Xi^{(\eta)}_{\beta,m,t}$ gives rise to a random measure on $\mathbf{R}^d$. In turn, this justifies regarding this BPS as a certain *mesure-valued stochastic process* with t being interpreted as the time variable. The integer-valued parameter $\eta \geq 1$ is understood as the initial number of identical and independent particles. For simplicity, they are assumed to be located at the origin at time $t = 0$. Each particle has the same mass m/η, where $m \in \mathbf{R}^1_+$ is fixed. Hence,

$$\mathbf{P}\{\Xi^{(\eta)}_{\beta,m,0}(\{0\}) = \eta \cdot (m/\eta)\} \;=\; \mathbf{P}\{\Xi^{(\eta)}_{\beta,m,0}(\{0\}) = m\} = 1. \tag{2.1}$$

In the sequel, η will approach infinity. Every particle immediately starts to perform a motion in $\mathbf{R}^d$ according to a certain probability law. Here, $d \geq 1$ is an integer. At an exponentially distributed time instant with mean $\eta^{-\beta}$, each particle splits into a random number of offspring with the mechanism of local branching given by (1.1)–(1.2). Every newly born particle is an identical replicate of its parent and immediately starts to perfom the same spatial motion. The motions, lifetimes and branchings of all these particles are independent of each other and of everything else.

Set

$$M^{(\eta)}_{\beta,m}(t) := \Xi^{(\eta)}_{\beta,m,t}(\mathbf{R}^d). \tag{2.2}$$

Due to a possibility of extinction, the stochastic process $M^{(\eta)}_{\beta,m}(t)$ (which represents the *total mass* of BPS $\Xi^{(\eta)}_{\beta,m,t}$) takes values in $[0, +\infty)$. It follows from the results of Sec. 1

of Dawson and Vinogradov [9] that for each fixed real $t > 0$,

$$M_{\beta,m}^{(\eta)}(t) \stackrel{d}{\to} M_{\beta,m}(t) \tag{2.3}$$

as $\eta \to \infty$. Hereinafter, the sign '$\stackrel{d}{\to}$' denotes weak convergence. The limiting non-negative r.v. $M_{\beta,m}(t)$ that emerges in (2.3) can be defined by virtue of its Laplace transform

$$\begin{aligned} \hbar_{M_{\beta,m}(t)}(u) &:= \mathbf{E}\exp\{-u \cdot M_{\beta,m}(t)\} \\ &= \exp\left\{-\frac{m \cdot u}{(1 + (\beta/(1+\beta)) \cdot t \cdot u^{\beta})^{1/\beta}}\right\}. \end{aligned} \tag{2.4}$$

Here, real $u \geq 0$. In turn, (2.4) yields the following scaling property. For every fixed positive real m and t,

$$M_{\beta,m}(t) \stackrel{\mathrm{d}}{=} m \cdot M_{\beta,1}(t/m^{\beta}), \tag{2.5}$$

where the sign '$\stackrel{\mathrm{d}}{=}$' is understood in the sense that the distributions of r.v.'s coincide.

Let $\Pi(\mu)$ denote a Poisson r.v. with mean $\mu \in \mathbf{R}_+^1$. Then it can be shown that r.v. $M_{\beta,m}(t)$ is compound Poisson with the value of Poisson parameter equal to

$$\mu_{\beta,m,t} := m \cdot (t \cdot \beta/(\beta+1))^{-1/\beta} \ (= m \cdot \mu_{\beta,1,t}). \tag{2.6}$$

Namely,

$$M_{\beta,m}(t) \stackrel{\mathrm{d}}{=} \frac{1}{\mu_{\beta,1,t}} \cdot \sum_{k=1}^{\Pi(\mu_{\beta,m,t})} \mathcal{Z}_{\beta}(k) \tag{2.7}$$

(compare to (2.11)). By convention, a compound Poisson r.v. equals zero in the case when the corresponding Poisson counting variable is zero. It is natural to say that in this case, the corresponding superprocess *becomes extinct by time* t. In view of (2.7),

$$\mathbf{P}\{M_{\beta,m}(t) = 0\} = \exp\{-\mu_{\beta,m,t}\}. \tag{2.8}$$

Also, the independent copies of the positive and absolutely continuous r.v. $\mathcal{Z}_{\beta}$ introduced by Zolotarev [32] are hereinafter denoted by $\{\mathcal{Z}_{\beta}(k), k \geq 1\}$. It is assumed that these r.v.'s do not depend on $\Pi(\mu_{\beta,m,t})$. Their common distribution function (or d.f.) $\mathcal{H}_{\beta}(\cdot)$ is continuous on $\mathbf{R}^1$. Also, it is positive on $\mathbf{R}_+^1$, where it has the completely monotone density hereinafter denoted by $p_{\beta}(x)$. This class of densities is defined by (3.1) and has a specific value. Their properties are considered in Section 3. In particular, function $p_{\beta}(x)$ blows up at zero as $\mathcal{C}onst \cdot x^{\beta-1}$ and decays at infinity as $\mathcal{C}onst \cdot x^{-(\beta+2)}$.

In view of the above, it is appropriate to refer to the laws of r.v.'s $\mathcal{Z}_{\beta}$ and $M_{\beta,m}(t)$ as *Zolotarev's* and *scaled Poisson-Zolotarev* distributions, respectively. The unboundedness of Zolotarev's density at zero causes serious technical difficulties. In particular, see Proposition 3.1.ii, Lemmas 4.1–4.3, and the proof of Theorem 2.1.

Alternatively, Zolotarev's r.v. $\mathcal{Z}_{\beta}$ can be characterized by its Laplace transform

$$\wp_{\beta}(u) = 1 - (1 + u^{-\beta})^{-1/\beta}. \tag{2.9}$$

Here, real $u > 0$. The subsequent differention of (2.9) implies that

$$\mathbf{E}\mathcal{Z}_{\beta} = 1. \tag{2.10}$$

Note that the Poisson r.v. $\Pi(\mu_{\beta,m,t})$ represents the number of surviving clusters of particles at time t, whereas r.v. $\mathcal{Z}_{\beta}(i)/\mu_{\beta,1,t}$ can be regarded as the total mass of an individual surviving cluster of age t. It is also of interest that the distribution of r.v. $(\beta/(\beta+1))^{1/\beta}\cdot\mathcal{Z}_\beta$ coincides with that of the total mass of a cluster that has an *infinitely small mass* (compare to pp. 231–234 of Dawson and Vinogradov [9]).

Subsequently, (2.7) is justified by a combination of formula (6.2.3) of Panjer and Willmot [26] with the following easy-to-check relationship:

$$\hbar_{M_{\beta,m}(t)}(u) = \exp\{\mu_{\beta,m,t}\cdot(\wp_\beta(u/\mu_{\beta,1,t}) - 1)\}\,. \tag{2.11}$$

An equivalent form of (2.11) in terms of distribution functions is given as formula (4.21).

The combination of (2.7) with Lévy representation for compound Poisson laws (cf., e.g., p. 12 of Bertoin [1]) and the above properties of Zolotarev's r.v. $\mathcal{Z}_\beta$ implies that the Lévy measure $\nu_{M_{\beta,m}(t)}$ of r.v. $M_{\beta,m}(t)$ has the density $\kappa_{\beta,m,t}(y)$ on $\mathbf{R}^1_+$ with respect to Lebesgue measure. Moreover, it follows from (2.7) that $\forall$ fixed $y\in\mathbf{R}^1_+$,

$$\kappa_{\beta,m,t}(y) = m\cdot\mu^2_{\beta,1,t}\cdot p_\beta(\mu_{\beta,1,t}\cdot y). \tag{2.12}$$

Recall that $\mu_{\beta,1,t}$ and $p_\beta(\cdot)$ are defined by formulas (2.6) and (3.1), respectively.

Without the loss of generality, we will mainly concentrate on the case when $m=1$, which is justified by the scaling property (2.5). Therefore, it is natural to denote $M_\beta(t) := M_{\beta,1}(t)$ (compare to the Introduction). It is known that the discontinuous real-valued stochastic process $M_\beta(t)$ is a martingale (cf., e.g., Th. 4.1.ii of Lambert [21]). In addition, Th. 5.2 of Lambert [21] stipulates that the stochastic process $M_\beta(t)$ is the unique càdlàg solution of the following stochastic differential equation:

$$dM_\beta(t) = M_\beta(t-)^{1/(1+\beta)}\cdot d\mathcal{R}_{1+\beta,\rho_1}(t). \tag{2.13}$$

Here, $\{\mathcal{R}_{1+\beta,\rho_1}(t),\ t\ge 0\}$ is the (Lévy) extreme stable process constructed starting from a particular extreme stable law with the index of stability $\alpha := 1+\beta$, skewness 1 and the scaling parameter $\rho_1 := 1/(1+\beta)\in\mathbf{R}^1_+$. This process does not perform negative jumps. Note in passing that the validity of (2.13) can also be derived from Th. 1.6 of Mytnik [25].

