Communications on Stochastic Analysis
Vol. 1, No. 3 (2007) 429-439

SCHRÖDINGER TYPE EQUATION ASSOCIATED WITH THE LÉVY AND VOLTERRA LAPLACIANS

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Abstract. In this paper, we discuss the application of white noise analysis to the Feynman path integral in the field theory. By introducing normal coordinates which permit us to describe the system as a set of independent harmonic oscillators, we calculate the Feynman path integral using white noise functionals. Moreover, we give an infinite dimensional Schrödinger type equation associated with the Lévy and Volterra Laplacians by introducing new renormalizations of the integral.

1. Introduction

A method of calculating the Feynman path integrals in terms of distribution theory was introduced in Streit, L. and Hida, T. [15]. This method has been studied by many authors in [4,9,12,14] and references cited therein.

This method is difficult to be applied to the quantum field theory without taking proper coordinates. We have to change the coordinates represented in the Lagrangian. In this paper, we use the best changing of coordinates called normal coordinates to calculate the Feynman path integrals in terms of an infinite dimensional stochastic analysis. With these calculations and by introducing new renormalizations, we give an infinite dimensional Schrödinger type equation.

This paper is organized as follows. In section 2, we make preparations to compute the Feynman path integral of the field theory using white noise functionals in the framework of the stochastic processes. This white noise formulation owes a great deal to Kuo, H.-H.[4].

Previously, in [14], when calculating the above path integral, we adopted the Brownian motion as a function of the linear combination of both time and space, as for a fluctuation of field function. In section 3, we consider a chain of many particles connected next to each other by springs, and assume that each particle moves according to a Brownian motion process. Then, following Feynman, R. P., Hibbs, A. R.[1], we introduce normal coordinates which permit us to describe the system as a set of independent harmonic oscillators. For each harmonic oscillator with a particular mode, we calculate the Feynman path integral using white noise functionals. Thus we have the result on the Feynman path integral for this whole system as a product of solutions for each path integral with one of the various modes. Then increasing the number of particles, and finally taking their limit to

2000 *Mathematics Subject Classification.* 60H40.
Key words and phrases. Infinite dimensional Schrödinger type equation, The Feynman path integral, The Lévy Laplacian, The Volterra Laplacian.

infinity, we expect to have the formula for the Feynman path integral of the field theory.

In this process, we give the relation between two kinds of Brownian motions for fluctuations. One Brownian motion is introduced for the fluctuation in the coordinate expressing the displacement of each particle and the other is for the fluctuation in the normal coordinate. We clarify the fact that one is just the inverse Fourier transform of the other.

Finally, by introducing the new renormalization $\mathcal{R}_\ell K(t,q)$, $t>0$, $q\in E[0,1]$, we give an infinite dimensional Schrödinger type equation expressed by Lévy's Laplacian Δ_L and Volterra's Laplacian Δ_V:

$$\frac{\partial \mathcal{R}_\ell K}{\partial t}(t,q)=\frac{i\hbar}{m}\left[\left(\int_0^1 f(t,x)dx\right)^{-1}\frac{\ell}{t}\Delta_L+\Delta_V\right]\mathcal{R}_\ell K(t,q)+\frac{1}{i\hbar}V(q)\cdot\mathcal{R}_\ell K(t,q),$$

(see Theorem 3.2.).

2. Standard setup of white noise calculus

Let $\mathbf{T}$ be an interval on $\mathbf{R}^d$ and let $L^2(\mathbf{T})\equiv L^2(\mathbf{T},m_L)$ be the Hilbert space of real valued square-integrable functions on $\mathbf{T}$ with inner product $(\cdot,\cdot)$, where m_L is the Lebesgue measure on $\mathbf{T}$.

We take a densely defined self-adjoint operator A on $L^2(\mathbf{T})$ with a domain $\mathrm{Dom}(A)$ satisfying the following conditions:

(A1) there exists an orthonormal basis $\{e_\nu;\nu\geq 0\}\subset\mathrm{Dom}(A)$ for $L^2(\mathbf{T})$ such that $Ae_\nu=\lambda_\nu e_\nu,\ \nu=0,1,2,\cdots$;

(A2) $1<\lambda_0<\lambda_1<\cdots$;

(A3) $\sum_{\nu=0}^\infty \lambda_\nu^{-2}<\infty$.

