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FUNCTIONAL LIMIT THEOREMS FOR TRACE PROCESSES IN A DYSON BROWNIAN MOTION

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Abstract. In this paper we study functional asymptotic behavior of p-trace processes of $n \times n$ Hermitian matrix valued Brownian motions, when n goes to infinity. For each $p \geq 1$ we establish uniform a.s. and L^q laws of large numbers and study the a.s. convergence of the supremum (respectively infimum) over a compact interval of the largest (respectively smallest) eigenvalue process. We also prove that the fluctuations around the limiting process, converge weakly to a one-dimensional centered Gaussian process Z_p, given as a Wiener integral with a deterministic Volterra kernel. This process depends on $Z_{p-1}, ..., Z_1$ and a Gaussian martingale of independent interest whose increasing process is explicitly derived. Our approach is based on stochastic analysis and semimartingales tools.

1. Introduction

For $n \geq 1$, let $\left\{B^{(n)}(t)\right\}_{t\geq 0} = \{(B_{jk}(t))\}_{t\geq 0}$ be an $n \times n$ Hermitian matrix-valued Brownian motion scaled by $1/\sqrt{n}$. That is, $(B_{jj}(t))_{j=1}^{n}, (\operatorname{Re} B_{jk}(t))_{j<k}$, $(\operatorname{Im} B_{jk}(t))_{j<k}$ is a set of n^2 independent one-dimensional Brownian motions with parameter $\frac{t}{2n}(1+\delta_{jk})$. For each $t > 0$, $B^{(n)}(t)$ is a Gaussian Unitary (GU) random matrix of parameter t/n ([11], [22]).

Let $\left\{\lambda^{(n)}(t)\right\}_{t\geq 0} = \{(\lambda_1^{(n)}(t), \lambda_2^{(n)}(t), ..., \lambda_n^{(n)}(t))\}_{t\geq 0}$ be the n-dimensional stochastic process of eigenvalues of $B^{(n)}$. In a pioneering and fundamental work, Dyson [9] showed that if the eigenvalues start at different positions ($\lambda_1^{(n)}(0) < \lambda_2^{(n)}(0) < ... < \lambda_n^{(n)}(0)$ *a.s.*), then they never meet at any time ($\lambda_1^{(n)}(t) < \lambda_2^{(n)}(t) < ... < \lambda_n^{(n)}(t)$ *a.s.* $\forall\, t > 0$) and furthermore they form a diffusion process satisfying the Itô Stochastic Differential Equation (SDE)

$$\mathrm{d}\lambda_i^{(n)}(t) = \frac{1}{\sqrt{n}}\mathrm{d}W_i^{(n)}(t) + \frac{1}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{\lambda_i^{(n)}(t) - \lambda_j^{(n)}(t)}, \quad t \geq 0, 1 \leq i \leq n, \tag{1.1}$$

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where $W_1^{(n)}, ..., W_n^{(n)}$ are independent one-dimensional standard Brownian motions. The stochastic process $\left\{\lambda^{(n)}(t)\right\}_{t\geq 0}$ is called the Dyson non-colliding Brownian motion corresponding to the Gaussian Unitary Ensemble (GUE), which we refer here as the (n-dimensional) *Dyson-Brownian model.*

From the diffusion processes point of view, the SDE (1.1) governs a system of interacting Brownian particles with a non-smooth drift coefficient.

An important object of study is the empirical measure process

$$\mu_t^{(n)} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j^{(n)}(t)}, \quad t \geq 0, \tag{1.2}$$

where δ_x is the unit mass at x. From the celebrated Wigner theorem in random matrices, one obtains that for each fix $t > 0$, $\mu_t^{(n)}$ converges a.s. to μ_t^{sc}, the Wigner semicircle distribution of parameter t, that is

$$\mu_t^{sc}(\mathrm{d}x) = \frac{1}{2\pi t}\sqrt{4t - x^2}1_{\left[-2\sqrt{t}, 2\sqrt{t}\right]}(x)\mathrm{d}x. \tag{1.3}$$

(See for example [11], [22], [27] or [28]). Also for a fixed $t > 0$, the behavior of the largest and smallest eigenvalues of GUE random matrices was established in [2], [3] (see also [11]). In the framework of dynamics and stochastic processes it is then natural to consider functional limit theorems for the process $\left(\mu_t^{(n)}\right)_{t\geq 0}$.

The study of the limit of interacting diffusions, but with smooth drift and diffusion coefficients, goes back to the pioneering work on *propagation of chaos* of McKean [21]. His result is a law of large numbers: the sequence of corresponding empirical measure processes $\mu_t^{(n)}$ converges to μ_t in probability, where $\mu_t(\mathrm{d}x)$ is the probability distribution of a real valued stochastic process satisfying an Itô SDE. The corresponding central limit theorem or limit of the fluctuations $S_n(t) = n^{1/2}(\mu_t^{(n)} - \mu_t)$ was considered by several authors in the Nineteen Eighties; see for example [14], [18], [23], [25], [26]. In particular, Hitsuda and Mitoma [14] have shown that the measure valued processes $S_n(t)$ converge weakly to a Gaussian process in the dual of a nuclear Fréchet space.