In general, constant ρ_1 can be replaced by an *arbitrary* $\rho\in\mathbf{R}^1_+$; the corresponding stable process is characterized by its *Laplace exponent*

$$\Psi_{1+\beta,\rho}(u) := \log\ \mathbf{E}\exp\{-u\cdot\mathcal{R}_{1+\beta,\rho}(1)\} = \rho\cdot u^{1+\beta}. \tag{2.14}$$

Here, real $u\ge 0$. The range of extreme stable r.v. $\mathcal{R}_{1+\beta,\rho}$ is the whole $\mathbf{R}^1$. It possesses a bounded density, which is hereinafter denoted by $\zeta_{1+\beta,\rho}(\cdot)$. A comprehensive description of its properties can be found in Ch. 2 of Zolotarev [33]. In particular,

$$\zeta_{1+\beta,\rho}(x) \sim \frac{\beta\cdot(\beta+1)}{\Gamma(1-\beta)}\ \cdot\rho\cdot x^{-(\beta+2)} \tag{2.15}$$

as $x\to+\infty$. Moreover, function $\zeta_{1+\beta,\rho}(x)$ admits an asymptotic expansion in negative powers of x as $x\to+\infty$ (compare to formula (3.2) that pertains to an analogous expansion for the density of a positive stable law with the index of stability $\beta\in(0,1)$). We will employ the existence of such expansion in the proof of Lemma 4.3, where another representative of class (2.14) is employed. In particular, see formula (4.20). At the same time, the explicit form will only be required for the principal term of such expansion. Its asymptotics is given on the right-hand side of (2.15).

Note that the lower tail of $\zeta_{1+\beta,\rho}(x)$ decays faster than that of a normal density at $-\infty$. At each fixed point of $\mathbf{R}^1$, it admits a convergent infinite series representation. We will use a specific representative of this class in the proof of Lemma 4.3.

The càdlàg process $M_\beta(t)$ is a time-homogeneous locally infinitely divisible Markov process in the sense of Sec. 5.2 of Freidlin and Wentzell [13]. In order to present its generator $\mathcal{B}_\beta$, it is convenient to introduce the analytic continuation of the gamma function onto $\mathbf{C} \setminus \{0; -1; -2; ...\}$. Hereinafter, this function is denoted by $\Gamma(z)$, whereas $\mathbf{C}$ stands for the complex plane. For non-positive integer z, function $\Gamma(z) := \infty$, since it possesses simple poles at such points. Subsequently, a modification of the formulas given on pp. 230–231 of Dawson and Vinogradov [9] stipulates that for a wide class of smooth functions $\mathcal{F}$,

$$\mathcal{B}_\beta\, \mathcal{F}(x) = \int_0^\infty \left(\mathcal{F}(x+v) - \mathcal{F}(x) - \mathcal{F}'(x)\cdot v\right) \cdot k_x^{(\beta)}(dv). \tag{2.16}$$

Here, $k_x^{(\beta)}(dv)$ is the following Lévy measure on $\mathbf{R}_+^1$:

$$k_x^{(\beta)}(dv) := -x \cdot \Gamma(-\beta)^{-1} \cdot v^{-(\beta+2)} \cdot dv. \tag{2.17}$$

In the 'continuous' case that corresponds to $\beta = 1$, one obtains a simpler analogue of representations (2.13) and (2.16)–(2.17). In particular, if $\beta = 1$ then the total mass process $M_1(t)$ of a super-Brownian motion is the unique solution of (2.13) in the space of continuous functions (with the understanding that $M_1(t-) = M_1(t)$ and that $\mathcal{R}_2(t)$ coincides with the standard Brownian motion in $\mathbf{R}^1$). It is a martingale and a time-homogeneous Markov diffusion process. It is also pertinent that the corresponding stochastic process is frequently called the *Feller diffusion* (with no drift). In this case, the generator is as follows:

$$\mathcal{B}_1\, \mathcal{F}(x) = (x/2) \cdot \mathcal{F}''(x). \tag{2.18}$$

Our first result provides the Lebesgue decomposition for univariate distributions of the total mass $M_\beta(t)$ of a discontinuous $(2, d, \beta)$-superprocess. This decomposition and properties of its components are interesting in their own right. In addition, this result will be employed in the proof of Theorem 2.2.i.

Theorem 2.1. *(i) For each fixed $t \in \mathbf{R}_+^1$, the Lebesgue decomposition of the distribution of non-negative r.v. $M_\beta(t)$ does not have a continuous-singular component. Its discrete component is comprised of the point mass $\exp\{-\mu_{\beta,1,t}\}$ at zero. The density $f_{\beta,t}(x)$ of the absolutely continuous component of $M_\beta(t)$ over $\mathbf{R}_+^1$ is a member of $\mathbf{C}(\mathbf{R}_+^1)$.*
(ii) The function $f_{\beta,t}(x)$ possesses the following asymptotics as $x \to \infty$:

$$f_{\beta,t}(x) \sim \frac{\beta}{\Gamma(1-\beta)} \cdot t \cdot x^{-(\beta+2)}. \tag{2.19}$$

(iii) The function $f_{\beta,t}(x)$ blows up at the right-hand neighborhood of zero with the polynomial rate. Namely,

$$f_{\beta,t}(x) \sim \frac{1}{\Gamma(1+\beta)} \cdot \exp\{-\mu_{\beta,1,t}\} \cdot \mu_{\beta,1,t}^{1+\beta} \cdot x^{\beta-1} \tag{2.20}$$

as $x \downarrow 0$.

It is relevant that the power asymptotics exhibited by the upper tail of d.f. of r.v. $M_\beta(t)$ which can be easily derived from (2.19) is consistent with the behavior of the corresponding Laplace transform at the neighborhood of zero that follows from (2.4).

Recall that BPS $\Xi^{(\eta)}_{\beta,m,t}$ is originated starting from the *non-random* number η of particles. However, it is also possible and often more natural to suppose that the initial number of particles of a BPS is *random*. Thus, the derivation of the '*high-density*' approximation becomes rather elegant under the assumption that the *initial number of particles is Poisson* distributed with mean η (cf., e.g., pp. 47–49 of [10]). This *Poisson* (or *quasi-stationary*) initial condition results in the appearance of a modified (or quasi-stationary) BPS $\Upsilon^{(\eta)}_{\beta,m,t}$ that was considered, among others, in author's joint works [7] and [8]. The idea of the poissonization of the initial field of particles can be traced back to p. 92 of [15].

It follows from [7] and [8] that $\forall t \in \mathbf{R}^1_+$, the total mass $\widetilde{M}^{(\eta)}_{\beta,m}(t)$ of the quasi-stationary BPS $\Upsilon^{(\eta)}_{\beta,m,t}$ approaches the same limit as that for the original BPS $\Xi^{(\eta)}_{\beta,m,t}$. Namely,

$$\widetilde{M}^{(\eta)}_{\beta,m}(t) \stackrel{d}{\to} M_{\beta,m}(t) \tag{2.21}$$

as $\eta \to \infty$ (compare to (2.3)). Note in passing that the analogues of the convergence results (2.3) and (2.21) also hold in the space of càdlàg functions equipped with the Skorokhod metric. However, we do not pursue this matter further concentrating on *fixed* times.

The Lévy measure of (compound Poisson) r.v. $\widetilde{M}^{(\eta)}_{\beta,m}(t)$ is given in Proposition 2.4.

For simplicity of notation, set $M^{(\eta)}_{\beta}(t) := M^{(\eta)}_{\beta,1}(t)$ and $\widetilde{M}^{(\eta)}_{\beta}(t) := \widetilde{M}^{(\eta)}_{\beta,1}(t)$.

It is relevant that one can derive (2.3) from (2.21) by evaluating the difference between the total masses $M^{(\eta)}_{\beta}(t)$ and $\widetilde{M}^{(\eta)}_{\beta}(t)$ with the use of the *compound Poisson approximation* bounds. To this end, we introduce the following quantity:

$$Q^{(\eta)}_{\beta,t} := (1 + t \cdot \eta^\beta \cdot \beta/(1+\beta))^{-1/\beta} \tag{2.22}$$

(compare to formula (1.12) of Dawson and Vinogradov [9]). The expression (2.22) can be interpreted as the *probability of survival* of descendents of an individual (or tagged) particle from their initial set by time t (Evidently, this is applicable to both BPS's $\Xi^{(\eta)}_{\beta,1,t}$ and $\Upsilon^{(\eta)}_{\beta,1,t}$.) See p. 227 of [9] for more detail. It is clear that

$$\mu^{(\eta)}_{\beta,1,t} := \eta \cdot Q^{(\eta)}_{\beta,t} \to \mu_{\beta,1,t} \tag{2.23}$$

as $\eta \to \infty$, where the latter quantity is defined by (2.6).

Subsequently, Theorem 1 of Le Cam [24] and formula (1.3) of Chen [4] stipulate that

$$\| \, \eta \cdot M^{(\eta)}_{\beta}(t) - \eta \cdot \widetilde{M}^{(\eta)}_{\beta}(t) \, \| \le \; 2\eta \cdot (Q^{(\eta)}_{\beta,t})^2 \;\sim\; 2 \cdot (\mu_{\beta,1,t})^2 \cdot \eta^{-1} \tag{2.24}$$

as $\eta \to \infty$. Hereinafter, $\| \, \mathcal{M} \, \|$ denotes the norm of a (generic) finite signed measure $\mathcal{M}$.