Obviously A^{-1} is extended to an operator of Hilbert-Schmidt class. Define the norm $|\cdot|_p$ by $|f|_p=|A^pf|_0$ for $f\in\mathrm{Dom}(A)$ and $p\in\mathbf{R}$, and let $E_p=E_p(\mathbf{T})$ be the completion of $\mathrm{Dom}(A)$ with respect to the norm $|\cdot|_p$. Where, $|\cdot|_0$ denotes the $L^2(\mathbf{T})$-norm. Then $E_p=E_p(\mathbf{T})$ is a real separable Hilbert space with the norm $|\cdot|_p$ and the dual space E_p' of E_p is the same as E_{-p} (see Ref.14). Let $E=E(\mathbf{T})$ be the projective limit space of $\{E_p;p\geq 0\}$ and let $E^*=E^*(\mathbf{T})$ be the dual space of E. Then we obtain a Gel'fand triple

$$E=E(\mathbf{T})\subset L^2(\mathbf{T})\subset E^*=E^*(\mathbf{T}).$$

Let μ be the measure on E^* satisfying

$$\int_{E^*}\exp\{i\langle x,\xi\rangle\}\,d\mu(x)=\exp\left(-\frac{1}{2}|\xi|_0^2\right),\ \xi\in E.$$

Here, $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E^*\times E$. We denote the complexifications of $L^2(\mathbf{T})$, E and E_p by $L^2_{\mathbf{C}}(\mathbf{T})$, $E_{\mathbf{C}}$ and $E_{\mathbf{C},p}$, respectively.

The space $(L^2)=L^2(E^*,\mu)$ of complex-valued square-integrable functionals defined on E^* admits the well-known Wiener-Itô decomposition:

$$(L^2)=\bigoplus_{n=0}^\infty H_n,$$

where H_n is the space of multiple Wiener integrals of order $n \in \mathbf{N}$ and $H_0 = \mathbf{C}$. Let $L^2_{\mathbf{C}}(\mathbf{T})^{\hat{\otimes} n}$ denote the n-fold symmetric tensor product of $L^2_{\mathbf{C}}(\mathbf{T})$. If $\varphi \in (L^2)$ is represented by $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n),\ f_n \in L^2_{\mathbf{C}}(\mathbf{T})^{\hat{\otimes} n}$, then the (L^2)-norm $\|\varphi\|_0$ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n! |f_n|_0^2 \right)^{1/2},$$

where $|\cdot|_0$ means also the norm of $L^2_{\mathbf{C}}(\mathbf{T})^{\hat{\otimes} n}$.

For $p \in \mathbf{R}$, let $\|\varphi\|_p = \|\Gamma(A)^p \varphi\|_0$, where $\Gamma(A)$ is the second quantization operator of A. If $p \geq 0$, let $(E)_p$ be the domain of $\Gamma(A)^p$. If $p < 0$, let $(E)_p$ be the completion of (L^2) with respect to the norm $\|\cdot\|_p$. Then $(E)_p$, $p \in \mathbf{R}$, is a Hilbert space with the norm $||\cdot||_p$. It is easy to see that for $p > 0$, the dual space $(E)_p^*$ of $(E)_p$ is given by $(E)_{-p}$. Moreover, for any $p \in \mathbf{R}$, we have the decomposition

$$(E)_p = \bigoplus_{n=0}^{\infty} H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{\mathbf{I}_n(f); f \in E_{\mathbf{C}}^{\hat{\otimes} n}\}$ with respect to $\|\cdot\|_p$. Here $E_{\mathbf{C}}^{\hat{\otimes} n}$ is the n-fold symmetric tensor product of $E_{\mathbf{C}}$. We also have

$$H_n^{(p)} = \{\mathbf{I}_n(f); f \in E_{\mathbf{C},p}^{\hat{\otimes} n}\}$$

for any $p \in \mathbf{R}$, where $E_{\mathbf{C},p}^{\hat{\otimes} n}$ is also the n-fold symmetric tensor product of $E_{\mathbf{C},p}$. The norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)_p$ is given by

$$\|\varphi\|_p = \left(\sum_{n=0}^{\infty} n! |f_n|_p^2 \right)^{1/2}, \ f_n \in E_{\mathbf{C},p}^{\hat{\otimes} n},$$

where the norm of $E_{\mathbf{C},p}^{\hat{\otimes} n}$ is denoted also by $|\cdot|_p$.

