

THE CHARACTERIZATION OF A CLASS OF PROBABILITY MEASURES BY MULTIPLICATIVE RENORMALIZATION

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ABSTRACT. We use the multiplicative renormalization method to characterize a class of probability measures on \mathbb{R} determined by five parameters. This class of probability measures contains the arcsine and the Wigner semi-circle distributions (the vacuum distributions of the field operators of interacting Fock spaces related to the Anderson model), as well as new nonsymmetric distributions. The corresponding orthogonal polynomials and Jacobi–Szegő parameters are derived from the orthogonal-polynomial generating functions. These orthogonal polynomials can be expressed in terms of the Chebyshev polynomials of the second kind.

1. Introduction

Let μ be a probability measure on \mathbb{R} with finite moments of all orders and not being supported by a finite set. Then we can apply the Gram-Schmidt process to the sequence $\{x^n\}_{n=0}^\infty$ to obtain an orthogonal sequence $\{P_n(x)\}_{n=0}^\infty$ in $L^2(\mathbb{R}, \mu)$ such that $P_0(x) = 1$ and $P_n(x)$ is a polynomial of degree n with leading coefficient 1. It is well known that this sequence $P_n(x), n \geq 0$, of orthogonal polynomials satisfies the recursion formula

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_{n-1}P_{n-1}(x), \quad n \geq 0, \quad (1.1)$$

where $\omega_{-1} = P_{-1} = 0$ by convention. The constants $\alpha_n, \omega_n, n \geq 0$, are called the Jacobi–Szegő parameters of the probability measure μ .

Here is a natural question: *Given such a probability measure μ on \mathbb{R} , how can one derive the associated orthogonal polynomials $\{P_n\}$ and the Jacobi–Szegő parameters $\{\alpha_n, \omega_n\}$?*

In a series of papers [8] [9] [10] [11] [12], Asai, Kubo, and Kuo have introduced the multiplicative renormalization method (to be briefly described in Section 2) to provide an answer to this question. This method starts with a function $h(x)$ to derive a generating function $\psi(t, x)$ which can then be used to produce the orthogonal polynomials $\{P_n(x)\}$ and the Jacobi–Szegő parameters $\{\alpha_n, \omega_n\}$. Two types of functions $h(x) = e^x$ and $h(x) = (1 - x)^c$ have been used to derive the classical orthogonal polynomials. This leads to the interesting problem:

“Suppose a function $h(x)$ is fixed. Find all probability measures for which the multiplicative renormalization method can be applied for the function $h(x)$.”

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Kubo [16] has solved this problem for the case $h(x) = e^x$. The resulting class of probability measures coincides with the Meixner class [1][19].

The purpose of the present paper is to solve the above problem for the case $h(x) = (1 - x)^{-1}$. The resulting class of probability measures contains the arcsine distribution, Wigner semi-circle distribution (the vacuum distributions related to the Anderson model [6] [14] [18]), the probability measures given by Bożejko and Bryc [13], and those new distributions in [17].

The results in this paper are somewhat related to those in a recent paper [7], which deals with the noncommutative case. In the paper [7] Anshelevich has used the resolvent-type generating function to derive the associated orthogonal polynomials. His generating function is different from our OP-generating function (see Definition 2.1). Moreover, he does not address the issues of Jacobi–Szegő parameters and the corresponding probability measures.

2. Multiplicative renormalization method

In this section we briefly describe the multiplicative renormalization method. Let μ be a probability measure as specified in Section 1. For convenience, we introduce the following term.

Definition 2.1. An *orthogonal polynomial generating* (OP-generating) *function* for μ is a function of the form

$$\psi(t, x) = \sum_{n=0}^{\infty} c_n P_n(x) t^n, \quad (2.1)$$

where P_n 's are the orthogonal polynomials given by Equation (1.1) and $c_n \neq 0$.

To find an OP-generating function for μ , we start with a function $h(x)$ on \mathbb{R} and define the functions:

$$\theta(t) = \int_{\mathbb{R}} h(tx) d\mu(x), \quad (2.2)$$

$$\tilde{\theta}(t, s) = \int_{\mathbb{R}} h(tx)h(sx) d\mu(x), \quad (2.3)$$

which are assumed to have power series expansion at the origin.

The next theorem gives a method, called *multiplicative renormalization method*, to find OP-generating functions for μ .

Theorem 2.2. ([8][10]) *Assume that $\rho(t)$ has a power series expansion at 0 with $\rho(0) = 0$ and $\rho'(0) \neq 0$. Then the function*

$$\Theta_{\rho}(t, s) = \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))} \quad (2.4)$$

is a function of the product ts if and only if the multiplicative renormalization

$$\psi(t, x) = \frac{h(\rho(t)x)}{\theta(\rho(t))} \quad (2.5)$$

is an OP-generating function for μ .