Systems of Itô SDE with non smooth drift coefficients arise naturally in the study of eigenvalue processes of matrix valued stochastic processes; see for example [6], [19], [20] and references therein. The asymptotic behavior of the empirical measure $\mu_t^{(n)}$ of eigenvalues of matrix valued Ornstein-Uhlenbeck (OU) processes (matrices whose entries are one-dimensional OU processes rather than Brownian motions) was initially considered by Chan [8] and Rogers and Shi [24] (see also [7]). They realized that $\mu_t^{(n)}$ converges weakly in the space of continuous probability measure valued processes to a deterministic law μ_t. Moreover, the limit has a unique stationary measure μ_∞ which follows a scaled semicircle law. Although formally their systems of eigenvalues processes contain the classical Dyson Brownian model (1.1), not all their proofs and results hold for this model.

From the stochastic realization point of view, it is not known if the family of semicircle laws governs a real Itô equation. Rather, it is well known that (1.3) is the (spectral) distribution of the so called free Brownian motion, the analogous in free probability of the classical Brownian motion ([4, Example 5.16]).

As for the central limit theorem, also in the general framework of eigenvalues systems of matrix-valued OU processes, Israelson [15] proved that the fluctuations $Y_n(t) = n(\mu_t^{(n)} - \mu_t)$ converge weakly to a Gaussian process in the dual of a nuclear Fréchet space, whose mean and covariance functions were explicitly derived recently by Bender [5]. These models include the case of the classical Dyson-Brownian motion (1.1), but no formulation nor proof for the corresponding result is given [15, Remark pp 27]. It is important to notice that the fluctuation process is considered with the scale factor n instead $\sqrt{n}$, as it is done for interacting diffusions with smooth coefficients and other classical cases.

In the present paper we are concerned with functional limit theorems for the p-moment or p-trace processes associated to $\mu_t^{(n)}$ in the Dyson-Brownian motion model (1.1), for any $p \geq 0$. Namely, we consider propagation of chaos and fluctuations for the one-dimensional stochastic processes $(\{M_{n,p}(t)\}_{t\geq 0}, n \geq 1), p \geq 0$, defined by the semimartingales

$$M_{n,p}(t) = \mathrm{Tr}(\left[B^{(n)}(t)\right]^p) = \int_{\mathbb{R}} x^p \mu_t^{(n)}(\mathrm{d}x) = \frac{1}{n}\sum_{j=1}^{n}\left[\lambda_j^{(n)}(t)\right]^p. \tag{1.4}$$

For a fixed $t > 0$, the importance of the study of moments for the GUE is illustrated, for example, in the works by Metha [22], Haagerup and Thorbjørnsen [11], Harer-Zagier [12] and Johanson [17], amongst others. An important role in those papers is played by the moments of the semicircle law μ_t^{sc} for fix $t > 0$. It is then natural to consider functional limit theorems for the dynamics of the p-trace stochastic processes, specially in the framework of stochastic analysis and semimartingales.

In the propagation of chaos direction, in Section 3 we show that for the Dyson Brownian model, the sequence of measure-valued processes $\mu_t^{(n)}$ converges weakly to μ_t^{sc} in the space of continuous functions from $\mathbb{R}_+$ into probability measures in $\mathbb{R}$, endowed with the uniform convergence on compact intervals of $\mathbb{R}_+$. We also prove uniform a.s. and in L^{2k} laws of large numbers. In proving these results we first show in Section 2 that the family $(\mu_t^{sc})_{t\geq 0}$ is characterized by the property that its Cauchy-Stieltjes transforms is the unique solution of an initial value problem, a result formally suggested from the work in [24]. In Section 3 we also prove the a.s. convergence of the supremum over the interval $[0,T]$ of the largest eigenvalue process $\left\{\lambda_n^{(n)}(t)\right\}_{t\geq 0}$ to $2\sqrt{T}$ as well as the corresponding result for the infimum of the smallest eigenvalue process $\left\{\lambda_1^{(n)}(t)\right\}_{t\geq 0}$.

In section 4 we address the question of weak convergence for the fluctuations of the moment processes $V_{n,p}(t) = \int x^p Y_t^{(n)}(\mathrm{d}x)$. It is shown that for each $p \geq 0$, $V_{n,p}$ converges to a one-dimensional Gaussian process Z_p given in terms of the previous $(p-1)$th limiting processes Z_1,...,Z_{p-1}, the Catalan numbers of order up to $(p-2)/2$ and a Gaussian martingale which is a $\frac{p+1}{2}-$self-similar process. The process Z_p is also written as a Wiener integral with a deterministic Volterra kernel.

2. Families of moments of semicircle laws

Consider the family $\{\mu_t^{sc}\}_{t>0}$ of Wigner semicircle laws given by (1.3). In this section we recall some useful properties of the moments $\mu_k^{sc}(t)$ for $t>0$ fixed and present new functional relations for the families $\{\mu_k^{sc}(t)\}_{t>0}$ and their corresponding Cauchy-Stieltjes transforms.

It is well known that for $t>0$ fixed, the odd moments $\mu_{2p+1}^{sc}(t)$ are 0 while the even moments are given by

$$\mu_{2p}^{sc}(t) = C_p t^p, p \geq 0, \tag{2.1}$$

where $C_p = \binom{2p}{p}/(p+1)$ are the so called Catalan numbers (see [10], [13]). We write $\mu_0^{sc} = \delta_0$.

The following functional recursive equation for the families of moments of semicircle laws holds.