The projective limit space (E) of spaces $(E)_p$, $p \in \mathbf{R}$ is a nuclear space. The inductive limit space $(E)^*$ of spaces $(E)_p, p \in \mathbf{R}$ is nothing but the dual space of (E). The space $(E)^*$ is called the space of *generalized white noise functionals* or *white noise distributions*. We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical bilinear form on $(E)^* \times (E)$. Then we have

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle$$

for any Φ and φ represented by

$$\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in (E)^*, \quad \varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E),$$

where the canonical bilinear form on $(E_{\mathbf{C}}^{\otimes n})^* \times (E_{\mathbf{C}}^{\otimes n})$ is denoted also by $\langle \cdot, \cdot \rangle$.

For any $\Phi \in (E)^*$, the S-transform on $(E)^*$ is defined by

$$S[\Phi](\xi) = \langle\langle \Phi, \varphi_\xi \rangle\rangle, \qquad \xi \in E_{\mathbf{C}},$$

where $\varphi_\xi \equiv \exp\{\langle \cdot, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle\} \in (E)$.

Theorem 2.1. ([4]) *A complex-valued function F on $E_{\mathbf{C}}$ is the S-transform of an element of $(E)^*$ if and only if F satisfies the conditions:*

1) *For every $\xi, \eta \in E_{\mathbf{C}}$, the function $z \to F(\xi + z\eta),\ z \in \mathbf{C}$, is an entire function of z.*
2) *There exist non-negative constants K, a and p such that*

$$|F(\xi)| \leq K \exp\{a|\xi|_p^2\},\ \xi \in E_{\mathbf{C}}.$$

Theorem 2.2. ([4]) *Let $\Phi_n \in (E)^*$ and set $F_n = S\Phi_n,\ n = 1, 2, \ldots$. Then Φ_n is storongly convergent in $(E)^*$ if and only if the following conditions are satisfied:*

1) *For each $\xi \in E_{\mathbf{C}}$, there exists $\lim_{n\to\infty} F_n(\xi)$.*
2) *There exist non-negative constants K, a, p such that*

$$|F_n(\xi)| \leq K \exp\{a|\xi|_p^2\},\ \forall n \in \mathbf{N}, \xi \in E_{\mathbf{C}}.$$

Let $d = 1$ and set $\mathbf{T} = [0, T]$. Take the white noise space (E^*, μ). For a real number $c < -1$ or $c > 0$, consider an informal expression

$$\exp\left[-\frac{1}{2c}\int_0^T \dot{B}(u)^2 du\right].$$

To renormalize this quantity, take an orthonormal basis $\{e_\nu; \nu \geq 0\}$ for $L^2([0, T])$ and define

$$\Phi_n(\dot{B}) \equiv \prod_{k=1}^{n}\left(\frac{1+c}{c}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2c}\langle \dot{B}, e_k\rangle^2\right].$$

The functional Φ_n is in (L^2). The S-transform of Φ_n is given by

$$S\Phi_n(\xi) = \exp\left[-\frac{1}{2(1+c)}\sum_{k=1}^{n}\langle \xi, e_k\rangle^2\right], \xi \in E_{\mathbf{C}}.$$

Hence, for each ξ, we have

$$\lim_{n\to\infty} S\Phi_n(\xi) = \exp\left[-\frac{1}{2(1+c)}\int_0^T \xi(u)^2 du\right].$$

It is easy to see that, for all $n \geq 1$,

$$|S\Phi_n(\xi)| \leq \exp\left[\frac{1}{2|1+c|}|\xi|_0^2\right].$$

Thus by Theorem 2.2, the sequence Φ_n converges strongly in $(E)^*$. The limit $\lim_{n\to\infty} \Phi_n$ in $(E)^*$ is denoted by $\mathcal{N} \exp\left[-\frac{1}{2c}\int_0^T \dot{B}(u)^2 du\right]$ and is called the renormalization of $\exp\left[-\frac{1}{2c}\int_0^T \dot{B}(u)^2 du\right]$. Then $\mathcal{N}$ can be written informally as

$$\mathcal{N} = \left(\sqrt{\frac{1+c}{c}}\right)^{\infty}.$$

The S-transform of the renormalization is given by

$$S\left(\mathcal{N}\exp\left[-\frac{1}{2c}\int_0^T \dot{B}(u)^2du\right]\right)(\xi) = \exp\left[-\frac{1}{2(1+c)}\int_0^T \xi(u)^2du\right],\ \xi \in E_{\mathbf{C}}.$$

Note that the right hand side of this equality is defined for any complex number $c \neq -1$. This is the S-transform of some generalized white noise functional.