Below are some examples:

μ	$h(x)$	$\theta(t)$	$\rho(t)$	$\psi(t, x)$
Gaussian $N(0, \sigma^2)$	e^x	$e^{\frac{1}{2}\sigma^2 t^2}$	t	$e^{tx - \frac{1}{2}\sigma^2 t^2}$
Poisson $Poi(\lambda)$	e^x	$e^{\lambda(e^t - 1)}$	$\ln(1 + t)$	$e^{-\lambda t}(1 + t)^x$
gamma $\Gamma(\alpha)$	e^x	$\frac{1}{(1-t)^\alpha}$	$\frac{t}{1+t}$	$(1 + t)^{-\alpha} e^{\frac{tx}{1+t}}$
uniform on $[-1, 1]$	$\frac{1}{\sqrt{1-x}}$	$\frac{2}{\sqrt{1+t} + \sqrt{1-t}}$	$\frac{2t}{1+t^2}$	$\frac{1}{\sqrt{1-2tx+t^2}}$
arcsine on $[-1, 1]$	$\frac{1}{1-x}$	$\frac{1}{\sqrt{1-t^2}}$	$\frac{2t}{1+t^2}$	$\frac{1-t^2}{1-2tx+t^2}$
semi-circle on $[-1, 1]$	$\frac{1}{1-x}$	$\frac{2}{1+\sqrt{1-t^2}}$	$\frac{2t}{1+t^2}$	$\frac{1}{1-2tx+t^2}$
beta on $[-1, 1]$ $\beta > -\frac{1}{2}, \beta \neq 0$	$\frac{1}{(1-x)^\beta}$	$\frac{2^\beta}{(1+\sqrt{1-t^2})^\beta}$	$\frac{2t}{1+t^2}$	$\frac{1}{(1-2tx+t^2)^\beta}$
negative binomial $r > 0, 0 < q < 1$	e^x	$\frac{(1-q)^r}{(1-qe^t)^r}$	$\ln \frac{1+t}{1+qt}$	$(1 + t)^x (1 + qt)^{-x-r}$
stochastic area	e^x	$\sec t$	$\tan^{-1} t$	$\frac{e^{x \tan^{-1} t}}{\sqrt{1+t^2}}$

For convenience, we make the following definition.

Definition 2.3. A probability measure μ is said to be *MRM-applicable* for a function $h(x)$ if there exists a function $\rho(t)$ having a power series expansion at 0 with $\rho(0) = 0$ and $\rho'(0) \neq 0$ such that the function $\psi(t, x)$ given by Equation (2.5) is an OP-generating function for μ , or equivalently, the function $\Theta_\rho(t, s)$ given by Equation (2.4) is a function of ts .

For example, from the above chart, we see that the Gaussian, Poisson, gamma, negative binomial, and stochastic area distributions are all MRM-applicable for the same function $h(x) = e^x$. Moreover, the arcsine and semi-circle distributions are MRM-applicable for the same function $h(x) = (1 - x)^{-1}$.

Suppose we have an OP-generating function $\psi(t, x)$ for μ , then we can expand it as a power series in t to obtain the orthogonal polynomials $P_n(x)$ and the constants c_n . Then we can use the following equalities (see Theorem 2.6 in [10]) to find the Jacobi–Szegő parameters $\{\alpha_n, \omega_n\}$:

$$\int_{\mathbb{R}} \psi(t, x)^2 d\mu(x) = \sum_{n=0}^{\infty} c_n^2 \lambda_n t^{2n}, \tag{2.6}$$

$$\int_{\mathbb{R}} x\psi(t, x)^2 d\mu(x) = \sum_{n=0}^{\infty} (c_n^2 \alpha_n \lambda_n t^{2n} + 2c_n c_{n-1} \lambda_n t^{2n-1}), \tag{2.7}$$

where $c_{-1} = 0$ by convention and $\lambda_0 = 1, \lambda_n = \omega_0 \omega_1 \cdots \omega_{n-1}, n \geq 1$. Note that we have $\omega_n = \frac{\lambda_{n+1}}{\lambda_n}, n \geq 0$.

As mentioned above, we can expand $\psi(t, x)$ as a power series in t to find the constants c_n . But it is often easier to use the next lemma.

Lemma 2.4. *Let μ be MRM-applicable for a function $h(x)$. Then the constants c_n 's in Equation (2.1) are given by*

$$c_n = \frac{h^{(n)}(0)\rho'(0)^n}{n!h(0)}, \quad n \geq 0, \tag{2.8}$$

where $\rho(t)$ is the function specified by Equation (2.5).

Proof. By Equations (2.1) and (2.5), we have

$$h(\rho(t)x) = \theta(\rho(t)) \sum_{n=0}^{\infty} c_n P_n(x) t^n.$$

Differentiate this equation n times and then evaluate at $t = 0$ to get

$$h^{(n)}(0)\rho'(0)^n x^n + \dots = \theta(0)n!c_n P_n(x),$$

which yields the value of c_n as given by Equation (2.8) since $\theta(0) = h(0) \neq 0$. \square

3. Derivation of OP-generating functions

From now on, we fix the function $h(x) = \frac{1}{1-x}$. The corresponding function $\theta(t)$ defined by Equation (2.2) is given by

$$\theta(t) = \int_{\mathbb{R}} \frac{1}{1-tx} d\mu(x). \tag{3.1}$$

Since $\theta(t)$ has a power series expansion at 0, the support S_μ of μ must be bounded. Thus $\theta(t)$ is analytic if $\frac{1}{t} \notin S_\mu$. Use the identity $\frac{1}{(1-tx)(1-sx)} = \frac{1}{t-s} \left(\frac{t}{1-tx} - \frac{s}{1-sx} \right)$ to write the function $\tilde{\theta}(t, s)$ in Equation (2.3) as

$$\tilde{\theta}(t, s) = \int_{\mathbb{R}} \frac{1}{(1-tx)(1-sx)} d\mu(x) = \frac{1}{t-s} \{t\theta(t) - s\theta(s)\}, \quad t \neq s, \tag{3.2}$$

and $\tilde{\theta}(t, t)$ can be easily checked to equal $\theta(t) + t\theta'(t)$.