The next two theorems are also among the main results of this paper. They can be regarded as local counterparts of the weak convergence results given by formulas (2.21) and (2.3), respectively. The parts (i) of these results provide the second-order local approximations for the laws of r.v.'s $\widetilde{M}^{(\eta)}_{\beta}(t)$ and $M^{(\eta)}_{\beta}(t)$, whereas parts (ii) pertain to the probabilities of extinction. These results are analogous to the local DeMoivre-Laplace theorem and complement *integral* theorems on weak convergence of BPS's to $(2, d, \beta)$-superprocesses under the fulfillment of (1.2). Hereinafter, $C_1(t, x) \leq C_2(t, x)$ denote certain positive real constants which may depend on both $t \in \mathbf{R}^1_+$ and $x \in \mathbf{R}^1_+$.

Theorem 2.2. *For fixed* $t \in \mathbf{R}^1_+$*, consider the total mass* $\widetilde{M}^{(\eta)}_\beta(t) = \Upsilon^{(\eta)}_{\beta,1,t}(\mathbf{R}^d)$ *of the quasi-stationary BPS* $\Upsilon^{(\eta)}_{\beta,1,t}$.
(i) We suppose that $x \in \mathbf{R}^1_+$ *is fixed and confine ourselves only to those values of the integer-valued parameter* η *for which the product* $x \cdot \eta \in \mathbf{N}$*. Then* x *belongs to the range of r.v.* $\widetilde{M}^{(\eta)}_\beta(t)$*. For such* x*, the probability function of r.v.* $\widetilde{M}^{(\eta)}_\beta(t)$ *admits the next upper bound for all sufficiently large admissible values of* η*:*

$$\left| \mathbf{P}\{\widetilde{M}^{(\eta)}_\beta(t) = x\} - \frac{1}{\eta} \cdot f_{\beta,t}(x) \right| \leq C_1(t, x)/\eta^2. \tag{2.25}$$

(ii) The discrepancy at the origin exhibits a slower decay and can be evaluated up to equivalence as $\eta \to \infty$*:*

$$\mathbf{P}\{\widetilde{M}^{(\eta)}_\beta(t) = 0\} - \exp\{-\mu_{\beta,1,t}\} \sim \frac{1}{\beta} \cdot (\mu_{\beta,1,t})^{1+\beta} \cdot \exp\{-\mu_{\beta,1,t}\} \cdot \eta^{-\beta}. \tag{2.26}$$

A similar but slightly less accurate result is also valid for the total mass $M^{(\eta)}_\beta(t)$ of the original BPS $\Xi^{(\eta)}_{\beta,1,t}$. The loss in accuracy is due to the method of proof, which is partly based on the Poisson approximation for the (random) number of clusters. This number has the binomial $\mathbf{B}(\eta, Q^{(\eta)}_{\beta,t})$ distribution with η trials and the probability of success $Q^{(\eta)}_{\beta,t}$ in a single trial given by (2.22). In particular, this approach implies that $\forall x > 0$, constant $C_2(t, x)$ that emerges in formula (2.27) below, exceeds $C_1(t, x)$ (compare to (2.25)). It is plausible that the method of proof of the following Theorem 2.3 can be generalized to incorporate other assumptions on the distribution of the initial number of particles. (A similar argument related to the case when $\beta = 1$ also seems to be pertinent to the method employed in Vinogradov [31].)

Theorem 2.3. *Fix* $t \in \mathbf{R}^1_+$.
(i) Suppose that $x \in \mathbf{R}^1_+$ *is fixed, and consider those values of* η *for which the product* $x \cdot \eta \in \mathbf{N}$*. Then* x *belongs to the range of r.v.* $M^{(\eta)}_{\beta,t}$*. For such* x*, one ascertains the following upper bound for all sufficiently large admissible values of* η*:*

$$\left| \mathbf{P}\{M^{(\eta)}_\beta(t) = x\} - \frac{1}{\eta} \cdot f_{\beta,t}(x) \right| \leq C_2(t, x)/\eta^2. \tag{2.27}$$

(ii) For $x = 0$ *and as* $\eta \to \infty$*,*

$$\mathbf{P}\{M^{(\eta)}_\beta(t) = 0\} - \mathbf{P}\{\widetilde{M}^{(\eta)}_\beta(t) = 0\} \sim -(\mu_{\beta,1,t})^2 \cdot \exp\{-\mu_{\beta,1,t}\} \cdot \frac{1}{2\eta}. \tag{2.28}$$

The comparison of (2.26) and (2.28) implies that in the case when $\eta \to \infty$, the discrepancy between the probabilities $\mathbf{P}\{M^{(\eta)}_\beta(t) = 0\}$ and $\mathbf{P}\{M_\beta(t) = 0\}$ (which is given by (2.8)) is of the same magnitude as for $\mathbf{P}\{\widetilde{M}^{(\eta)}_\beta(t) = 0\}$. Moreover, (2.28) implies that the order of decay in the 'integral' theorem, that emerges from (2.24), is sharp.

Let us clarify the relationships between Theorems 2.2 and 2.3 as well as their connections with the other results in this area by means of the next proposition and remarks. By analogy to [9], we say that an initial (tagged) particle gives rise to a *cluster of its offspring*, which are alive at time t. Of course, such cluster will be empty with probability $1-Q^{(\eta)}_{\beta,t}$.

Denote the *mass* of the above *cluster* by $S^{(\eta)}_{\beta,t}(i)$, where $1 \le i \le \eta$ stands for the index of this tagged particle. It is clear that

$$M^{(\eta)}_{\beta}(t) \overset{\mathrm{d}}{=} \sum_{i=1}^{\eta} S^{(\eta)}_{\beta,t}(i). \tag{2.29}$$

The r.v.'s $\{S^{(\eta)}_{\beta,t}(i), 1 \le i \le \eta\}$ which emerge in (2.29) are independent copies of r.v. $S^{(\eta)}_{\beta,t}$. By pp. 231–232 of Dawson and Vinogradov [9], the range and Laplace transform of r.v. $S^{(\eta)}_{\beta,t}$ are the set $\eta^{-1}\cdot\mathbf{Z}_+$ and the following function, respectively:

$$g_{\beta,t,\eta}(u) := \mathbf{E}\exp\{-u\cdot S^{(\eta)}_{\beta,t}\} = 1-((1-e^{-u/\eta})^{-\beta}+t\cdot\eta^{\beta}\cdot\beta/(1+\beta))^{-1/\beta}. \tag{2.30}$$

Here, real $u \ge 0$. The following assertion is of a particular value.

Proposition 2.4. *The non-negative lattice r.v. $\widetilde{M}^{(\eta)}_{\beta,m}(t)$, which represents the total mass of quasi-stationary BPS $\Upsilon^{(\eta)}_{\beta,m,\cdot}$ at time t, is compound Poisson and hence, infinitely divisible. Its Lévy measure $\widetilde{\nu}^{(\eta)}_{\beta,m,t}(\cdot)$ is concentrated on the set $\eta^{-1}\cdot\mathbf{N}$ such that $\forall s \in \eta^{-1}\cdot\mathbf{N}$,*

$$\widetilde{\nu}^{(\eta)}_{\beta,m,t}(\{s\}) = m\cdot\eta\cdot\mathbf{P}\{S^{(\eta)}_{\beta,t} = s\}. \tag{2.31}$$

Let us explain the similarity between Theorems 2.2–2.3 (which involve the poissonization) and Gnedenko's method. (Note that in contrast to Gnedenko, we have not centered the original r.v.) It is relevant that this method was originally developed for the proof of general integral theorems on weak convergence of triangular arrays of independent r.v.'s to infinitely divisible laws. Its version (adapted to the semimartingale setting) can also be found on p. 403 of Jacod and Shiryaev [17].

Remark 2.5. The idea of the (modified) Gnedenko's method consists in approximating each term of the sum (2.29) of i.i.d.r.v.'s by a compound Poisson r.v. whose Lévy measure coincides with the probability function of the original r.v. (i.e., $S^{(\eta)}_{\beta,t}(i)$) everywhere except the origin. But setting $m = 1/\eta$ in (2.31) stipulates that the appoximating infinitely divisible r.v. is given by an independent copy of compound Poisson r.v. $\widetilde{M}^{(\eta)}_{\beta,1/\eta}(t)$. Hence, Gnedenko-type approximation for the total sum over η that emerges on the right-hand side of (2.29) should be represented as the sum of η i.i.d. copies of r.v. $\widetilde{M}^{(\eta)}_{\beta,1/\eta}(t)$.

Now, let $\{\widetilde{M}^{(\eta)}_{\beta,1/\eta}(t)_j, 1 \le j \le \eta\}$ denote such i.i.d. copies of r.v. $\widetilde{M}^{(\eta)}_{\beta,1/\eta}(t)$. A subsequent application of Proposition 2.4 implies that the Gnedenko-type approximation for $M^{(\eta)}_{\beta}(t)$ is given by the following sum of i.i.d.r.v.'s:

$$\widetilde{M}^{(\eta)}_{\beta,1/\eta}(t)_1 + \ldots + \widetilde{M}^{(\eta)}_{\beta,1/\eta}(t)_\eta \overset{\mathrm{d}}{=} \widetilde{M}^{(\eta)}_{\beta}(t).$$

But this is consistent with the poissonization of the initial field of particles.