The following integrand $\mathcal{F}(T,x)$ of the Feynman path integral for a harmonic oscillator has been verified to be a generalized white noise functional in $(E)^*$ by Streit, L. and Hida, T. [15], see also Kuo, H.-H.[4].

$$\mathcal{F}(T,x) = \left(\mathcal{N}\exp\left[\frac{1}{2}(\frac{im}{\hbar}+1)\int_0^T \dot{B}(u)^2du\right]\right)\cdot$$
$$\cdot\exp\left[-\frac{i}{\hbar}\int_0^T V(x-B(u))du\right]\ \delta_x(B(t))$$

Here, $V(x) = \frac{1}{2}m\omega^2x^2$ is the potential energy for the harmonic oscillator and $\delta_x(B(t))$ is Donsker's delta function.

In section 3, we try to compute the S-transform of this generalized white noise functional $\mathcal{F}(T,x)$. This calculation itself is related to that of the Feynman path integral in the field theory directly.

3. Application to Quantum Field Theory

We now establish the mathematical formulation for the Feynman path integral in the quantum field theory using a stochastic process as fluctuation.

Let $T > 0$ and $X > 0$ be fixed. We consider the path integral

$$I(T,X) \equiv \mathcal{M}\int\exp\left[\frac{i}{\hbar}\int_0^T\int_0^X\left(\frac{m}{2}\left(\frac{\partial\widetilde{\phi}(t,x)}{\partial t}\right)^2 - \frac{k}{2}\left(\frac{\partial\widetilde{\phi}(t,x)}{\partial x}\right)^2\right)dxdt\right]\cdot$$
$$\cdot\,\delta(\widetilde{\phi}(T,X))\mathcal{D}\widetilde{\phi}, \tag{3.1}$$

where, $\mathcal{M}$ is a renormalizing factor.

We divide the interval $[0,X]$ into N equal parts and put $\widetilde{\phi}_n(t) = \widetilde{\phi}(t,\rho n)$ $(n = 1,2,...,N)$. Namely, we consider the system consisting of N mass points and $\widetilde{\phi}_n(t)$ representing the displacement of the n-th mass point located at the distance of ρn from the origin.

For example, let us consider the model in which some mass points are connected next to each other by a set of springs of equal length ρ, in a one-dimensional array. Here we consider that all mass points and springs have the same mass and the same spring modulus, respectively. The mass and the spring modulus are also denoted by m and k, respectively. Naturally, we should take the limit as $\rho \to 0$. For simplicity we assume $\rho N = X = 1$ and after some calculations we take the limit as $N \to \infty$. In this case, the potential energy term in (3.1) can be written as

$$V = \sum_{n=1}^N \frac{k}{2}\left(\widetilde{\phi}_{n+1}(t) - \widetilde{\phi}_n(t)\right)^2.$$

Thus the Lagrangian L in (3.1) becomes

$$L=\sum_{n=1}^{N}\frac{m}{2}\left(\frac{\partial\widetilde{\phi}_n(t)}{\partial t}\right)^2-\sum_{n=1}^{N}\frac{k}{2}\left(\widetilde{\phi}_{n+1}(t)-\widetilde{\phi}_n(t)\right)^2. \tag{3.2}$$

Within the stochastic framework, the path integral with the Lagrangian L as in (3.2) now becomes

$$I_N(T,\phi)\equiv\mathcal{M}\int_{E^*[0,T]}\exp\left[\frac{i}{\hbar}\int_0^T\sum_{n=1}^{N}\left(\frac{m}{2}\left(\frac{\partial\widetilde{\phi}_n(t)}{\partial t}\right)^2\right.\right.$$
$$\left.\left.-\frac{k}{2}\left(\widetilde{\phi}_{n+1}(t)-\widetilde{\phi}_n(t)\right)^2\right)dt\right]\prod_{n=1}^{N}\delta(\phi_n-B_n(T))\ \mathcal{D}\widetilde{\phi}_n(t), \tag{3.3}$$

where $\phi=(\phi_1,\ldots,\phi_N)\in\mathbf{R}^N$ and $\widetilde{\phi}_n(t)=\phi_n-B_n(t)$ with the Brownian motion $B_n(t)$ of the n-th mass point. Besides, the Donsker delta function $\delta(\phi_n-B_n(T))$ and $\mathcal{D}\widetilde{\phi}_n(t)$ can be written informally as

$$\delta(\phi_n-B_n(T))=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[iz_n\left(\phi_n-B_n(T)\right)\right]dz_n,$$

$$\mathcal{D}\widetilde{\phi}_n(t)=(2\pi)^{\infty}\exp\left[\frac{1}{2}\int_0^T\dot{B}_n(t)^2dt\right]d\mu(\dot{B}_n),$$

respectively. Note that $\dot{B}_n$ is a variable in $E^*[0,T]$.