Lemma 3.1. *Assume that the function $\theta(t)$ in Equation (3.1) has a power series expansion at 0 and $\rho(t)$ is a function having a power series expansion at 0 with $\rho(0) = 0$ and $\rho'(0) \neq 0$ such that the function*

$$\Theta_\rho(t, s) = \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))} \tag{3.3}$$

is a function $J(ts)$ of the product ts with $J'(0) \neq 0$ and $J''(0) \neq 0$. Then $\rho(t)$ and $\theta(\rho(t))$ must be given by

$$\rho(t) = \frac{2t}{\alpha + 2\beta t + \gamma t^2}, \tag{3.4}$$

$$\theta(\rho(t)) = \frac{1}{1 - (b + at)\rho(t)}, \tag{3.5}$$

where α, β, γ, b , and a are constants.

Remark 3.2. From the proof below, $\alpha \neq 0$. It will follow from Equation (4.2) that $a \neq 0$ and $\gamma \neq 0$ because μ is assumed to have infinite support.

Proof. By assumption $\Theta_\rho(t, s) = J(ts)$. Then by Equations (3.2) and (3.3),

$$J(ts) = \frac{1}{\rho(t) - \rho(s)} \left[\frac{\rho(t)}{\theta(\rho(s))} - \frac{\rho(s)}{\theta(\rho(t))} \right]. \tag{3.6}$$

Differentiate this equation with respect to s to get

$$\begin{aligned} {}_tJ'(ts) &= \frac{\rho'(s)}{(\rho(t) - \rho(s))^2} \left[\frac{\rho(t)}{\theta(\rho(s))} - \frac{\rho(s)}{\theta(\rho(t))} \right] \\ &+ \frac{1}{\rho(t) - \rho(s)} \left[\frac{-\rho(t)\theta'(\rho(s))\rho'(s)}{\theta(\rho(s))^2} - \frac{\rho'(s)}{\theta(\rho(t))} \right]. \end{aligned} \tag{3.7}$$

Put $s = 0$ and note that $\theta(0) = 1$ to obtain

$${}_tJ'(0) = \frac{\rho'(0)}{\rho(t)} - \theta'(0)\rho'(0) - \frac{\rho'(0)}{\rho(t)\theta(\rho(t))},$$

which can be solved for $\theta(\rho(t))$,

$$\theta(\rho(t)) = \frac{1}{1 - (b + at)\rho(t)},$$

where $b = \theta'(0)$ and $a = J'(0)/\rho'(0)$. Thus Equation (3.5) is proved. Differentiate both sides of this equation to get $\theta'(\rho(t))\rho'(t)$. Then put $\theta(\rho(t))$ and $\theta'(\rho(t))\rho'(t)$ into Equation (3.7) to show that

$$\begin{aligned} {}_tJ'(ts) &= \frac{\rho'(s)}{(\rho(t) - \rho(s))^2} [\rho(t) - \rho(s) + a(t - s)\rho(t)\rho(s)] \\ &- \frac{1}{\rho(t) - \rho(s)} [a\rho(t)\rho(s) + \rho'(s) + a(s - t)\rho(t)\rho'(s)]. \end{aligned}$$

Finally, differentiate both sides of this equation with respect to s (straightforward, but very tedious) and then put $s = 0$ to get

$$t^2 J''(0) = 2a\rho'(0)\frac{t}{\rho(t)} + a\rho''(0)t - 2a\rho'(0),$$

which yields the function

$$\rho(t) = \frac{2t}{\alpha + 2\beta t + \gamma t^2},$$

where $\alpha = \frac{2}{\rho'(0)}$, $\beta = -\frac{\rho''(0)}{2\rho'(0)^2}$, and $\gamma = \frac{J''(0)}{J'(0)\rho'(0)}$. This proves Equation (3.4). \square

Theorem 3.3. *Let μ be a probability measure on \mathbb{R} with the associated functions $\theta(t)$ and $\rho(t)$ satisfying the assumption in Lemma 3.1. Then μ is MRM-applicable for $h(x) = \frac{1}{1-x}$ and has an OP-generating function given by*

$$\psi(t, x) = \frac{1 - (b + at)\rho(t)}{1 - \rho(t)x} = \frac{\alpha + 2(\beta - b)t + (\gamma - 2a)t^2}{\alpha - 2t(x - \beta) + \gamma t^2}, \tag{3.8}$$

which satisfies the equalities:

$$\int_{\mathbb{R}} \psi(t, x)^2 d\mu(x) = 1 + \frac{2at^2}{\alpha - \gamma t^2}, \quad (3.9)$$

$$\int_{\mathbb{R}} x\psi(t, x)^2 d\mu(x) = b + 2at \frac{\alpha + \beta t}{\alpha - \gamma t^2}, \quad (3.10)$$

where $\rho(t)$, α , β , γ , a , and b are given by Equations (3.4) and (3.5).