Lemma 2.1. *For each $r \geq 2$ and $t > 0$*

$$\mu_r^{sc}(t) = \frac{r}{2}\sum_{j=0}^{r-2}\int_0^t \mu_{r-2-j}^{sc}(s)\mu_j^{sc}(s)\mathrm{d}s. \tag{2.2}$$

Proof. The result is trivial if r is odd. For $r=2p$, $p\geq 1$, using the fact that the odd moments are 0 (from the RHS of (2.2), relation (2.1) and the following well known formula for the Catalan numbers

$$C_{k+1} = \sum_{j=0}^{k} C_j C_{k-j},$$

it is easily seen that (2.2) is satisfied. □

We next present a characterization of the family of distributions $(\mu_t^{sc})_{t\geq 0}$ in terms of an initial valued problem for the corresponding Cauchy-Stieltjes transforms. Recall that for a finite non-negative measure ν on $\mathbb{R}$, its *Cauchy-Stieltjes transform* is defined by

$$G^\nu(z) = \int_{\mathbb{R}} \frac{\nu(\mathrm{d}x)}{x-z},$$

for all $z\in\mathbb{C}$ with $\mathrm{Im}(z)\neq 0$. It is well known that G^ν is analytic in $\mathbb{C}\setminus\mathbb{R}$, $\overline{G^\nu(z)} = G^\nu(\bar{z})$, $G^\nu : \mathbb{C}^+ \to \mathbb{C}^+$, where $\mathbb{C}^+ := \{z : \mathrm{Im}(z) > 0\}$ and $\lim_{\eta\to\infty}\eta\,|G^\nu(i\eta)| < \infty$ (see for example [13]).

For the semicircle law μ_t^{sc}, writing $G_t^{sc} = G^{\mu_t^{sc}}$, we have the relation

$$G_t^{sc}(z) = \frac{1}{2t}\left(\sqrt{z^2-4t} - z\right), t>0, \mathrm{Im}(z)\neq 0. \tag{2.3}$$

where $\sqrt{z^2-4t}$ denotes the branch that has the asymptotic behavior

$$\sqrt{z^2-4\sigma^2} = z + O(|z|^{-1}),\ z\to\infty.$$

Lemma 2.2. *The family $(\mu_t^{sc})_{t\geq 0}$ is characterized by the property that its Cauchy-Stieltjes transforms is the unique solution of the initial value problem*

$$\begin{cases} \frac{\partial G_t(z)}{\partial t} = & G_t(z)\frac{\partial G_t(z)}{\partial z}, \quad t>0, \\ G_0(z) = & -\frac{1}{z}, \qquad\qquad z\in\mathbb{C}^+, \end{cases} \tag{2.4}$$

which satisfies $G_t(z) \in \mathbb{C}^+$ *for* $z \in \mathbb{C}^+$ *and*

$$\lim_{\eta \to \infty} \eta \left| G_t(i\eta) \right| < \infty, \text{ for each } t > 0. \tag{2.5}$$

Proof. It is easily seen that (2.3) satisfies (2.4). To show the uniqueness of the solution we proceed as in [24, Section 4]. Consider the following differential equation in $\mathbb{C}^+$,

$$\dot{z}_t = -G_t(z_t), z_0 = h \in \mathbb{C}^+. \tag{2.6}$$

From (2.4) it follows that $\ddot{z}_t = 0$, that is $\dot{z}_t = \dot{z}_0 = -\ G_0(h) = \frac{1}{h}$ and hence $z_t = \frac{t}{h} + h$.

For fixed $t_0 > 0, \theta \in \mathbb{C}^+$, choose for (2.6) the initial condition $h = h(t_0, \theta) \in \mathbb{C}^+$ such that

$$z_{t_0}(\theta) = \frac{t_0}{h(t_0,\theta)} + h(t_0,\theta) = \theta,$$

i.e.,

$$h(t_0,\theta) = \frac{\theta \pm \sqrt{\theta^2 - 4t_0}}{2}.$$

Then we obtain that

$$G_{t_0}(\theta) = -\dot{z}_{t_0} = -\dot{z}_0 = \frac{1}{h(t_0,\theta)} = \frac{1}{2t_0}\left(\sqrt{\theta^2 - 4t_0} \mp \theta\right) \in \mathbb{C}^+.$$

But only in the case $G_{t_0}(\theta) = \frac{1}{2t_0}\left(\sqrt{\theta^2 - 4t_0} - \theta\right)$ the condition (2.5) is satisfied. □

3. Propagation of chaos for the moments

In this section we prove the weak convergence of $\left(\mu_t^{(n)}\right)_{t \geq 0}$ to a measure valued process satisfying an evolution equation and prove uniform a.s. and L^q laws of large numbers for the moment processes $M_{n,p}(t) = \mathrm{Tr}(\left[B^{(n)}(t)\right]^p), t \geq 0$, given by (1.4).

Let $\mathrm{Pr}(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$ endowed with the topology of weak convergence and let $C(\mathbb{R}_+, \mathrm{Pr}(\mathbb{R}))$ be the space of continuous functions from $\mathbb{R}_+$ into $\mathrm{Pr}(\mathbb{R})$, endowed with the topology of uniform convergence on compact intervals of $\mathbb{R}_+$. As it is usual, for a probability measure μ and a μ-integrable function f we use the notation $\langle \mu_t, f \rangle = \int f(x)\mu(\mathrm{d}x)$.

Proposition 3.1. *Assume that* $\mu_0^{(n)}$ *converges weakly to* δ_0.