We tried to calculate the path integral (3.3) directly in the same way as in section 2. In fact, we have carried out the calculation up to $N=5$. However, by using this method, it is difficult to obtain the general calculating result of this integral for any number N.

Therefore, following Feynman, R. P., Hibbs, A. R.[1], we analyse the system in another way using the normal coordinates. The solutions of the classical equations of motion for the Lagrangian L as in (3.2) are well known in general. We express them with the normal coordinates following [1]. Hereafter, for the simplicity, we sometimes abbreviate the notation for the time dependence of functions. For example, ϕ_n is written by $\varphi_n(t)$ with $\varphi_n(0)=\phi_n,\ n=1,2,\ldots$.

The classical equations of motion for the Lagrangian L as in (3.2) are the classical Lagrangian equations of motion. They also become

$$m\ddot{\varphi}_n=k\left(\varphi_{n+1}-2\varphi_n+\varphi_{n-1}\right),\ n=1,2,\ldots.$$

The solutions of these equations are of the form,

$$\varphi_n=Ae^{-i(n\beta-\omega t)},\ n=1,2,\ldots,$$

where β is a constant taking on a discrete set of values. Thus we must solve

$$-\omega^2\varphi_n=\frac{k}{m}\left(\varphi_{n+1}-2\varphi_n+\varphi_{n-1}\right),\ n=1,2,\ldots.$$

The frequency is given by

$$-\omega^2=\frac{k}{m}\left(e^{i\beta}-2+e^{-i\beta}\right)=-4\frac{k}{m}\sin^2\frac{\beta}{2}.$$

This gives the values of ω in terms of β, but not all values of β are allowed. The periodic boundary condition implies that

$$\beta = \frac{2\pi\alpha}{N},\ \alpha = 1, 2, \ldots.$$

We denote the frequency ω corresponding to $\beta = \frac{2\pi\alpha}{N}$ by ω_α.

We are now in a position to represent the various modes with their normal coordinates as

$$Q_\alpha(t) = \sum_{n=1}^{N} \frac{1}{\sqrt{N}} \varphi_n(t) e^{in\frac{2\pi\alpha}{N}}.$$

Then the Lagrangian can be expressed in the form:

$$L = \sum_{n=1}^{N} \left(\frac{m}{2}\dot{\varphi}_n^2 - \frac{k}{2}(\varphi_{n+1} - \varphi_n)^2 \right) = \frac{m}{2}\sum_{\alpha=1}^{N} \left(\dot{Q}_\alpha^* \dot{Q}_\alpha - \omega_\alpha^2 Q_\alpha^* Q_\alpha \right).$$

These coordinates are complex, but if one prefers real coordinates, one can define instead two real quantities Q_α^c, Q_α^s as coordinates by

$$Q_\alpha^c = \frac{1}{\sqrt{2}}(Q_\alpha + Q_\alpha^*), Q_\alpha^s = \frac{i}{\sqrt{2}}(Q_\alpha - Q_\alpha^*).$$

Then $Q_\alpha Q_\alpha^*$ is expressed in real values as

$$Q_\alpha Q_\alpha^* = Q_\alpha Q_{-\alpha} = \frac{1}{2}\left[(Q_\alpha^c)^2 + (Q_\alpha^s)^2 \right].$$

The Lagrangian is given by

$$L = \frac{m}{4}\sum_{\alpha=1}^{N} \left(\left[(\dot{Q}_\alpha^c)^2 + (\dot{Q}_\alpha^s)^2 \right] - \omega_\alpha^2 \left[(Q_\alpha^c)^2 + (Q_\alpha^s)^2 \right] \right).$$