Proof. Put Equations (3.2), (3.4), and (3.5) into Equation (3.3) to derive

$$\Theta_{\rho}(t, s) = 1 + \frac{2ats}{\alpha - \gamma ts}. \quad (3.11)$$

This shows that $\Theta_{\rho}(t, s)$ is a function of ts . Thus by Theorem 2.2 the probability measure μ is MRM-applicable for the function $h(x) = \frac{1}{1-x}$ and

$$\psi(t, x) = \frac{h(\rho(t)x)}{\theta(\rho(t))} = \frac{1 - (b + at)\rho(t)}{1 - \rho(t)x} \quad (3.12)$$

is an OP-generating function for μ . Next observe that

$$\int_{\mathbb{R}} \psi(t, x)^2 d\mu(x) = \int_{\mathbb{R}} \frac{h(\rho(t)x)^2}{\theta(\rho(t))^2} d\mu(x) = \frac{\tilde{\theta}(\rho(t), \rho(t))}{\theta(\rho(t))^2} = \Theta_{\rho}(t, t).$$

Then apply Equation (3.11) with $s = t$ to get Equation (3.9). Note that

$$\int_{\mathbb{R}} x\psi(t, x)^2 d\mu(x) = \frac{(1 - (b + at)\rho(t))^2}{\rho(t)} \int_{\mathbb{R}} \left[\frac{1}{(1 - \rho(t)x)^2} - \frac{1}{1 - \rho(t)x} \right] d\mu(x)$$

Hence we can use Equations (3.2), (3.4), and (3.5) to derive Equation (3.10). \square

4. Jacobi–Szegő parameters and orthogonal polynomials

Let μ be MRM-applicable for the function $h(x) = \frac{1}{1-x}$. We now derive the associated Jacobi–Szegő parameters $\{\alpha_n, \omega_n\}$ and orthogonal polynomials $\{P_n\}$ as given by Equation (1.1).

First note that by Lemma 3.1, the associated functions $\rho(t)$ and $\theta(\rho(t))$ must be given by Equations (3.4) and (3.5), respectively. Moreover, there are five constants α , β , γ , b , and a in these two equations.

Theorem 4.1. *Let μ be MRM-applicable for the function $h(x) = \frac{1}{1-x}$. Then the Jacobi–Szegő parameters are given by*

$$\alpha_n = \begin{cases} b, & \text{if } n = 0, \\ \beta, & \text{if } n \geq 1, \end{cases} \quad (4.1)$$

$$\omega_n = \begin{cases} \frac{a\alpha}{2}, & \text{if } n = 0, \\ \frac{\alpha\gamma}{4}, & \text{if } n \geq 1, \end{cases} \quad (4.2)$$

where α , β , γ , b , and a are given by Equations (3.4) and (3.5).

Proof. First we can use Lemma 2.4 to find that

$$c_n = \left(\frac{2}{\alpha}\right)^n, \quad n \geq 0. \tag{4.3}$$

Then use Equations (2.6) and (3.9) to get the following equality:

$$1 + \frac{2at^2}{\alpha - \gamma t^2} = \sum_{n=0}^{\infty} c_n^2 \lambda_n t^{2n}.$$

Expand the left-hand side as a power series in t and use Equation (4.3) to show that $\lambda_0 = 1$ and

$$\lambda_n = \frac{a\alpha}{2} \left(\frac{\alpha\gamma}{4}\right)^{n-1}, \quad n \geq 1, \tag{4.4}$$

which yields Equation (4.2) since $\omega_n = \frac{\lambda_{n+1}}{\lambda_n}$, $n \geq 0$. Next we can use Equations (2.7) and (3.10) to get the equality:

$$b + 2at \frac{\alpha + \beta t}{\alpha - \gamma t^2} = \sum_{n=0}^{\infty} (c_n^2 \alpha_n \lambda_n t^{2n} + 2c_n c_{n-1} \lambda_n t^{2n-1}).$$

By expanding the left-hand side as a power series in t and then compare the coefficients of t^{2n} , we obtain the values of α_n 's as given by Equation (4.1). \square

Recall that $\omega_n = \lambda_{n+1}/\lambda_n$ and μ has infinite support. Hence $\omega_n > 0$ for all $n \geq 0$. Then by Equation (4.2), a, α , and γ must be of the same sign.