Next, observe that the proof of *local* Theorem 2.3 employs a similar idea of approximating r.v. $M^{(\eta)}_{\beta}(t)$ by compound Poisson r.v. $\widetilde{M}^{(\eta)}_{\beta}(t)$. By (2.29), the problem of the verification/refutation of the infinite divisibility of the original r.v. is reduced to that of the mass $S^{(\eta)}_{\beta,t}$ of an individual cluster. The latter problem is open. Note that in the case of the critical binary branching for which $\beta = 1$, Th. 2.1.i of Vinogradov [30] reveals the existence of the threshold value for the fulfillment of this property.

We conclude this section with two additional remarks aimed at comparing the main results of this section with those pertinent to the case of convergence to the *continuous* super-Brownian motion.

Remark 2.6. It is easy to show that $\mathbf{E}S_{\beta,t}^{(\eta)} = \mathbf{E}\widetilde{M}_{\beta,1/\eta}^{(\eta)}(t) = 1/\eta$. The analogous result also holds if $\beta = 1$. However, in the latter case, the variances of the counterparts of r.v.'s which emerge in the above formula are different (compare to Remark 2.8.ii of Vinogradov [31]). Evidently, the property of the finiteness of variance(s) does not hold under our main assumption that the mechanism of local branching is given by formula (1.2) with $\beta < 1$.

Remark 2.7. For the case of the critical *binary* branching, which is characterized by $\beta = 1$ in formula (1.1), the counterparts of the above Theorems 2.2–2.3 are given in Vinogradov [31] (see Th. 2.9 therein). The accuracy of approximation given by parts (i) of Theorems 2.2–2.3, which pertain to the case when $x > 0$, is the same as in the case when $\beta = 1$. In contrast, the accuracy of convergence for the probabilities of extinction that can be derived from parts (ii) of Theorems 2.2–2.3, is much worse than in the case of attraction to a continuous super-Brownian motion. This is because for $\beta = 1$, the density of the absolutely continuous component of the total mass $M_1(t)$ of the super-Brownian motion is bounded at zero as opposed to the case when $\beta < 1$ (compare Theorem 2.1 with formula (2.19) of Vinogradov [31]).

3. Properties of relevant distributions with polynomial tails

The main assertions of this section rely on properties of r.v. $\mathcal{Z}_\beta$ introduced by Zolotarev [32]. Its d.f. is denoted by $\mathcal{H}_\beta(\cdot)$, whereas the Laplace transform $\wp_\beta(\cdot)$ is given by (2.9). Lamperti [22] noted that Zolotarev's r.v. has a density (see p. 232 therein).

It is straightforward to derive starting from the integral representation for $\mathcal{H}_\beta(\cdot)$ given on p. 253 of Zolotarev [32] that $\forall x \in \mathbf{R}_+^1$,

$$\begin{aligned} \exists\, p_\beta(x) &:= \mathcal{H}'_\beta(x) \\ &= \frac{\beta^2}{\Gamma(1/\beta)} \cdot x^{\beta-1} \cdot \int_{0+}^{\infty} e^{-(x/u)^\beta} \cdot u^{-(\beta+1)} \cdot q_{\beta,1}(u) \cdot du. \end{aligned} \tag{3.1}$$

Hereinafter, $q_{\beta,1}(u)$ denotes the probability density of the positive stable r.v. $\mathcal{ST}_{\beta,1}$, which is characterized by the index of stability $\beta \in (0,1)$, skewness 1 and the unit value of the scaling parameter. The characteristic function of this r.v. equals $\exp\{-(-it)^\beta\}$. It is relevant that one can derive the following asymptotic representation by employing the results of Sec. 2.5 of Zolotarev [33]: $\forall$ fixed integer $\mathcal{N} \geq 1$,

$$\begin{aligned} q_{\beta,1}(u) = \beta \cdot \sum_{k=1}^{\mathcal{N}} \frac{(-1)^{k+1}}{(k-1)!} \cdot \frac{1}{\Gamma(1-k\beta)} \cdot u^{-(k\beta+1)} + \mathcal{O}(u^{-((\mathcal{N}+1)\beta+1)}) \\ \left(\sim \frac{\beta}{\Gamma(1-\beta)} \cdot u^{-(\beta+1)} \right) \end{aligned} \tag{3.2}$$

as $u \to \infty$ (compare to formula (3.2.4') of Vinogradov [28]). In addition,

$$q_{\beta,1}(u) \sim \frac{(u/\beta)^{-(2-\beta)/(2\cdot(1-\beta))}}{\sqrt{2\pi\cdot\beta\cdot(1-\beta)}} \cdot \exp\{-(1-\beta)\cdot(u/\beta)^{-\beta/(1-\beta)}\} \tag{3.3}$$

as $u \downarrow 0$ (cf., e.g., formula (2.5.18) of Zolotarev [33]).

By (3.2)–(3.3), all the moments of r.v. $\mathcal{ST}_{\beta,1}$ of order less then β (including all *negative* moments) are *finite*. In contrast, all the moments of order greater than or equal to β are *infinite*. It is also relevant that the explicit expressions for all the *finite* moments of r.v. $\mathcal{ST}_{\beta,1}$ are available (cf., e.g., (4.7)). Due to the presence of the exponential factor on the right-hand side of (3.3), $q_{\beta,1}(x)$ decays at the neighborhood of zero faster than any power of x. In addition, a combination of (3.1) with the well-known basic properties of stable densities yields that $\forall x \in \mathbf{R}^1_+$, function $p_\beta(x) > 0$, and that

$$p_\beta \in \mathbf{C}(\mathbf{R}^1_+). \tag{3.4}$$

The first result of this section concerns previously unknown properties of r.v. $\mathcal{Z}_\beta$.

Proposition 3.1. *(i) The density $p_\beta(x)$ of r.v. $\mathcal{Z}_\beta$ is completely monotone on $\mathbf{R}^1_+$.*
(ii) Zolotarev's density exhibits the following asymptotics:

$$p_\beta(u) \sim \frac{1+\beta}{\Gamma(1-\beta)} \cdot u^{-(\beta+2)} \tag{3.5}$$

as $u \to \infty$, whereas

$$p_\beta(u) \sim \frac{1}{\Gamma(1+\beta)} \cdot u^{\beta-1} \tag{3.6}$$

as $u \downarrow 0$.

In turn, part (i) of this proposition implies

Corollary 3.2. *Zolotarev's r.v. $\mathcal{Z}_\beta$ is infinitely divisible.*

Remark 3.3. The upper tail of the density $p_\beta(x)$ admits an asymptotic expansion in the powers of $x^{-(k\beta+2)}$ as $x \to \infty$. This can be relatively easily derived by substituting the finite-series representation (3.2) for $q_{\beta,1}(x)$ into (3.1). In the sequel, we will require to employ the *existence* of such expansion only. This is because we will need to use the explicit form just for its *principal term* that is given by (3.5).

The 'transition' from the (scaled) density $p_\beta(\cdot)$ to the density $f_{\beta,t}(\cdot)$ of the absolutely continuous component of r.v. $M_\beta(t)$ involves the use of the Lebesgue decomposition of $M_\beta(t)$ that is given by Theorem 2.1. In addition, it employs the cut-off of an infinite series comprised of the densities of 'scaled' convolutions of $\mathcal{H}_\beta(\cdot)$ with the Poisson weights.

This idea is straightforward. Its analogues have already been successfully used. Thus, in the case when the (generic or original) density is *bounded* and positive only on $[A, \infty)$ (with some real $A \geq 0$), and provided that it regularly varies at infinity in a certain sense, the results of Sec. 3 of Klüppelberg [20] stipulate that the density of the absolutely continuous component of the distribution of a Poisson sum is equivalent at infinity to the original density multiplied by the value of the Poisson parameter. Moreover, all the subsequent results in this area the author is aware about also utilize the same condition of the *boundedness* of the original density. Hence, it is the fulfillment of the blow-up property (3.6) that necessitates one to modify some of Klüppelberg's techniques and develop new ideas in order to confirm the validity of the same relationship between the given and 'randomized' densities.

Following Klüppelberg [20], suppose that two generic d.f.'s $\mathbf{F}$ and $\mathbf{G}$ are concentrated on $\mathbf{R}_+^1$, where they possess the probability densities $\phi(\cdot)$ and $\gamma(\cdot)$, respectively. Then their convolution $\mathbf{F} * \mathbf{G}$ is absolutely continuous on $\mathbf{R}_+^1$ such that $\forall x > 0$, its density equals

$$\phi \otimes \gamma(x) := \int_0^x \phi(x-y) \cdot \gamma(y) \cdot dy = \int_0^x \gamma(x-y) \cdot \phi(y) \cdot dy. \tag{3.7}$$

Also, we denote the density of the n-fold convolution $\mathbf{F}^{n*}$ by $\phi^{n\otimes}(x)$, where $x \in \mathbf{R}_+^1$. It is straightforward to extend the operation $\otimes$ given by (3.7) to wider classes of non-negative integrable functions with domain $[0, \infty)$. In the sequel, we will need the next two definitions, which are due to Klüppelberg [20] (see Def.'s 3 and 4 therein, respectively).