Then the family of measure-valued processes $\left(\mu_t^{(n)}\right)_{t \geq 0}$ *converges weakly in* $C(\mathbb{R}_+, \mathrm{Pr}(\mathbb{R}))$ *to the unique continuous probability-measure valued function satisfying that for each* $f \in C_b^2(\mathbb{R})$

$$\langle \mu_t, f \rangle = f(0) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y). \tag{3.1}$$

Moreover, the unique continuous solution of (3.1) is the family of semicircle laws $(\mu_t^{sc})_{t \geq 0}$.

Proof. An application of Itô's formula to (1.1) gives that for $f \in C_b^2$,

$$\left\langle \mu_t^{(n)}, f \right\rangle = \left\langle \mu_0^{(n)}, f \right\rangle + \frac{1}{n^{\frac{3}{2}}} \sum_{j=1}^{n} \int_0^t f'\left(\lambda_j^{(n)}(s)\right) \mathrm{d}W_j^{(n)}(s)$$

$$+\frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \frac{f'(x)-f'(y)}{x-y} \mu_s^{(n)}(\mathrm{d}x)\mu_s^{(n)}(\mathrm{d}y), t \geq 0. \tag{3.2}$$

The proof of tightness of the sequence of processes $\left\{\left(\mu_t^{(n)}\right)_{t\geq 0}\right\}_{n\geq 1}$ in the space $C\left(\mathbb{R}_+, \Pr(\mathbb{R})\right)$ is the same as in [24, Section 3].

By Doob's inequality, for any $\varepsilon, T > 0$, we have

$$\sum_n P\left(\sup_{0\leq t\leq T}\left|\frac{1}{n^{\frac{3}{2}}}\sum_{j=1}^n \int_0^t f'\left(\lambda_j^{(n)}(s)\right)\mathrm{d}W_j^{(n)}(s)\right| > \varepsilon\right)$$

$$\leq \frac{4}{\varepsilon^2}\sum_n \frac{1}{n^3}\sum_{j=1}^n \int_0^T \left[f'\left(\lambda_j^{(n)}(s)\right)\right]^2 \mathrm{d}s \leq K\sum_n \frac{1}{n^2} < \infty,$$

and therefore

$$\sup_{0\leq t\leq T}\left|\frac{1}{n^{\frac{3}{2}}}\sum_{j=1}^n \int_0^t f'\left(\lambda_j^{(n)}(s)\right)\mathrm{d}W_j^{(n)}(s)\right| \xrightarrow{a.s.} 0, \text{ as } n\to\infty.$$

Now, it is clear that any weak limit $(\mu_t)_{t\geq 0}$ of a subsequence $\left(\mu_t^{(n_k)}\right)_{t\geq 0}$ should satisfy (3.1). Applying (3.1) to the determining sequence of functions

$$f_j(x) = \frac{1}{x - z_j}, \quad z_j \in (\mathbb{Q}\times\mathbb{Q}) \cap \mathbb{C}^+,$$

and using a continuity argument, we get that the Cauchy-Stieltjes transform $(G_t)_{t\geq 0}$ of $(\mu_t)_{t\geq 0}$ satisfies the integral equation

$$G_t(z) = -\frac{1}{z} + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \frac{\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y)}{(x-z)(y-z)^2}, \quad t\geq 0, z\in\mathbb{C}^+. \tag{3.3}$$

From (3.3) it easily seen that (2.4) is satisfied and consequently $(\mu_t)_{t\geq 0}$ is the family $(\mu_t^{sc})_{t\geq 0}$. Therefore all limits of subsequences of $\left\{\left(\mu_t^{(n)}\right)_{t\geq 0}\right\}_{n\geq 1}$ coincide with $(\mu_t^{sc})_{t\geq 0}$ and thus the sequence $\left(\mu_t^{(n)}\right)_{t\geq 0}$ converges weakly to $(\mu_t^{sc})_{t\geq 0}$.

The above reasoning also shows that $(\mu_t^{sc})_{t\geq 0}$ is the unique continuous solution of (3.1). □

Remark 3.2. The Ornstein-Uhlenbeck case when $\left\{\left(\lambda_1^{(n)}(t), ..., \lambda_n^{(n)}(t)\right)\right\}_{t\geq 0}$ satisfy the SDE

$$\mathrm{d}\lambda_i^{(n)}(t) = \frac{\sigma}{\sqrt{n}}\mathrm{d}W_i^{(n)}(t) - \theta\lambda_i^{(n)}(t)\mathrm{d}t + \frac{\alpha}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{\lambda_i^{(n)}(t) - \lambda_j^{(n)}(t)}, t\geq 0, \tag{3.4}$$

with $\sigma, \theta, \alpha > 0$, is considered by [8] and [24]. The method used to prove Proposition 3.2 is similar as in [24].

The next goal is to prove uniform a.s. and L^{2k}, for each $k \geq 1$, laws of large numbers for the trace processes $M_{n,p}$. The first part of the next result gives useful recursive equations systems for the semimartingales $M_{n,p}$ in terms of the martingales

$$X_{n,p}(t) = \frac{1}{n^{\frac{3}{2}}} \sum_{j=1}^{n} \int_0^t \left[\lambda_j^{(n)}(s)\right]^p \mathrm{d}W_j^{(n)}(s), t \geq 0, \tag{3.5}$$

whose increasing processes are given by

$$\langle X_{n,p} \rangle_t = \frac{1}{n^2} \int_0^t M_{n,2p}(s) \mathrm{d}s, t \geq 0, \tag{3.6}$$

for any $p \geq 0$ and $n \geq 1$.