Theorem 4.2. *Let μ be MRM-applicable for the function $h(x) = \frac{1}{1-x}$. Then the associated orthogonal polynomials P_n are given by*

$$P_n(x) = \left(\frac{\sqrt{\alpha\gamma}}{2}\right)^n U_n\left(\frac{x-\beta}{\sqrt{\alpha\gamma}}\right) + (\beta-b) \left(\frac{\sqrt{\alpha\gamma}}{2}\right)^{n-1} U_{n-1}\left(\frac{x-\beta}{\sqrt{\alpha\gamma}}\right) + \frac{\alpha(\gamma-2a)}{4} \left(\frac{\sqrt{\alpha\gamma}}{2}\right)^{n-2} U_{n-2}\left(\frac{x-\beta}{\sqrt{\alpha\gamma}}\right), \quad n \geq 0, \tag{4.5}$$

where $U_{-2} = U_{-1} = 0$ by convention, α, β, γ, b , and a are given by Equations (3.4) and (3.5), and $U_n(x)$ are the Chebyshev polynomials of the second kind, i.e.,

$$U_n(x) = \frac{\sin[(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} 2^{n-2k} x^{n-2k}, \quad n \geq 0.$$

Proof. From Equation (1.17) in [17], we have

$$\frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n,$$

Replace x by $\frac{x-\beta}{\sqrt{\alpha\gamma}}$ and t by $\sqrt{\frac{\gamma}{\alpha}}t$ to get

$$\frac{\alpha}{\alpha-2t(x-\beta)+\gamma t^2} = \sum_{n=0}^{\infty} \left(\sqrt{\frac{\gamma}{\alpha}}\right)^n U_n\left(\frac{x-\beta}{\sqrt{\alpha\gamma}}\right) t^n. \tag{4.6}$$

Now write the OP-generating function $\psi(t, x)$ in Equation (3.8) as the sum

$$\frac{\alpha}{\alpha-2t(x-\beta)+\gamma t^2} + \frac{2(\beta-b)t}{\alpha-2t(x-\beta)+\gamma t^2} + \frac{(\gamma-2a)t^2}{\alpha-2t(x-\beta)+\gamma t^2}$$

and then use Equation (4.6) to obtain Equation (4.5). □

5. Derivation of probability measures (a special case)

Consider a probability measure μ with density function $f(x)$. Suppose μ is MRM-applicable for the function $h(x) = \frac{1}{1-x}$. As pointed out in the beginning of Section 3, the support S_μ of μ is bounded and $\theta(t)$ is analytic for $\frac{1}{t} \notin S_\mu$. Moreover, $\theta(t)$ can be analytically extended to $\mathbb{C} \setminus S_\mu$ and is essentially singular on S_μ . The Hilbert transform of f is given by

$$(\mathcal{H}f)(t) = \text{p.v.} \int_{\mathbb{R}} \frac{f(x)}{\pi(t-x)} dx,$$

where and hereafter “p.v.” denotes the principal value of an integral. Hence the function $(\mathcal{H}f)(t)$ can be analytically extended to $\mathbb{C} \setminus S_\mu$ and

$$(\mathcal{H}f)(t) = \frac{1}{\pi t} \theta\left(\frac{1}{t}\right), \quad t \in \mathbb{R} \setminus S_\mu,$$

where the function $\theta(t)$ is defined by Equation (3.1). Moreover, the extension $\tilde{\theta}(t)$ of $\theta(t)$ to \mathbb{R} is given by

$$\tilde{\theta}(t) = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \{ \theta(t + i\epsilon) + \theta(t - i\epsilon) \}, \quad t \in \mathbb{R}. \tag{5.1}$$

By the inverse Hilbert transform, we have

$$f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{\tilde{\theta}(s)}{\pi^2(1-xs)} ds. \tag{5.2}$$

Now consider the special case with $\alpha = \gamma = 1$ and $\beta = 0$ in Equation (3.4), namely, $\rho(t)$ is given as follows:

$$\rho(t) = \frac{2t}{1+t^2}.$$

Let $s = \rho(t) = \frac{2t}{1+t^2}$. Then $t = \frac{1-\sqrt{1-s^2}}{s}$. Hence by Equation (3.5),

$$\theta(s) = \frac{1}{1-a-bs+a\sqrt{1-s^2}}, \quad |s| < 1.$$

Then by Equation (5.1) its extension $\tilde{\theta}$ to \mathbb{R} is given by

$$\tilde{\theta}(s) = \begin{cases} \frac{1}{1-a-bs+a\sqrt{1-s^2}}, & \text{if } |s| < 1, \\ \frac{1-a-bs}{(a^2+b^2)s^2-2b(1-a)s+1-2a}, & \text{if } |s| \geq 1. \end{cases} \tag{5.3}$$

This shows, in particular, that the support of μ is $[-1, 1]$.