Definition 3.4. A function $\psi : \mathbf{R}^1 \to [0, \infty)$ such that $\psi(x) > 0$ only on $[A, \infty)$ with some real $A \geq 0$, and for which the 2-fold 'convolution' $\psi^{2\otimes}$ is well defined, is said to belong to class $\mathcal{SD}(\tau)$ with real $\tau \geq 0$ if

$$\exists \lim_{x\to\infty} \psi^{2\otimes}(x)/\psi(x) =: 2 \cdot \Delta < \infty, \tag{3.8}$$

and $\forall y \in \mathbf{R}^1$,

$$\exists \lim_{x\to\infty} \psi(x-y)/\psi(x) = e^{\tau \cdot y}. \tag{3.9}$$

If $\tau = 0$ in (3.9), one obtains that $\Delta = 1$ in (3.8). The subclass $\mathcal{SD}(0)$ should be regarded as the local counterpart of the class of *subexponential* distributions.

A wider auxilliary class of functions is described in the following

Definition 3.5. A function $\psi : \mathbf{R}^1 \to [0, \infty)$ such that $\psi(x) > 0$ only on $[A, \infty)$ with some real $A \geq 0$, and for which the 2-fold 'convolution' $\psi^{2\otimes}$ is well defined, is said to belong to class $\mathcal{LD}(\tau)$ with real $\tau \geq 0$ if $\forall y \in \mathbf{R}^1$ relationship (3.9) holds.

The next result is quite natural. Its 'integral' version is easy. However, the following assertion does not seem to be available in the literature. This is mainly due to the unboundedness of function $p_\beta(x)$ at the origin. The following property of Zolotarev's density will be used in the proof of Lemma 4.3.

Proposition 3.6. *The density* $p_\beta(x) \in \mathcal{SD}(0)$.

It is plausible that a straightforward and self-contained proof of the above proposition can be found. However, the author elected to give its alternative derivation that involves an application of known results. He believes that such proof is of some methodological value, since it stresses the importance of the techniques of exponential tilting.

The other new property of r.v. $M_\beta(t)$ is given in part (i) of the next lemma. It stipulates a representation of the distribution of *integer-valued* r.v. $\eta \cdot \widetilde{M}_\beta^{(\eta)}(t)$ in terms of that of *mixed* r.v. $M_{\beta,\eta}(t \cdot \eta^\beta)$ ($\stackrel{\mathrm{d}}{=} \eta \cdot M_\beta(t)$). Recall that the former r.v. represents the total number of particles of the quasi-stationary BPS $\Upsilon_{\beta,1,t}^{(\eta)}$ alive at time t. In addition, we remind that $f_{\beta,t}(\cdot)$ is the density of the absolutely continuous component of r.v. $M_\beta(t)$ and refer the reader to Subsec. 8.3.2 of Johnson et al. [18] on the basics on Poisson mixtures.

Lemma 3.7. *(i) The r.v.* $\eta \cdot \widetilde{M}_{\beta}^{(\eta)}(t)$ *can be represented as the Poisson mixture with the unit value of the Poisson parameter and mixing distribution being that of r.v.* $M_{\beta,\eta}(t \cdot \eta^{\beta})$*. (ii) For each fixed* $x > 0$ *and those* η *for which* $x \cdot \eta$ *takes on an integer value,*

$$\mathbf{P}\{\eta \cdot \widetilde{M}_{\beta}^{(\eta)}(t) = x \cdot \eta\} = \frac{1}{(x \cdot \eta)!} \cdot [\, \mathbf{P}\{M_{\beta,\eta}(t \cdot \eta^{\beta}) = 0\} + \frac{1}{\eta} \cdot \int_{0+}^{\infty} e^{-z} \cdot z^{x \cdot \eta} \cdot f_{\beta,t}(z/\eta) \cdot dz \,]. \tag{3.10}$$

Remark 3.8. It is plausible that analogue(s) of the Poisson mixture representation given by Lemma 3.7.i can be discovered for the corresponding measure-valued stochastic process $\Upsilon^{(\eta)}_{\beta,\eta,\cdot}$ or in the context of mixed Poisson processes.

4. Technical proofs

Proof of Proposition 3.1. (i) It follows from (3.1) that $\forall x \in \mathbf{R}^1_+$,

$$\exists\, p'_{\beta}(x) = -\frac{\beta^2}{\Gamma(1/\beta)} \cdot \{(1-\beta) \cdot x^{\beta-2} \cdot \int_{0+}^{\infty} e^{-(x/u)^{\beta}} \cdot \frac{q_{\beta,1}(u)}{u^{\beta+1}} \cdot du + \beta \cdot x^{2(\beta-1)} \cdot \int_{0+}^{\infty} e^{-(x/u)^{\beta}} \cdot \frac{q_{\beta,1}(u)}{u^{2\beta+1}} \cdot du\} < 0. \tag{4.1}$$

The successive differentiation of (4.1) with respect to x ascertains that the density function $p_{\beta} \in \mathbf{C}^{\infty}(\mathbf{R}^1_+)$. Starting from (4.1), it is easy to derive by induction that the n^{th} derivative $p^{(n)}_{\beta}(x)$ of Zolotarev's density equals the product of the expression $(-1)^n \cdot \beta^2/\Gamma(1/\beta)$ and a finite sum of the specific 2^n terms which all have the following type:

$$C_{1,\ell}(\beta) \cdot x^{-C_{2,\ell}(\beta)} \cdot \int_{0+}^{\infty} e^{-(x/u)^{\beta}} \cdot \frac{q_{\beta,1}(u)}{u^{C_{3,\ell}(\beta)}} \cdot du.$$

Here, $\{C_{i,\ell}(\beta),\ 1 \le i \le 3,\ 1 \le \ell \le 2^n\}$ are certain positive constants which depend on β. By (3.2)–(3.3), the above integrals are finite and positive. The proof of part (i) is completed by making a comparison with formula (III.10.10) of Steutel and van Harn [27]. (ii) In order to derive (3.5), fix $\epsilon \in (0,1)$ and split the domain of the integral that emerges in (3.1) into two parts. Namely, this integral is represented as

$$\mathcal{I}_1 + \mathcal{I}_2. \tag{4.2}$$

Here,

$$\mathcal{I}_1 = \int_{0+}^{x^{1-\epsilon}} e^{-(x/u)^{\beta}} \cdot u^{-(\beta+1)} \cdot q_{\beta,1}(u) \cdot du \tag{4.3}$$

and

$$\mathcal{I}_2 = \int_{x^{1-\epsilon}}^{\infty} e^{-(x/u)^{\beta}} \cdot u^{-(\beta+1)} \cdot q_{\beta,1}(u) \cdot du. \tag{4.4}$$

A combination of (3.2) and (3.3) stipulates that the function $|\ q_{\beta,1}(u)/u^{\beta+1}\ |$ is bounded from above. Therefore,

$$\mathcal{I}_1 \le Const \cdot x^{1-\epsilon} \cdot \exp\{-x^{\epsilon \cdot \beta}\}. \tag{4.5}$$

In order to evaluate the integral that emerges in (4.4), we employ approximation (3.2) and make the change of variables $z := x/u$. One ascertains that

$$\mathcal{I}_2 \sim \frac{\beta}{\Gamma(1-\beta)} \cdot x^{-(2\beta+1)} \cdot \int_{0+}^{x^{\epsilon}} z^{2\beta} \cdot \exp\{-z^{\beta}\} \cdot dz \sim \frac{\beta}{\Gamma(1-\beta)} \cdot \frac{(1+\beta) \cdot \Gamma(1/\beta)}{\beta^3} \cdot x^{-(2\beta+1)} \tag{4.6}$$

as $x \to \infty$. The proof of (3.5) is then completed by combining (3.1) with (4.2)–(4.6).

Next, in order to derive (3.6) observe that the integral which emerges in (3.1) does not exceed

$$\mathbf{E}(\mathcal{ST}^{\ -(1+\beta)}) = \frac{\Gamma(1/\beta)}{\beta^3 \cdot \Gamma(\beta)} \tag{4.7}$$

(cf., e.g., formula (2.3) of Brockwell and Brown [2]). It turns out that this integral is in fact equivalent to the latter constant as $x \downarrow 0$. To show this, fix real $\delta > 0$ and split the domain of this integral into two parts. Thus,

$$\int_{0+}^{\infty} e^{-(x/u)^{\beta}} \cdot u^{-(\beta+1)} \cdot q_{\beta,1}(u) \cdot du = \mathcal{I}_3 + \mathcal{I}_4, \tag{4.8}$$

where $\mathcal{I}_3$ and $\mathcal{I}_4$ correspond to the integration over $(0, x^{1+\delta}]$ and $(x^{1+\delta}, \infty)$, respectively. It is obvious that

$$\mathcal{I}_3 = \mathcal{O}(x^{1+\delta}) \tag{4.9}$$

as $x \downarrow 0$. On the other hand,

$$\mathcal{I}_4 \geq \int_{x^{1+\delta}}^{\infty} \exp\{-x^{-\delta\cdot\beta}\} \cdot u^{-(\beta+1)} \cdot q_{\beta,1}(u) \cdot du \sim \mathbf{E}(\mathcal{ST}^{\ -(1+\beta)}) \tag{4.10}$$

as $x \downarrow 0$. To conclude, it remains to combine formulas (3.1) and (4.7)–(4.10). □

Proof of Proposition 3.6. Given real $\theta > 0$, consider the *exponential tilting* of Zolotarev's density:

$$p_{\beta}^{(\theta)}(x) := \wp_{\beta}(\theta)^{-1} \cdot e^{-\theta x} \cdot p_{\beta}(x). \tag{4.11}$$

Recall that the Laplace transform $\wp_{\beta}(\cdot)$ is given by (2.9).