Theorem 3.3. *(i) The following relations hold for $n \geq 1, r \geq 1$ and $t \geq 0$*

$$M_{n,r}(t) = M_{n,r}(0) + rX_{n,r-1}(t) + \frac{r}{2} \sum_{j=0}^{r-2} \int_0^t M_{n,r-2-j}(s) M_{n,j}(s) \mathrm{d}s, t \geq 0, \tag{3.7}$$

where $X_{n,p}(t)$ is the martingale given by (3.5).

(ii) Assume that for each $p \geq 1, k \geq 1$,

$$\sup_n E\left(M_{n,2p}^{2k}(0)\right) < \infty, \tag{3.8}$$

$$M_{n,2p}(0) \xrightarrow{a.s.} 0 \text{ as } n \to \infty. \tag{3.9}$$

Then for every $T > 0$ there exist a constant $K(p,k,T)$ such that

$$\sup_n E\left(\sup_{0 \leq t \leq T} M_{n,2p}^{2k}(t)\right) \leq K(p,k,T) < \infty, \tag{3.10}$$

and

$$\sup_{0 \leq t \leq T} \left|M_{n,2p}(t) - \mu_{2p}^{sc}(t)\right| \xrightarrow{a.s.} 0 \text{ as } n \to \infty, \tag{3.11}$$

$$E\left(\sup_{0 \leq t \leq T} \left|M_{n,2p}(t) - \mu_{2p}^{sc}(t)\right|^{2k}\right) \longrightarrow 0 \text{ as } n \to \infty. \tag{3.12}$$

Proof. (i) The relation (3.7) follows from (3.2) with $f(x) = x^r$.

(For example, from (3.7) we have

$$\begin{aligned} M_{n,0}(t) &= M_{n,0}(0) + 1, \\ M_{n,1}(t) &= M_{n,1}(0) + X_{n,0}(t), \\ M_{n,2}(t) &= M_{n,2}(0) + 2X_{n,1}(t) + t, \text{ etc.}) \end{aligned} \tag{3.13}$$

(ii) The Harer-Zagier formula recursion formula for the moments $M_{n,2q}(t) = \mathrm{Tr}(\left[B^{(n)}(t)\right]^{2q})$ for $t > 0$ fixed (see [11, Corollary 4.2], [12, pp 460], [22, pp 117-120]) and the equality in law between $B_{ij}^{l,n}(t)$ and $\sqrt{t}B_{ij}^{l,n}(1)$, $l = 1, 2$, imply the equality

$$E\left(M_{n,2q}(t)\right)=\left(\sum_{j=0}^{\left[\frac{q}{2}\right]}\alpha_j(q)n^{-2j}\right)t^q,\ q\geq 0,n\geq 1, \tag{3.14}$$

where

$$\alpha_j(q)=\begin{cases}=0, & j\geq\left[\frac{q}{2}\right]+1,\\ =C_q, & j=0,q\geq 0\end{cases},$$

and

$$\alpha_j(q+1)=\frac{4q+2}{q+2}\alpha_j(q)+\frac{q(4q^2-1)}{q+2}\alpha_{j-1}(q-1),\ q,j\geq 1.$$

Then, from the above relations and Jensen's inequality we have the estimate

$$E\left(M_{n,2p}^{2k}(t)\right)\leq E\left(M_{n,4kp}(t)\right)\leq K(p,k)t^{2kp}. \tag{3.15}$$

Next, by Burkholder's inequality and using (3.6) we obtain

$$\begin{aligned} E\left(\sup_{0\leq t\leq T}|X_{n,p}(t)|^{2k}\right) &\leq \frac{K(T,p,k)}{n^{2k}}E\left(\left|\int_0^T M_{n,2p}(s)\mathrm{d}s\right|^k\right)\\ &\leq \frac{K_1(T,p,k)}{n^{2k}}\int_0^T E\left(M_{n,2p}^k(s)\right)\mathrm{d}s \end{aligned} \tag{3.16}$$

and then, by (3.15),

$$E\left(\sup_{0\leq t\leq T}|X_{n,p}(t)|^{2k}\right)\leq\frac{K_2(T,p,k)}{n^{2k}}\leq K_2(T,p,k)<\infty. \tag{3.17}$$

Hence, (3.10) follows using the Minkovski and Hölders inequalities, (3.8), (3.15) and (3.17) in (3.7).

On the other hand, using Chebyshev inequality and (3.17) we have that for each $\varepsilon>0$

$$\sum_n P\left(\sup_{0\leq t\leq T}|X_{n,p}(t)|>\varepsilon\right)\leq K\sum_n\frac{1}{n^{2k}}<\infty,$$

and thus

$$\sup_{0\leq t\leq T}|X_{n,p}(t)|\overset{a.s.}{\longrightarrow}0 \text{ as } n\to\infty. \tag{3.18}$$

The almost surely convergence in (3.11) follows from (3.9), (3.18) and (3.7) by an induction argument, since the family $(\mu_r^{sc}(t))_{t,r\geq 0}$ satisfies uniquely the relation (2.2). Now (3.10) and (3.11) yield (3.12). □

Approximating the continuous functions with compact support by polynomials we obtain the following consequences of the above theorem.