Lemma 5.1. *The constants a and b satisfy the condition: $a > 0$, $|b| \leq 1 - a$.*

Proof. As pointed out in the paragraph prior to Theorem 4.2, the constants a, α , and γ must be of the same sign. But we assume that $\alpha = \gamma = 1$ in this section. Hence $a > 0$. Moreover, note that the denominator $1 - a - bs + a\sqrt{1 - s^2}$ in Equation (5.3) must be positive for $|s| < 1$. Hence its values at $s = 1, -1$ are positive. This implies that $|b| \leq 1 - a$. □

Lemma 5.2. *The following equality holds:*

$$\text{p.v.} \int_{\mathbb{R}} \frac{\tilde{\theta}(s)}{\pi^2(1-xs)} ds = L(x) \left(\pi\sqrt{1-x^2} 1_{(-1,1)}(x) - (1-a-bx)\{L_1-L_2\} \right), \quad (5.4)$$

where $L(x), L_1,$ and L_2 are given by

$$\begin{aligned} L(x) &= \frac{a}{\pi^2[a^2 + b^2 - 2b(1-a)x + (1-2a)x^2]}, \\ L_1 &= \text{p.v.} \int_0^\infty \frac{2}{(1-a-b)u^2 + 2au + 1-a+b} du, \\ L_2 &= \text{p.v.} \int_{|s|>1} \frac{a}{(a^2 + b^2)s^2 - 2b(1-a)s + 1-2a} ds. \end{aligned} \quad (5.5)$$

Proof. Use Equation (5.3) to write $\text{p.v.} \int_{\mathbb{R}} \frac{\tilde{\theta}(s)}{\pi^2(1-xs)} ds$ as the sum of the following two integrals:

$$\begin{aligned} I_1 &= \text{p.v.} \int_{-1}^1 \frac{1}{\pi^2(1-xs)(1-a-bs+a\sqrt{1-s^2})} ds, \\ I_2 &= \text{p.v.} \int_{|s|>1} \frac{1-a-bs}{\pi^2(1-xs)[(a^2 + b^2)s^2 - 2b(1-a)s + 1-2a]} ds. \end{aligned}$$

For the integral $I_1,$ let $u = \sqrt{\frac{1+s}{1-s}}$ and then use the integral formulas in Section 8 to derive the equality

$$\begin{aligned} I_1 &= L(x) \left(\pi\sqrt{1-x^2} 1_{(-1,1)}(x) + \frac{1}{a} [(1-a)x - b] \ln \left| \frac{(1+x)(1-a-b)}{(1-x)(1-a+b)} \right| \right. \\ &\quad \left. - \text{p.v.} \int_0^\infty \frac{2(1-a-bx)}{(1-a-b)u^2 + 2au + 1-a+b} du \right), \end{aligned}$$

where $L(x)$ is given by Equation (5.5). Similarly, we can use the integral formulas in Section 8 to show that

$$\begin{aligned} I_2 &= L(x) \left(-\frac{1}{a} [(1-a)x - b] \ln \left| \frac{(1+x)(1-a-b)}{(1-x)(1-a+b)} \right| \right. \\ &\quad \left. + \text{p.v.} \int_{|s|>1} \frac{a(1-a-bx)}{(a^2 + b^2)s^2 - 2b(1-a)s + 1-2a} ds \right). \end{aligned}$$

Finally, sum up I_1 and I_2 to obtain Equation (5.4). □

Theorem 5.3. *Let μ be a probability measure with density function $f(x)$. Then μ is MRM-applicable for $h(x) = \frac{1}{1-x}$ with $\rho(t) = \frac{2t}{1+t^2}$ if and only if its density function $f(x)$ is given by*

$$f(x) = \begin{cases} \frac{a\sqrt{1-x^2}}{\pi[a^2 + b^2 - 2b(1-a)x + (1-2a)x^2]}, & \text{if } |x| < 1, \\ 0, & \text{otherwise,} \end{cases} \quad (5.6)$$

where $a > 0$ and $|b| \leq 1 - a$. Moreover, the resulting OP-generating function is

$$\psi(t, x) = \frac{1 - 2bt + (1 - 2a)t^2}{1 - 2tx + t^2}.$$

Proof. Suppose $d\mu = f(x) dx$ is MRM-applicable for $h(x) = \frac{1}{1-x}$ with $\rho(t) = \frac{2t}{1+t^2}$. Then $f(x)$ is given by Equation (5.2). Thus by Lemma 5.2 we only need to show that $L_1 = L_2$ in order to obtain Equation (5.6). Consider the following three cases and use the integral formulas in Section 8 to evaluate the integrals L_1 and L_2 in Lemma 5.2.

Case 1. For $a > 0$, $b^2 + 2a < 1$, we have

$$L_1 = L_2 = \begin{cases} \frac{1}{\sqrt{1-2a-b^2}} \left(\pi - \tan^{-1} \frac{2a\sqrt{1-2a-b^2}}{1-2a-a^2-b^2} \right), & \text{if } 2a + a^2 + b^2 < 1, \\ \frac{1}{\sqrt{1-2a-b^2}} \left(-\tan^{-1} \frac{2a\sqrt{1-2a-b^2}}{1-2a-a^2-b^2} \right), & \text{if } 2a + a^2 + b^2 > 1. \end{cases}$$

Case 2. For $a > 0$, $b^2 + 2a > 1$, $|b| \leq 1 - a$, we have

$$L_1 = L_2 = \frac{1}{\sqrt{b^2 + 2a - 1}} \ln \left| \frac{a - \sqrt{b^2 + 2a - 1}}{a + \sqrt{b^2 + 2a - 1}} \right|.$$