Here, to establish the Feynman path integral for each harmonic oscillator in terms of white noise functionals, we can regard the classical quantities Q_α^c and Q_α^s as quantum quantities

$$\widetilde{Q}_\alpha^c(t) = q_\alpha^c - B_\alpha^c(t) \text{ and } \widetilde{Q}_\alpha^s(t) = q_\alpha^s - B_\alpha^s(t), \tag{3.4}$$

respectively. Thus, with this Lagrangian, the path integral (3.1) can be rewritten in the stochastic form as

$$\begin{aligned} &K_{0,N}(T,q) \equiv \\ &\mathcal{M}\int_{E^*[0,T]} \exp\left[\frac{im}{4\hbar}\sum_{\alpha=1}^{N}\int_0^T \left(\left[(\dot{\widetilde{Q}}_\alpha^c)^2 + (\dot{\widetilde{Q}}_\alpha^s)^2 \right] - \omega_\alpha^2 \left[(\widetilde{Q}_\alpha^c)^2 + (\widetilde{Q}_\alpha^s)^2 \right] \right) dt \right] \cdot \\ &\quad \cdot \prod_{\alpha=1}^{N} \delta\left(q_\alpha^c - B_\alpha^c(T) \right)\ \delta\left(q_\alpha^s - B_\alpha^s(T) \right)\ \mathcal{D}\widetilde{Q}_\alpha^c\ \mathcal{D}\widetilde{Q}_\alpha^s \\ &= \mathcal{M}\prod_{\alpha=1}^{N}\int_{E^*[0,T]} \exp\left[\frac{im}{4\hbar}\int_0^T \left((\dot{\widetilde{Q}}_\alpha^c)^2 - \omega_\alpha^2 (\widetilde{Q}_\alpha^c)^2 \right) dt \right] \delta\left(q_\alpha^c - B_\alpha^c(T) \right)\ \mathcal{D}\widetilde{Q}_\alpha^c \cdot \\ &\quad \cdot \int_{E^*[0,T]} \exp\left[\frac{im}{4\hbar}\int_0^T \left((\dot{\widetilde{Q}}_\alpha^s)^2 - \omega_\alpha^2 (\widetilde{Q}_\alpha^s)^2 \right) dt \right] \delta\left(q_\alpha^s - B_\alpha^s(T) \right)\ \mathcal{D}\widetilde{Q}_\alpha^s, \end{aligned} \tag{3.5}$$

where $\widetilde{Q} = (\widetilde{Q}^c_\alpha, \widetilde{Q}^s_\alpha;\ \alpha = 1, 2, \ldots, N)$ and $q = (q^c_\alpha, q^s_\alpha;\ \alpha = 1, 2, \ldots, N)$. Note that the whole system is described as a set of independent harmonic oscillators with a particular mode. We can regard $I(T, X)$ as a limit of some renormalization of $K_{0,N}(T, q)$ as $N \to \infty$.

Remark. Here, we would like to point out a relation between two kinds of coordinates with fractuations of Brownian motions. That is, the relation between $\widetilde{Q}_\alpha(t)$ and $\widetilde{\phi}_n(t)$. The setting in (3.4) remains the same by putting $\widetilde{\phi}_n(t) = \phi_n - B_n(t)$, i.e.,

$$\widetilde{\phi}_n(t) = \phi_n - B_n(t) = \frac{1}{\sqrt{N}} \sum_{\alpha=1}^{N} \widetilde{Q}_\alpha(t) e^{-in\frac{2\pi\alpha}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{\alpha=1}^{N} \frac{q^c_\alpha - iq^s_\alpha}{\sqrt{2}} e^{-i\frac{2\pi\alpha}{N}n} - \frac{1}{\sqrt{N}} \sum_{\alpha=1}^{N} \frac{B^c_\alpha(t) - iB^s_\alpha(t)}{\sqrt{2}} e^{-i\frac{2\pi\alpha}{N}n}. \tag{3.6}$$

On the most right-hand side, the first term equals to ϕ_n and the second term is nothing but the inverse discrete Fourier transform of the complex Brownian motion. On the other hand, the discrete Fourier transforms of ϕ_n and $B_n(t)$ can be written as,

$$q^c_\alpha = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \phi_n \cos(\frac{2\pi\alpha}{N}n),\ \ q^s_\alpha = -\sqrt{\frac{2}{N}} \sum_{n=1}^{N} \phi_n \sin(\frac{2\pi\alpha}{N}n),$$

and

$$B^c_\alpha(t) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} B_n(t) \cos(\frac{2\pi\alpha}{N}n),\ \ B^s_\alpha(t) = -\sqrt{\frac{2}{N}} \sum_{n=1}^{N} B_n(t) \sin(\frac{2\pi\alpha}{N}n),$$

respectively.