It follows from (3.5) that function $p_{\beta}^{(\theta)}(\cdot) \in \mathcal{LD}(\theta)$, since relationship (3.9) is obviously fulfilled. Also, it is convenient to say that the probability density $p_{\beta}^{(\theta)}(\cdot)$ is that of a (positive) *exponentially tilted Zolotarev's r.v.* $\mathcal{Z}_{\beta}^{(\theta)}$.

Subsequently, a combination of (3.5) and (4.11) with Laplace's method stipulates that function

$$\overline{\mathbf{F}}_{\beta}^{(\theta)}(x) := \mathbf{P}\{\mathcal{Z}_{\beta}^{(\theta)} > x\} \sim \frac{1+\beta}{\Gamma(1-\beta)} \cdot \frac{1}{\theta \cdot \wp_{\beta}(\theta)} \cdot e^{-\theta x} \cdot x^{-(\beta+2)} \tag{4.12}$$

as $x \to \infty$ (compare to the derivation of formula (5.1.11) of Vinogradov [28]). It is also relevant that the above representation (4.12) is a special case of condition (0.16) therein. Our generic reference to Laplace's method is Ch. 4 of de Bruijn [3].

The laws which satisfy (4.12) are well studied. Thus, most of Ch. 5 of Vinogradov [28] is devoted to the distributions whose tails possess similar asymptotics. A subsequent combination of (4.12) with Th. 4 of Cline [5] implies that function $\overline{\mathbf{F}}_{\beta}^{(\theta)}$ belongs to the well-known class $\mathcal{S}(\theta)$ that was considered, among others, in Def. 1.4.9 of Embrechts et al. [11]. (This is an 'integral' analogue of the class $\mathcal{SD}(\theta)$ described in Definition 3.4.)

Next, apply Cor. 2.2.b of Klüppelberg [20] to obtain that function $p_\beta^{(\theta)} \in \mathcal{SD}(\theta)$. To conclude, it remains to utilize an argument provided on p. 267 of Klüppelberg [20]. □

Given $\ell \in \mathbf{N}$, let us define the conditioned r.v. $\mathcal{X}_{\beta,t}(\ell)$ as follows:

$$\mathcal{X}_{\beta,t}(\ell) \stackrel{\mathrm{d}}{=} (M_\beta(t) \ \mid \ \Pi(\mu_{\beta,1,t}) > \ell), \tag{4.13}$$

where the latter two r.v.'s are related by virtue of formula (2.7). It is easy to see that the Laplace transform $\lambda_{\mathcal{X}_{\beta,t}(\ell)}(\cdot)$ of r.v. $\mathcal{X}_{\beta,t}(\ell)$ admits the next representation:

$$\begin{aligned} &\lambda_{\mathcal{X}_{\beta,t}(\ell)}(u) = (\mathbf{P}\{\Pi(\mu_{\beta,1,t}) > \ell\})^{-1} \\ &\cdot \left[\hbar_{M_\beta(t)}(u) - \exp\{-\mu_{\beta,1,t}\} \cdot \left(1 + \sum_{k=1}^{\ell} \frac{\mu_{\beta,1,t}^k}{k!} \cdot \wp_\beta(u/\mu_{\beta,1,t})^k \right) \right], \end{aligned} \tag{4.14}$$

where $u \geq 0$. Set

$$\ell_\beta := \min\{\ell \in \mathbf{N} : \ell > 1/\beta - 1\}. \tag{4.15}$$

Now, we present three lemmas which will be used in the proof of Theorem 2.1.

Lemma 4.1. *(i) For each fixed $\ell \in \mathbf{N}$,*

$$\lambda_{\mathcal{X}_{\beta,t}(\ell)}(u) = \mathcal{O}(u^{-(\ell+1)\cdot\beta}) \tag{4.16}$$

as $u \to \infty$.
(ii) The r.v. $\mathcal{X}_{\beta,t}(\ell_\beta)$ defined by (4.13)–(4.15) has a bounded continuous density on $\mathbf{R}^1$.

Proof. (i) It is straightforward and relies on the use of representations (2.11) and (2.9).
(ii) It easily follows by a combination of (4.16) with the inversion formula (cf., e.g., Th. XV.3 of Feller [12]). □

The following lemma demonstrates that the convolution properties of function $p_\beta(x)$ have some resemblance with those of the exponential density.

Lemma 4.2. *(i) For each fixed $\ell \in \mathbf{N}$, the ℓ-fold 'convolution' $p_\beta^{\ell\otimes}(\cdot)$ exhibits the following power-law behavior at the origin:*

$$p_\beta^{\ell\otimes}(x) \sim \frac{1}{\beta^\ell \cdot \Gamma(\ell\cdot\beta)} \cdot x^{\ell\cdot\beta-1} \tag{4.17}$$

as $x \downarrow 0$.
(ii) For $\ell < 1/\beta$, function $p_\beta^{\ell\otimes}(x)$ blows up at zero, whereas in the case when $\ell \geq \ell_\beta + 1$, the density $p_\beta^{\ell\otimes}(x)$ is bounded on $\mathbf{R}^1$.
(iii) $\forall \ell \in \mathbf{N}$, function $p_\beta^{\ell\otimes}(x) \in \mathbf{C}(\mathbf{R}_+^1)$.

Proof. (i) The proof is carried out by induction in ℓ. The induction base is given by (3.6). The proof of the induction step is easily obtained by substituting $\phi(x) := p_\beta(x)$, $\gamma(x) := p_\beta^{\ell\otimes}(x)$ into (3.7) with the subsequent application of asymptotic representations (3.6) and (4.17).
(ii) The first statement immediately follows from (4.17). The proof of the second assertion is similar to that of Lemma 4.1.
(iii) See (3.4) for $\ell = 1$. It is generally true that the convolution improves regularity

properties. However, the rigorous proof of this folk theorem seems to be available only for the probability laws which are continuous on the whole $\mathbf{R}^1$ (or $\mathbf{R}^k$). It is unfortunate that due to the presence of a spike at the origin (see (3.6)), these results do not appear to be applicable in our setting. However, in the case when $\beta \in (1/2, 1)$ the proof is almost identical to that of Lemma 4.1.ii. For the remaining values of $\beta \in (0, 1/2]$, there is a direct straightforward proof by induction. It relies on the combination of standard $\epsilon - \delta$ arguments with the power-law behavior given by (3.6) and (4.17). The details are left to the reader. □

Let us combine formulas (2.10) and (3.5) with Th. 5 of Sec. 35 by Gnedenko and Kolmogorov [14] to get that

$$(\mathcal{Z}_\beta(1) + ... + \mathcal{Z}_\beta(n) - n)/n^{1/(1+\beta)} \stackrel{d}{\to} \mathcal{R}_{1+\beta,\rho_2}(1) \tag{4.18}$$

as $n \to \infty$. Here, $\rho_2 := 1/\beta$, and $\mathcal{R}_{1+\beta,\rho}(1)$ is the stable r.v. with density $\zeta_{1+\beta,\rho}(\cdot)$ (see (2.14) and (2.15)). Set $\mathcal{P}_n := \mathbf{P}\{\Pi(\mu_{\beta,1,t}) = n\}$. The proof of the next statement relies on a refinement of the local counterpart of (4.18).

Lemma 4.3. *Suppose that $y \to \infty$. Then*

$$\mathcal{Q} := \sum_{n=1}^{\infty} \mathcal{P}_n \cdot \ | \ p_\beta^{n\otimes}(y) - n \cdot p_\beta(y) \ | = o(y^{-(\beta+2)}).$$

Proof. Let us split $\mathcal{Q}$ into two parts. The first is the sum over $1 \le n \le \ell_\beta$; the other series corresponds to the values of $n \ge \ell_\beta + 1$. By Proposition 3.6, the first sum is $o(y^{-(\beta+2)})$ as $y \to \infty$. It is evident that the second (infinite) series

$$\begin{aligned} &\sum_{n=\ell_\beta+1}^{\infty} \mathcal{P}_n \cdot \ | \ p_\beta^{n\otimes}(y) - n \cdot p_\beta(y) \ | \\ \le &\sum_{n=\ell_\beta+1}^{\infty} \mathcal{P}_n \cdot | \ n \cdot p_\beta(y) - n^{-1/(1+\beta)} \cdot \zeta_{1+\beta,\rho_2}((y-n)/n^{1/(1+\beta)}) \ | \\ + &\sum_{n=\ell_\beta+1}^{\infty} \mathcal{P}_n \cdot \ | \ p_\beta^{n\otimes}(y) - n^{-1/(1+\beta)} \cdot \zeta_{1+\beta,\rho_2}((y-n)/n^{1/(1+\beta)}) \ | \ . \end{aligned} \tag{4.19}$$

It is relatively easy to show that the first sum that emerges on the right-hand side of (4.19) is $o(y^{-(\beta+2)})$ as $y \to \infty$. This follows from the scaling and asymptotic properties of stable density $\zeta_{1+\beta,\rho_2}(\cdot)$ and its derivatives (see (2.15) and the reference above that formula).