Case 3. For $a > 0$, $b^2 + 2a = 1$, we have

$$L_1 = L_2 = \frac{2}{a}.$$

Conversely, consider the function $f(x)$ defined by Equation (5.6). Note that

$$a^2 + b^2 - 2b(1-a)x + (1-2a)x^2 = a^2(1-x^2) + [b - (1-a)x]^2.$$

Hence $f(x)$ is nonnegative. We can rewrite $f(x)$ as

$$f(x) = W_0 \frac{\sqrt{1-x^2}}{\pi(1-px)(1-qx)}, \quad (5.7)$$

where X_0 , p , and q are given by

$$\begin{aligned} W_0 &= \frac{a}{a^2 + b^2}, \\ p &= \frac{b(1-a) + a\sqrt{b^2 + 2a - 1}}{a^2 + b^2}, \\ q &= \frac{b(1-a) - a\sqrt{b^2 + 2a - 1}}{a^2 + b^2}. \end{aligned}$$

Decompose $f(x)$ as

$$f(x) = \frac{W_0}{\pi(p-q)} \left(p \frac{\sqrt{1-x^2}}{1-px} - q \frac{\sqrt{1-x^2}}{1-qx} \right)$$

and then use Formula 4 in Section 8 to show that $\int_{-1}^1 f(x) dx = 1$. Hence $f(x)$ is a density function.

Next we compute $\theta(t) = \int_{-1}^1 \frac{f(x)}{1-tx} dx$. Decompose the integrand as

$$\frac{f(x)}{1-tx} = \frac{W_0}{\pi} \left(A \frac{\sqrt{1-x^2}}{1-tx} + B \frac{\sqrt{1-x^2}}{1-px} + C \frac{\sqrt{1-x^2}}{1-qx} \right)$$

and again use Formula 4 in Section 8 to derive the equality

$$\theta(t) = \frac{1}{1 - a - bt + a\sqrt{1 - t^2}}, \quad |t| < 1.$$

Then use Equation (3.2) to find

$$\tilde{\theta}(t, s) = \frac{1 - a + a\frac{t+s}{t\sqrt{1-s^2} + s\sqrt{1-t^2}}}{(1 - a - bt + a\sqrt{1 - t^2})(1 - a - bs + a\sqrt{1 - s^2})}, \quad |t|, |s| < 1.$$

Therefore, we have

$$\frac{\tilde{\theta}(t, s)}{\theta(t)\theta(s)} = 1 - a + a\frac{t + s}{t\sqrt{1 - s^2} + s\sqrt{1 - t^2}}.$$

Observe that this function is independent of b . Thus by the proof of Lemma 2.3 in [17] (the case when $b = 0$), we can find a function $\rho(t)$ such that

$$\Theta_\rho(t, s) = \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))} = 1 - a + a\frac{\rho(t) + \rho(s)}{\rho(t)\sqrt{1 - \rho(s)^2} + \rho(s)\sqrt{1 - \rho(t)^2}}$$

is a function of the product ts . Such a function is given by

$$\rho(t) = \frac{2t}{1 + t^2}.$$

Hence by Theorem 2.2 the probability measure $d\mu = f(x) dx$ is MRM-applicable for $h(x) = \frac{1}{1-x}$ and the resulting OP-generating function is given by

$$\psi(t, x) = \frac{h(\rho(t)x)}{\theta(\rho(t))} = \frac{1 - 2bt + (1 - 2a)t^2}{1 - 2tx + t^2}.$$

This completes the proof of the theorem. □

6. Characterization of a class of probability measures

In this section we will characterize the class of probability measures that are MRM-applicable for the function $h(x) = \frac{1}{1-x}$. Let μ be such a probability measure. There are five parameters α, β, γ, b , and a in the associated functions $\rho(t)$ and $\theta(\rho(t))$ in Equations (3.4) and (3.5), respectively.

The value of λ_n in Equation (4.4) implies that $\lim_{n \rightarrow \infty} \lambda_n^{-1/2n} = 2/\sqrt{\alpha\gamma} \neq 0$. Hence $\sum_{n=1}^\infty \lambda_n^{-1/2n} = \infty$. Thus by Theorem 1.11 in [20] the probability measure μ is uniquely determined by its moments. This implies, in view of Equation (3.5), that μ is the unique probability measure satisfying the equation

$$\int_{\mathbb{R}} \frac{1}{1 - \rho(t)x} d\mu(x) = \frac{1}{1 - (b + at)\rho(t)} \tag{6.1}$$

with $\rho(t) = \frac{2t}{\alpha + 2\beta t + \gamma t^2}$. Recall that a, α , and γ are of the same sign, which without loss of generality can be taken to be positive.

Make the following changes of variables and constants:

$$x = \sqrt{\alpha\gamma}y + \beta, \quad d\nu(y) = d\mu(\sqrt{\alpha\gamma}y + \beta), \tag{6.2}$$

$$A = \frac{a}{\gamma}, \quad B = \frac{b - \beta}{\sqrt{\alpha\gamma}}, \quad t = \sqrt{\frac{\alpha}{\gamma}} \frac{z}{1 + \sqrt{1 - z^2}}. \tag{6.3}$$

Then Equation (6.1) becomes the following equivalent equation

$$\int_{\mathbb{R}} \frac{1}{1 - zy} d\nu(y) = \frac{1}{1 - A - Bz + A\sqrt{1 - z^2}}, \tag{6.4}$$

which holds for all small z . By replacing $d\nu(y)$ with $d\nu(-y)$, if necessary, we may assume that $B \geq 0$.