Corollary 3.4. *Assume (ii) in Theorem 3.3. Then*

a) For any bounded continuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ we have

$$\lim_{n\to\infty}\sup_{0\leq t\leq T}\left|\int f(x)\mu_t^{(n)}(\mathrm{d}x)-\int f(x)\mu_t^{sc}(\mathrm{d}x)\right|=0 \ a.s. \tag{3.19}$$

b) For any interval $(a,b) \subset \mathbb{R}$

$$\lim_{n\to\infty} \max_{0\le t\le T} \left| \frac{1}{n} \# \left\{ 1 \le j \le n : \lambda_j^{(n)}(t) \in (a,b) \right\} - \mu_t^{sc}((a,b)) \right| = 0 \ a.s. \tag{3.20}$$

Remark 3.5. As a consequence of the above theorem we obtain Arnold's extension of the Wigner semicircle law (see [1], [11]), since the moments determines uniquely the semicircle law.

Remark 3.6. The recursive relation (3.7) suggests that the moment processes $M_{n,2p}$ are continuous functionals for the martingales $X_{n,0}$, $X_{n,1}$, ..., $X_{n,2p-1}$. An explicit formula of $M_{n,2p}$ would be useful (in particular to obtain (3.14)).

The behavior of the largest and smallest eigenvalues of GUE random matrices was established in [2], [3] (see also [11]). In the next theorem we extend these results for the supremum of the largest eigenvalue process as well as for the infimum of the smallest eigenvalue process from a Dyson Brownian motion.

Theorem 3.7. *Assume (ii) of Theorem 3.3. Then, for each* $T > 0$ *we have*

$$\max_{0\le t\le T} \lambda_n^{(n)}(t) \xrightarrow{a.s.} 2\sqrt{T} \ \text{as } n \to \infty, \tag{3.21}$$

$$\min_{0\le t\le T} \lambda_1^{(n)}(t) \xrightarrow{a.s.} -2\sqrt{T} \ \text{as } n \to \infty. \tag{3.22}$$

Proof. From the estimate (3.6) of [11] we have

$$E\left[\exp\left(\alpha\lambda_n^{(n)}(t)\right)\right] \le n \exp\left(\frac{\alpha^2 t}{2n} + 2\alpha\sqrt{t}\right), \ \forall \alpha, t > 0. \tag{3.23}$$

Next, if $t_1 < t_2$ from (1.1) and the fact that

$$\exp\left(\frac{\alpha}{\sqrt{n}} W_n^{(n)}(t) - \frac{\alpha^2 t}{2n}\right)$$

is a martingale, we have

$$E\left[\exp\left(\alpha\lambda_n^{(n)}(t_2)\right) \mid B^{(n)}(s) : s \le t_1\right]$$

$$= E\left[\exp\left(\frac{\alpha}{\sqrt{n}} W_n^{(n)}(t_2) + \frac{\alpha}{n}\sum_{j=1}^{n-1}\int_0^{t_2} \frac{ds}{\lambda_n^{(n)}(s) - \lambda_j^{(n)}(s)}\right) \mid B^{(n)}(s) : s \le t_1\right]$$

$$\ge \exp\left(\frac{\alpha}{n}\sum_{j=1}^{n-1}\int_0^{t_1} \frac{ds}{\lambda_n^{(n)}(s) - \lambda_j^{(n)}(s)}\right) E\left[\exp\left(\frac{\alpha}{\sqrt{n}} W_n^{(n)}(t_2)\right) \mid B^{(n)}(s) : s \le t_1\right]$$

$$= \exp\left(\frac{\alpha^2 t_2}{2n} + \frac{\alpha}{n}\sum_{j=1}^{n-1}\int_0^{t_1} \frac{ds}{\lambda_n^{(n)}(s) - \lambda_j^{(n)}(s)}\right)$$

$$\times E\left[\exp\left(\frac{\alpha}{\sqrt{n}} W_n^{(n)}(t_2) - \frac{\alpha^2 t_2}{2n}\right) \mid B^{(n)}(s) : s \le t_1\right]$$

$$= \exp\left(\frac{\alpha^2 (t_2 - t_1)}{2n} + \frac{\alpha}{n}\sum_{j=1}^{n-1}\int_0^{t_1} \frac{ds}{\lambda_n^{(n)}(s) - \lambda_j^{(n)}(s)}\right) \exp\left(\frac{\alpha}{\sqrt{n}} W_n^{(n)}(t_1)\right)$$

$$\geq \exp\left(\alpha\lambda_n^{(n)}(t_1)\right),$$

i.e., $\exp\left\{\alpha\lambda_n^{(n)}(t)\right\}_t$ is a submartingale.