A bound of the same magnitude for the rightmost sum in (4.19) also holds. This follows from the *non-uniform* estimate (4.20) of the remainder, which takes into account large deviations. Numerous non-uniform estimates of such character in 'integral' limit theorems are given in author's monograph [28] (see also references therein). However, the derivation of similar bounds in *local* theorems is a less developed subject.

Next, a relatively straightforward adaptation of the method developed by Inzhevitov [16] stipulates that $\forall \epsilon > 0$, $\exists$ constant $\mathcal{C}_\epsilon \in \mathbf{R}^1_+$ such that $\forall n \geq \ell_\beta + 1$ and $\forall x \in \mathbf{R}^1$,

$$\begin{aligned} &\mid n^{1/(1+\beta)} \cdot p_\beta^{n\otimes}(x \cdot n^{1/(1+\beta)} + n) - \zeta_{1+\beta,\rho_2}(x) - \Delta_\beta(x,n) \mid \\ &\qquad\qquad \leq \frac{\mathcal{C}_\epsilon}{n^{2/(1+\beta)} \cdot (1+ \mid x \mid)^{\beta+3-\epsilon}} \, . \end{aligned} \tag{4.20}$$

Here, $\Delta_\beta(x,n)$ is a finite functional series comprised of the refinining terms of the asymptotic expansion of the scaled density $n^{1/(1+\beta)} \cdot p_\beta^{n\otimes}(x \cdot n^{1/(1+\beta)} + n)$. All its members are asymptotically negligible compare to the principal term $\zeta_{1+\beta,\rho_2}(x)$. It is pertinent that the refining terms which comprise quantity $\Delta_\beta(x,n)$ that emerged in (4.20) are analogous to those which appeared in 'integral' theorems on stable convergence derived in author's monograph [28].

The boundedness of density $p_\beta^{n\otimes}(\cdot)$ $\forall n \geq \ell_\beta + 1$ was established in Lemma 4.2.ii. Hence, n^{th} term of the rightmost series in (4.19) does not exceed

$$\mathcal{P}_n \cdot \big(\, o(n^{-\beta/(1+\beta)} \cdot y^{-(\beta+2)}) + \mathcal{C}_\epsilon \cdot n^{(\beta-\epsilon)/(1+\beta)} \cdot (n^{1/(1+\beta)} + \mid y - n \mid)^{-(\beta+3-\epsilon)} \, \big) \, .$$

The rest is straightforward and left to the reader. □

Proof of Theorem 2.1. (i) It follows from the combination of Proposition 3.1 with (2.12), Prop. III.4.11.ii of Steutel and van Harn [27] and the well-known relations between different forms of the canonical representation of infinitely divisible laws that the difference $\mathbf{P}\{M_\beta(t) \leq x\} - \mathbf{P}\{M_\beta(t) = 0\}$ is absolutely continuous on $\mathbf{R}^1_+$. Fix $x \in \mathbf{R}^1_+$ and rewrite formulas (2.7) and (2.11) in terms of d.f.'s:

$$\begin{aligned} &\mathbf{P}\{M_\beta(t) \leq x\} - \exp\{-\mu_{\beta,1,t}\} \cdot \mathbf{E}_0(x) \\ &\qquad\qquad = \sum_{k=1}^{\infty} \mathcal{P}_k \cdot \mathbf{P}\{\mathcal{Z}_\beta(1) + ... + \mathcal{Z}_\beta(k) \leq x \cdot \mu_{\beta,1,t}\}. \end{aligned} \tag{4.21}$$

Here, $\mathbf{E}_0(\cdot)$ denotes d.f. of the degenerate law concentrated at the origin. The subsequent combination of (4.13)–(4.14) with (4.21) ascertains that

$$\begin{aligned} \mathbf{P}\{0 < M_\beta(t) \leq x\} &= \sum_{k=1}^{\ell_\beta} \mathcal{P}_k \cdot \mathbf{P}\{\mathcal{Z}_\beta(1) + ... + \mathcal{Z}_\beta(k) \leq x \cdot \mu_{\beta,1,t}\} \\ &\qquad + \mathbf{P}\{\Pi(\mu_{\beta,1,t}) > \ell_\beta\} \cdot \mathbf{P}\{\mathcal{X}_{\beta,t}(\ell_\beta) \leq x\}. \end{aligned} \tag{4.22}$$

Since the expression on the right-hand side of (4.22) is a finite linear combination of the specific functions, it can be differentiated term-wise. The rest follows from Lemmas 4.1.ii and 4.2.iii.

(iii) A combination of the arguments presented in the proof of part (i) with the formula given on p. 265 of Klüppelberg [20] implies that

$$\begin{aligned} f_{\beta,t}(x) = \; &\frac{d}{dx} \, \mathbf{P}\{0 < M_\beta(t) \leq x\} = \mu_{\beta,1,t} \cdot \sum_{k=1}^{\infty} \mathcal{P}_k \cdot p_\beta^{k\otimes}(x \cdot \mu_{\beta,1,t}) \\ &= \mu_{\beta,1,t} \cdot \sum_{k=1}^{\ell_\beta} \mathcal{P}_k \cdot p_\beta^{k\otimes}(x \cdot \mu_{\beta,1,t}) + \mathbf{P}\{\Pi(\mu_{\beta,1,t}) > \ell_\beta\} \cdot \mathbf{F}'_{\mathcal{X}_{\beta,t}}(x). \end{aligned} \tag{4.23}$$

Here, $\mathbf{F}_{\mathcal{X}_{\beta,t}}(\cdot)$ denotes d.f. of $\mathcal{X}_{\beta,t}$. The rest easily follows by combining (4.17), (4.23) and Lemma 4.1.ii.
(ii) The validity of (2.19) follows by combining (2.6), (3.5), (4.23) and Lemma 4.3. □

Proof of Lemma 3.7. (i) The Poisson mixture representation follows from the combination of formulas (6.2.3) and (8.2.3) by Panjer and Willmot [26], formula (1.18) of Dawson and Vinogradov [9], and the above formula (2.4).
(ii) The validity of (3.10) is obtained by combining part (i) with scaling property (2.5). □

Proof of Theorem 2.2. It is based on an application of representation (3.10) of Lemma 3.7 and the properties of function $f_{\beta,t}(x)$ established in Theorem 2.1. It is also relevant that the method of proof is similar to that of Th. 2.9.i of Vinogradov [31].
(ii) In order to derive (2.26), observe that the initial number of particles of BPS $\Upsilon^{(\eta)}_{\beta,1,t}$ which have alive descendents at time t constitutes a *Rao damage process*. Hence, it is Poisson distributed with mean $\mu^{(\eta)}_{\beta,1,t}$ (compare to p. 259 of Vinogradov [29]). The reader is referred to Sec. 9.2 of Johnson et al. [18] for the properties of such processes. Hence, $\mathbf{P}\{\widetilde{M}^{(\eta)}_{\beta}(t) = 0\} = \exp\{-\mu^{(\eta)}_{\beta,1,t}\}$. The rest is a slight refinement of (2.23).
(i) To prove (2.25), fix real $x > 0$. Consider those $\eta \to \infty$ for which $x \cdot \eta \in \mathbf{N}$, and apply Lemma 3.7.ii. By (2.5), $\mathbf{P}\{M_{\beta,\eta}(t \cdot \eta^{\beta}) = 0\} = \exp\{-\mu_{\beta,1,t}\}$.