Similaly as in (3.3), the Donsker delta function δ and $\mathcal{D}\widetilde{Q}_\alpha(t)$ in (3.4) can be written informally as

$$\delta\left(q_\alpha - B_\alpha(T)\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[iz_\alpha\left(q_\alpha - B_\alpha(T)\right)\right] dz_\alpha,$$

$$\mathcal{D}\widetilde{Q}_\alpha(t) = (2\pi)^\infty \exp\left[\frac{1}{2}\int_0^T \dot{B}_\alpha(t)^2 dt\right] d\mu(\dot{B}_\alpha),$$

respectively. Note that $\dot{B}_\alpha$ denotes an element of $E^*[0, T]$ and the integral is over the white noise space $E^*[0, T]$ as before. As mentioned in section 2, the integrand of the Feynman path integral for each harmonic oscillator has been verified to be a generalized white noise functional in $(E)^*$ with the help of the S-transform.

Thus we have the result of the Feynman path integral for this whole system as a product of the solutions for each path integral with one of the various modes as follows,

$$K_{0,N}(T, q) = \prod_{\alpha=1}^{N} \left(\frac{m\omega_\alpha}{2\pi i\hbar \sin\omega_\alpha T}\right)^{1/2} \exp\left(\frac{im\omega_\alpha}{4\hbar \tan\omega_\alpha T}\left[(q^c_\alpha)^2 + (q^s_\alpha)^2\right]\right),$$

where

$$\omega_\alpha = 2\sqrt{\frac{k}{m}} \sin\frac{\pi\alpha}{N}.$$

Then taking $N \to \infty$, we expect to have the formula for the Feynman path integral in the field theory in the stochastic framework.

Define

$$K_{\ell,N}(T,q) = \prod_{\alpha=\ell+1}^{N} \left(\frac{m\omega_\alpha}{2\pi i\hbar \sin\omega_\alpha T}\right)^{1/2} \exp\left(\frac{im\omega_\alpha}{4\hbar\tan\omega_\alpha T}\left[(q_\alpha^c)^2 + (q_\alpha^s)^2\right]\right).$$

Then for each $\ell \in \{0,1,\ldots,N-1\}$, the renormalization of $\lim_{N\to\infty} K_{0,N}(T,q)$ is expressed as

$$\mathcal{R}_\ell K(T,q) \equiv \lim_{N\to\infty} \frac{K_{0,N}(T,q)}{K_{\ell,N}(T,0)}$$

$$= C_\ell(T)\exp\left(\frac{im}{4\hbar}\int_0^1 f(T,x)\left[q^c(x)^2 + q^s(x)^2\right]dx\right)$$

where, $\mathcal{R}_\ell K(T,q) = \mathcal{R}_\ell K(T,q^c,q^s)$,

$$C_\ell(T) \equiv \left(\frac{m}{2\pi i T\hbar}\right)^{\ell/2},\ \ell \in \{0\}\cup\mathbf{N},\ f(T,x) = \frac{2\sqrt{\frac{k}{m}}\sin(\pi x)}{\tan\left(2T\sqrt{\frac{k}{m}}\sin(\pi x)\right)},$$

and

$$q^c(x) = \sqrt{2}\sum_{n=1}^{\infty}\phi_n\cos(2\pi nx),\ q^s(x) = \sqrt{2}\sum_{n=1}^{\infty}\phi_n\sin(2\pi nx).$$

We may take $(\phi_n)_{n=1}^{\infty}$ such that q^c and q^s are in $E[0,1]$.

Let F be an element in the image $S[(E)^*]$ of the S-transform on $(E)^*$. ¿From Theorem 2.2, for any $\xi,\eta \in E_{\mathbf{C}} = E_{\mathbf{C}}[0,1]\times E_{\mathbf{C}}[0,1]$, the function $F(\xi+z\eta)$ is an entire function of $z \in \mathbf{C}$. Therefore, it can be expanded in series as

$$F(\xi+z\eta) = \sum_{n=0}^{\infty}\frac{z^n}{n!}F^{(n)}(\xi)(\eta,\ldots,\eta),$$

where $F^{(n)}(\xi) : E_{\mathbf{C}}\times\cdots\times E_{\mathbf{C}} \to \mathbf{C}$ is a continuous n-linear functional.