From (3.23) and Doob's inequality we obtain

$$P\left(\max_{0\leq t\leq T}\lambda_n^{(n)}(t) > \varepsilon + 2\sqrt{T}\right)$$

$$\leq P\left(\max_{0\leq t\leq T}\exp\left(\alpha\lambda_n^{(n)}(t)\right) > \exp(\alpha\left(\varepsilon + 2\sqrt{T}\right))\right)$$

$$\leq \exp\left(-\alpha\left(\varepsilon+2\sqrt{T}\right)\right)E\left(\exp\left(\alpha\lambda_n^{(n)}(T)\right)\right) \leq n\exp\left(-\alpha\varepsilon+\frac{\alpha^2T}{2n}\right),$$

and the function

$$\alpha \longrightarrow \exp\left(-\alpha\varepsilon+\frac{\alpha^2T}{2n}\right),\ \alpha>0,$$

attains its minimum for $\alpha = n\varepsilon$, and replacing above, we get the inequality

$$P\left(\max_{0\leq t\leq T}\lambda_n^{(n)}(t) > \varepsilon + 2\sqrt{T}\right) \leq n\exp\left(-\frac{n\varepsilon^2}{2}\right). \tag{3.24}$$

Hence from (3.24) and Borel-Cantelli lemma

$$\limsup_{n\longrightarrow\infty}\max_{0\leq t\leq T}\lambda_n^{(n)}(t) \leq 2\sqrt{T}, \quad \text{a.s.}. \tag{3.25}$$

Next, from (3.20) we have

$$\max_{0\leq t\leq T}\left|\frac{1}{n}\#\left\{1\leq j\leq n : \lambda_j^{(n)}(t)\in[a,b]\right\}\right| \xrightarrow{a.s.} \max_{0\leq t\leq T}\mu_t^{sc}([a,b]) \text{ as } n\to\infty,$$

and then

$$\max_{0\leq t\leq T}\#\left\{1\leq j\leq n : \lambda_j^{(n)}(t)\in\left[2\sqrt{T}-\varepsilon, 2\sqrt{T}\right]\right\} \xrightarrow{a.s.} \infty \text{ as } n\to\infty,$$

and consequently

$$\liminf_{n\longrightarrow\infty}\max_{0\leq t\leq T}\lambda_n^{(n)}(t) \geq 2\sqrt{T}, \quad \text{a.s.}. \tag{3.26}$$

From (3.25), (3.26) we obtain (3.21). Finally, (3.22) follows from (3.21) applied to $\left(-B^{(n)}(t)\right)_{t\geq 0}$. □

4. Fluctuations of the moments

In this section we consider the asymptotic fluctuations of the moments processes $\{M_{n,p}(t)\}_{t\geq 0}$ around the corresponding moments $\left\{\mu_p^{sc}(t)\right\}_{t\geq 0}$ of the semicircle distribution. Let

$$Y_t^{(n)} = n\left(\mu_t^{(n)} - \mu_t^{sc}\right), \tag{4.1}$$

$V_{n,0}(t) = 0$ and for $p\geq 1$

$$V_{n,p}(t) = \int x^p Y_t^{(n)}(\mathrm{d}x) = n\left(M_{n,p}(t) - \mu_p^{sc}(t)\right). \tag{4.2}$$

The relation (3.1) and (3.2) imply that for $f \in C^2$ and $t \geq 0$,

$$\begin{aligned}\left\langle Y_t^{(n)}, f\right\rangle =& \left\langle Y_0^{(n)}, f\right\rangle \\ &+ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_0^t f'\left(\lambda_j^{(n)}(s)\right) \mathrm{d}W_j^{(n)}(s) - \int_0^t \int x f'(x) Y_s^{(n)}(\mathrm{d}x)\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \frac{f'(x)-f'(y)}{x-y}\left[\mu_s^{(n)}(\mathrm{d}x)Y_s^{(n)}(\mathrm{d}y) + \mu_s^{sc}(\mathrm{d}x)Y_s^{(n)}(\mathrm{d}y)\right].\end{aligned} \tag{4.3}$$

The martingales

$$N_{n,p}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_0^t \left[\lambda_j^{(n)}(s)\right]^p \mathrm{d}W_j^{(n)}(s), \quad t \geq 0, p \geq 0, \tag{4.4}$$

play an important role in the Dyson Brownian model (1) and in particular in the sequel. In the next result we prove their weak convergence to an additive Gaussian processes.

Proposition 4.1. *The martingale $N_{n,p}$ converges weakly in $C(\mathbb{R}_+, \mathbb{R})$, when n goes to infinity, to a centered Gaussian martingale N_p with covariance function*

$$E\left(N_p(s)N_p(t)\right) = \frac{C_p}{p+1}\left(s \wedge t\right)^{p+1} \tag{4.5}$$

and increasing process

$$\left\langle N_p\right\rangle_t = \int_0^t \mu_{2p}^{sc}(s)\mathrm{d}s = \frac{C_p}{p+1} t^{p+1}. \tag{4.6}$$

Proof. Since

$$\left\langle N_{n,p}\right\rangle_t = \int_0^t M_{n,2p}(s)\mathrm{d}s,$$

by Burkholder's inequality and (3.15) we have that for $t_1 < t_2 \leq T$,

$$\begin{aligned}E\left(\left|N_{n,p}(t_1) - N_{n,p}(t_2)\right|^4\right) &\leq K_1 E\left(\left|\int_{t_1}^{t_2} M_{n,2p}(s)\mathrm{d}s\right|^2\right) \\ &\leq K_2\left(t_2 - t_1\right)\int_{t_1}^{t_2} E\left(M_{n,2p}^2(s)\mathrm{d}s\right) \\ &\leq K(T,p)\left(t_2 - t_1\right)^2\end{aligned} \tag{4.7}$$

and thus the sequence $\left(N_{n,p}\right)_n$ is tight in $C(\mathbb{R}_+, \mathbb{R})$.