In order to evaluate the integral that emerges on the right-hand side of (3.10), we make the change of variables $v := z/\eta$ and use Laplace's method (cf., e.g., Ch. 4 of de Bruijn [3]). In addition, apply Stirling's formula with the estimate of remainder. This stipulates the following *second-order approximation*:

$$\mathbf{P}(\eta \cdot \widetilde{M}^{(\eta)}_{\beta}(t) = x \cdot \eta) = \frac{1}{(x \cdot \eta)!} \cdot [\, \exp\{-\mu_{\beta,1,t}\} + \eta^{x \cdot \eta} \cdot \int_{0+}^{\infty} e^{-\eta(v - x \log\, v)} \cdot f_{\beta,t}(v) \cdot dv \,] = \frac{1}{\eta} \cdot f_{\beta,t}(x) + \mathcal{O}(1/\eta^2) \tag{4.24}$$

as $\eta \to \infty$. The latter integral is well defined due to the properties of function $f_{\beta,t}(v)$ established in Theorem 2.1. It is also pertinent that although the continuous function $f_{\beta,t}(v)$ blows up at the right-hand neighborhood of zero, but the rate of its growth given by Theorem 2.1.iii makes the contribution of such neighborhood negligible. □

Proof of Proposition 2.4. It is true that $\forall$ real $t > 0$,

$$\widetilde{M}^{(\eta)}_{\beta,m}(t) \stackrel{\mathrm{d}}{=} \sum_{k=1}^{\Pi(m \cdot \mu^{(\eta)}_{\beta,1,t})} \widehat{S}^{(\eta)}_{\beta,t}(k). \tag{4.25}$$

The validity of (4.25) can be established by following the arguments of the proof of Prop. 1.10 of Dawson and Vinogradov [9]. These arguments are similar to those given on p. 259 of Vinogradov [29] and in Sec. 2 of Vinogradov [30] in the context of convergence to the total mass of the *continuous* super-Brownian motion. Here, r.v.'s $\{\widehat{S}^{(\eta)}_{\beta,t}(k), k \geq 1\}$ which emerge in (4.25) are independent copies of *positive* r.v. $\widehat{S}^{(\eta)}_{\beta,t}$. It is defined by conditioning as follows: $\forall s \in \eta^{-1} \cdot \mathbf{N}$,

$$\mathbf{P}\{\widehat{S}^{(\eta)}_{\beta,t} = s\} = \mathbf{P}\{S^{(\eta)}_{\beta,t} = s \mid S^{(\eta)}_{\beta,t} > 0\} = \mathbf{P}\{S^{(\eta)}_{\beta,t} = s\}/Q^{(\eta)}_{\beta,t}, \tag{4.26}$$

where r.v. $S^{(\eta)}_{\beta,t}$ is characterized by its Laplace transform (2.30).

By (4.25), $\widetilde{M}^{(\eta)}_{\beta,m}(t)$ is compound Poisson. The validity of (2.31) follows by combining (4.25)–(4.26) with the above-quoted results given on p. 12 of Bertoin [1]. □

Proof of Theorem 2.3. The method of proof is similar to that of Th. 2.9.ii of Vinogradov [31]. It employs the Poisson approximation and Theorem 2.2.
(ii) The validity of (2.28) is obtained by employing formula (1.13') of Dawson and Vinogradov [9]: $\mathbf{P}\{M^{(\eta)}_{\beta}(t) = 0\} = (1 - Q^{(\eta)}_{\beta,t})^{\eta}$. The rest follows from formula (2.24).
(i) The proof of (2.27) is of the same character as that of formula (2.32) of Vinogradov [31]. In addition, it employs representations (4.25)–(4.26). Fix real $x > 0$ and consider those $\eta \to \infty$ for which $x \cdot \eta \in \mathbf{N}$. It is relatively easy to demonstrate that

$$\mathbf{P}\{\eta \cdot M^{(\eta)}_{\beta}(t) = x \cdot \eta\} = \mathbf{P}\left\{ \eta \cdot \sum_{i=1}^{\mathbf{B}(\eta, Q^{(\eta)}_{\beta,t})} \widehat{S}^{(\eta)}_{\beta,t}(i) = x \cdot \eta \right\}. \tag{4.27}$$

Recall that $\mathbf{B}(\eta, Q^{(\eta)}_{\beta,t})$ is a binomial r.v. with the corresponding values of parameters. This r.v. is assumed to be independent of the sequence $\{\widehat{S}^{(\eta)}_{\beta,t}(i),\ i \geq 1\}$ of r.v.'s. The latter ones are independent copies of r.v. $\widehat{S}^{(\eta)}_{\beta,t}$ whose law is given by (4.26).

Next, we approximate binomial r.v. $\mathbf{B}(\eta, Q^{(\eta)}_{\beta,t})$ with a Poisson r.v. $\Pi(\mu^{(\eta)}_{\beta,1,t})$. The latter variable is also assumed to be independent of the above sequence $\{\widehat{S}^{(\eta)}_{\beta,t}(i),\ i \geq 1\}$. The remainder of this approximation will be estimated by employing sharp upper bounds in the local Poisson theorem given in Karymov [19].

By analogy to the arguments used in the proof of Th. 2.9.ii of Vinogradov [31], one obtains that the probability that emerges on the right-hand side of (4.27) admits the following representation:

$$\begin{aligned} &\sum_{k=1}^{\infty} \mathbf{P}\{\eta \cdot \sum_{i=1}^{k} \widehat{S}^{(\eta)}_{\beta,t}(i) = x \cdot \eta \mid \Pi(\mu^{(\eta)}_{\beta,1,t}) = k\} \cdot \mathbf{P}\{\Pi(\mu^{(\eta)}_{\beta,1,t}) = k\} \\ &\qquad - \sum_{k=\eta+1}^{\infty} \mathbf{P}\{\eta \cdot \sum_{i=1}^{k} \widehat{S}^{(\eta)}_{\beta,t}(i) = x \cdot \eta\} \cdot \mathbf{P}\{\Pi(\mu^{(\eta)}_{\beta,1,t}) = k\} \\ &+ \sum_{k=1}^{\eta} \mathbf{P}\{\eta \cdot \sum_{i=1}^{k} \widehat{S}^{(\eta)}_{\beta,t}(i) = x\eta\} \cdot (\mathbf{P}\{\mathbf{B}(\eta, Q^{(\eta)}_{\beta,t}) = k\} - \mathbf{P}\{\Pi(\mu^{(\eta)}_{\beta,1,t}) = k\}). \end{aligned} \tag{4.28}$$

We will demonstrate below that the middle and rightmost sums which emerge in formula (4.28) are asymptotically negligible. Also, the leftmost sum that appears in (4.28) pertains to a Poisson random sum of i.i.d.r.v.'s. It is relatively easy to see that this sum equals

$$\mathbf{P}\{\eta \cdot \widetilde{M}^{(\eta)}_{\beta}(t) = x \cdot \eta\} \tag{4.29}$$

(compare to (4.25)). Hence, the evaluation of (4.29) is reduced to Theorem 2.2.i.

Next, it is obvious that the absolute value of the middle sum that emerges in formula (4.28) does not exceed $\mathbf{P}\{\Pi(\mu^{(\eta)}_{\beta,1,t}) > \eta\}$. The latter probability is easily estimated by virtue of the exponential Chebyshev inequality. It decays towards zero faster than any negative power of η. The details are left to the reader.

It remains to estimate the rightmost sum that emerges in (4.28). To this end, we will utilize the following upper bound for the tail probabilities of partial sums of i.i.d.r.v.'s. It is assumed that the tails of the individual terms have a power type, which is justified by (2.30). Then Prop. 1.1.1 of Vinogradov [28] stipulates that $\exists$ constant $\mathcal{C}_\beta \in \mathbf{R}^1_+$ such that $\forall\ 1 \le k \le x \cdot \eta$,

$$\mathbf{P}\left\{ \eta \cdot \sum_{i=1}^{k} \widehat{S}^{(\eta)}_{\beta,t}(i) = x \cdot \eta \right\} \le \mathcal{C}_\beta \cdot k \cdot (x \cdot \eta)^{-(1+\beta)}. \tag{4.30}$$

Also, the fact that each $\eta \cdot \widehat{S}^{(\eta)}_{\beta,t}(i) \ge 1$ implies that $\forall$ integer $k > x \cdot \eta$, the probability that emerges on the left-hand side of (4.30) equals zero.

At this stage, we decompose the rightmost sum over k that emerges in (4.28) into two parts. The first sum $\sum_1$ includes the values of $k \le$ *Const*, whereas the second sum $\sum_2$ pertains to the values of index k which tend to infinity with η. In order to estimate the absolute value of $\sum_1$, we combine (4.30) with the *uniform* upper bound in the local Poisson theorem (cf., e.g., Cor. 1 of Karymov [19]). One easily derives that

$$\mid \Sigma_1 \mid = \mathcal{O}(1/\eta^{2+\beta}) \tag{4.31}$$

as $\eta \to \infty$. In addition, the sum over $r(\eta) \le k \le \min(x \cdot \eta, \eta)$ is estimated by the use of the *nonuniform* upper bound in the local Poisson theorem given by Th. 4 of Karymov [19]. Here, $r(\eta)$ is a certain (non-random) numerical sequence that tends to infinity as $\eta \to \infty$. A combination of this bound with (4.30) ascertains that

$$\mid \Sigma_2 \mid \le \frac{D_1}{\eta^{2+\beta}} \cdot \sum_{\ell=r(\eta)}^{\infty} \frac{\ell \cdot D_2^\ell}{(\ell-2)^{\ell-2}}\ . \tag{4.32}$$

Here, D_1 and D_2 are certain positive constants which depend on β, t and x but do not depend on η. The rest is trivial, since the sum that emerges on the right-hand side of (4.32) constitutes the tail of a convergent series. To conclude, combine (4.30)–(4.32). □

Proof of Corollary 3.2. It is easily obtained by a combination of Proposition 3.1.i with Th. III.10.7 of Steutel and van Harn [27]. □

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