Let Z be a finite interval on $\mathbf{R}$. Then we can take an orthonormal basis $\{\zeta_n\}_{n=0}^{\infty}(\subset E)$ for $L^2(Z)$, which satisfies properties of the equal density and the uniform boundedness. If

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F''(\cdot)(\zeta_n,\zeta_n)$$

exists in $S[(E)^*]$ for each $\xi \in E_{\mathbf{C}}$, then this limit is called the *Lévy Laplacian depending on* Z of F, and denoted by $\Delta_L^Z F(\cdot)$.

If functional derivatives F', F_L'', F_V'' exist, and satisfy the following conditions

1) $F'(\xi)(\eta) = \int_{\mathbf{R}} F'(\xi;t)\eta(t)\,dt$,
2) $F''(\xi)(\eta,\zeta) = \int_{\mathbf{R}} F_L''(\xi;t)\eta(t)\zeta(t)\,dt + \int_{\mathbf{R}^2} F_V''(\xi;s,t)\eta(s)\zeta(t)\,dsdt$,

3) $F'(\xi;\cdot) \in L^1_{loc}(\mathbf{R}), F''_L(\xi;\cdot) \in L^1_{loc}(\mathbf{R}), F''_V(\xi;\cdot,\cdot) \in L^2(\mathbf{R}^2)$ for each $\xi \in E$,

4) $F''_V(\xi)$ is a trace class bilinear functional for each ξ and $trF''_V(\cdot) \in S[(E)^*]$,

then the Volterra Laplacian $\Delta_V F$ of F is defined by

$$\Delta_V F(\xi) = trF''_V(\xi),\ \xi \in E_{\mathbf{C}}.$$

The Lévy Laplacian $\Delta_L = \Delta_L^{[0,1]}$, the Volterra Laplacian Δ_V for $\mathcal{R}_\ell K(t,q)$ and $\frac{\partial \mathcal{R}_\ell K}{\partial t}(t,q)$ can be calculated respectively as

$$\Delta_L \mathcal{R}_\ell K(t,q) = \mathcal{R}_\ell K(t,q) \frac{im}{\hbar} \int_0^1 f(t,x)dx;$$

$$\Delta_V \mathcal{R}_\ell K = \mathcal{R}_\ell K(t,q) (\frac{im}{2\hbar})^2 \int_0^1 f(t,x)^2 \left[q^c(x)^2 + q^s(x)^2\right] dx;$$

and

$$\frac{\partial \mathcal{R}_\ell K}{\partial t}(t,q) =$$

$$-\mathcal{R}_\ell K(t,q)\Big(\frac{\ell}{t} + \frac{im}{4\hbar}\int_0^1 f(t,x)^2\Big[1+\tan^2\Big(2t\sqrt{\frac{k}{m}}\sin(\pi x)\Big)\Big][q^c(x)^2+q^s(x)^2]dx\Big).$$

The potential energy term given before becomes now

$$V(q) = \frac{m}{4}\int_0^1 \Big(2\sqrt{\frac{k}{m}}\sin(\pi x)\Big)^2 \left[q^c(x)^2 + q^s(x)^2\right] dx.$$

Lemma 3.1. *Let c be a positive number in $(0,\pi)$. Then, the integral*

$$\int_0^1 \frac{c\sin(\pi x)}{\tan(c\sin(\pi x))}dx$$

exists.

Proof. The proof follows from the fact that $\frac{c\sin(\pi x)}{\tan(c\sin(\pi x))}$ is bounded on $[0,1]$ when $0 < c < \pi$. □

Therefore, if we put together all these terms and compare them with the above lemma, we have an infinite dimensional Schrödinger type equation as follows.

Theorem 3.2. *For each $\ell \in \mathbf{N}$, $\mathcal{R}_\ell K(t,q)$ satisfies the equation:*

$$\frac{\partial \mathcal{R}_\ell K}{\partial t}(t,q) = \frac{i\hbar}{m}\left[\left(\int_0^1 f(t,x)dx\right)^{-1}\frac{\ell}{t}\Delta_L + \Delta_V\right]\mathcal{R}_\ell K(t,q) + \frac{1}{i\hbar}V(q)\cdot\mathcal{R}_\ell K(t,q) \tag{3.7}$$

Remark. The functional $\mathcal{R}_0 K$ is the *normalization*

$$\lim_{N\to\infty}\frac{K_{0,N}(t,q)}{E[K_{0,N}(t,q)]}$$

of the formal limit $\lim_{N\to\infty} K_{0,N}(t,q)$ in the usual sence in the quantum physics. For this normalization, (3.7) has only the Volterra Laplacian.

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