Let $\left(N_{n_k,p}\right)_k$ be a weakly convergent subsequence to a limit N_p. By [16, Corollary 1.19, pp 486] it follows that N_p is a continuous local martingale (in fact is a martingale) and by [16, Corollary 6.6, pp 342] the vector $(N_{n_k,p}, \langle N_{n_k,p}\rangle)$ converges weakly to a limit $\left(N_p, \langle N_p\rangle\right)$.

Since

$$\left\langle N_{n_k,p}\right\rangle_t = \int_0^t M_{n_k,2p}(s)\mathrm{d}s,$$

by using (3.11) we get (4.6). Then, from [16, Theorem 4.4 pp 102] we have that that N_p is a Gaussian martingale, whose covariance is given by (4.5). Therefore all weak limits are the same and consequently the sequence $N_{n,p}$ converges weakly to N_p. □

Remark 4.2. It is clear that the additive centered Gaussian process is such that

$$N_p(t) \stackrel{L}{=} C_p^{\frac{1}{2}} \int_0^t s^{\frac{p}{2}} \mathrm{d}W_s, \tag{4.8}$$

where W is a Brownian motion and N_p is $\frac{p+1}{2}$−self-similar.

In the final result of this paper we show for each $p \geq 1$, the fluctuation processes $V_{n,p}$ converge weakly to a one dimensional Gaussian process Z_p, which is given by a recursive expression that involves $Z_{p-1}, ..., Z_1$, the Gaussian martingale N_{p-1} and the families of moments $\{\mu_k^{sc}(t)\}_{t\geq 0}, k = 1, ..., p-2$. It is mentioned in Israelson [15, Remark pp 27], that the proof of a result from which the following theorem is obtained, can be adapted from a general result ($\theta > 0$) in [15]. For the sake of the reader convenience, we present here a simpler and direct proof and in addition we identify the limiting trace processes explicitly.

Theorem 4.3. *Assume that for each* $p, k \geq 1$

$$\sup_n E\left(V_{n,p}^{2k}(0)\right) < \infty, \tag{4.9}$$

and that $V_{n,p}(0)$ *converges weakly to* $V_p^{(0)} \in \mathbb{R}$ *as* $n \to \infty$. *Then* $V_{n,p}$ *converges weakly in* $C(\mathbb{R}_+, \mathbb{R})$ *to the centered Gaussian process* Z_p *satisfying* $Z_0 = 0$,

$$\begin{aligned} Z_p(t) + p\int_0^t Z_p(s)\mathrm{d}s = V_p^{(0)} + \frac{p}{2}\int_0^t \{2\left[\mu_{p-2}^{sc}(s) + \mu_{p-3}^{sc}(s)Z_1(s)\right. \\ \left. +... + \mu_1^{sc}(s)Z_{p-3}(s)\right] + Z_{p-2}(s)\}\,\mathrm{d}s + pN_{p-1}(t), \end{aligned} \tag{4.10}$$

where N_p *is given by (4.8).*

Remark 4.4. Two alternative expression for the process Z_p is given as follows.
a)Write

$$\begin{aligned} a_p(t) = V_p^{(0)} + \frac{p+1}{2}\int_0^t \{2\left[\mu_{p-1}^{sc}(s) + \mu_{p-2}^{sc}(s)Z_1(s)\right. \\ \left. +... + \mu_1^{sc}(s)Z_{p-2}(s)\right] + Z_{p-1}(s)\}\,ds + (p+1)N_p(t), p \geq 1, \end{aligned}$$

then

$$Z_p(t) = a_{p-1}(t) - p\int_0^t e^{-p(t-s)} a_{p-1}(s)\mathrm{d}s. \tag{4.11}$$

b) There is a measurable deterministic Volterra kernel K_p such that

$$Z_p(t) = \int_0^t K_p(t, s)dW_s. \tag{4.12}$$

*Proof. of Theorem 4.3***.** Taking $f(x) = x^p$ in (4.3) we obtain the equality

$$V_{n,p}(t) = V_{n,p}(0) - p\int_0^t V_{n,p}(s)\mathrm{d}s$$

$$+\frac{p}{2}\int_0^t \left\{M_{n,p-2}(s) + \mu_{p-2}^{sc}(s) + \left[M_{n,p-3}(s) + \mu_{p-3}^{sc}(s)\right]V_{n,1}(s)\right.$$

$$\left. + ... + \left[M_{n,1}(s) + \mu_1^{sc}(s)\right]V_{n,p-3}(s) + V_{n,p-2}(s)\right\}\mathrm{d}s + pN_{n,p-1}(t). \tag{4.13}$$

By the Skorohod representation of the weak convergence (eventually in a new probability space) we can assume that

$$\left((V_{n,k}(0)_{1\le k\le p}), (M_{n,k})_{1\le k\le p-2}), (N_{n,k})_{0\le k\le p-1})\right)$$

converges almost surely in $\mathbb{R}^p \times C(\mathbb{R}_+, \mathbb{R}^{2(p-1)})$ to

$$\left(\left(V_k^{(0)}\right)_{1\le k\le p}, (\mu_k^{sc})_{1\le k\le p-2}, (N_k)_{0\le k\le p-1}\right),$$

and still (4.13) is satisfied.

Then, by induction we deduce that $V_{n,p}$ converges almost surely to Z_p given by (4.10). From (4.11) and by induction again it is easily seen that (4.12) holds and consequently Z_p is Gaussian. □

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