Theorem 6.1. *Let $A > 0$ and $B \geq 0$. Then the unique probability measure ν that satisfies Equation (6.4) is given by*

$$d\nu(y) = W_0 \frac{\sqrt{1 - y^2}}{\pi(1 - py)(1 - qy)} 1_{(-1,1)}(y) dy + W_1 d\delta_{\frac{1}{p}}(y) + W_2 d\delta_{\frac{1}{q}}(y), \tag{6.5}$$

where $d\delta_r(y)$ denotes the Dirac delta measure at r and p, q, W_0, W_1, W_2 are the following numbers,

$$p = \frac{B(1 - A) + A\sqrt{B^2 + 2A - 1}}{A^2 + B^2}, \tag{6.6}$$

$$q = \frac{B(1 - A) - A\sqrt{B^2 + 2A - 1}}{A^2 + B^2}, \tag{6.7}$$

$$W_0 = \frac{A}{A^2 + B^2},$$

$$W_1 = \frac{1 - A - pB - A\sqrt{1 - p^2}}{p(q - p)(A^2 + B^2)},$$

$$W_2 = \frac{1 - A - qB - A\sqrt{1 - q^2}}{q(p - q)(A^2 + B^2)}.$$

Remark 6.2. Here are three important comments about this theorem.

- (1) In view of Equations (5.7) and (6.5), and the numbers W_0, p, q defined there and here, there might be some confusion. In fact, there is a crucial difference between Theorems 5.3 and 6.1, namely, we do not assume the condition $|B| \leq 1 - A$, i.e., $|b| \leq 1 - a$ as in Theorem 5.3. This is the reason for the occurrence of the Dirac delta measures in Equation (6.5).
- (2) When $B^2 + 2A - 1 < 0$, we have complex numbers p and q . But in this case, $W_1 = W_2 = 0$. Hence μ has a density function.
- (3) When $B^2 + 2A - 1 = 0$, we have $p = q$. In this case, we use the convention that $W_1 = W_2 = 0$. Hence μ has a density function.

Proof. As pointed out above, a probability measure satisfying Equation (6.4) is unique. Hence it suffices to show that ν as defined by Equation (6.5) indeed satisfies Equation (6.4). To this end, we define p and q by Equations (6.6) and (6.7), respectively. Then we will find W_0, W_1 , and W_2 for Equation (6.5) so that the resulting ν is a probability measure satisfying Equation (6.4).

We have the partial fraction decomposition

$$\begin{aligned} & \frac{1}{(1 - zy)(1 - py)(1 - qy)} \\ &= \frac{z^2}{(z - p)(z - q)} \frac{1}{1 - zy} + \frac{p^2}{(p - z)(p - q)} \frac{1}{1 - py} + \frac{q^2}{(q - z)(q - p)} \frac{1}{1 - qy}. \end{aligned}$$

Then use Formula 4 in Section 8 to show that

$$\int_{\mathbb{R}} \frac{1}{1-zy} d\nu(y) \\ = W_0 \left[\frac{1 - \sqrt{1-z^2}}{(z-p)(z-q)} + \frac{1 - \sqrt{1-p^2}}{(p-z)(p-q)} + \frac{1 - \sqrt{1-q^2}}{(q-z)(q-p)} \right] + W_1 \frac{p}{p-z} + W_2 \frac{q}{q-z}.$$

Moreover, it is easy to verify that

$$\frac{1}{1-A-Bz+A\sqrt{1-z^2}} = \frac{1-A-Bz-A\sqrt{1-z^2}}{(A^2+B^2)(z-p)(z-q)}.$$

Therefore, Equation (6.4) is equivalent to the following equation:

$$W_0 \left[\frac{1 - \sqrt{1-z^2}}{(z-p)(z-q)} + \frac{1 - \sqrt{1-p^2}}{(p-z)(p-q)} + \frac{1 - \sqrt{1-q^2}}{(q-z)(q-p)} \right] + W_1 \frac{p}{p-z} + W_2 \frac{q}{q-z} \\ = \frac{1-A-Bz-A\sqrt{1-z^2}}{(A^2+B^2)(z-p)(z-q)}.$$

Multiply both sides by $(A^2+B^2)(z-p)(z-q)$ to get

$$(A^2+B^2)W_0 \left[1 - \sqrt{1-z^2} + \frac{z-q}{q-p}(1 - \sqrt{1-p^2}) + \frac{z-p}{p-q}(1 - \sqrt{1-q^2}) \right] \\ + p(A^2+B^2)W_1(q-z) + q(A^2+B^2)W_2(p-z) \\ = 1 - A - Bz - A\sqrt{1-z^2}. \quad (6.8)$$

Comparing the coefficients of $\sqrt{1-z^2}$ yields that

$$W_0 = \frac{A}{A^2+B^2}.$$

Put this value of W_0 into Equation (6.8) to get rid of the terms involving $\sqrt{1-z^2}$. Then put $z=p$ and $z=q$ to find the values of W_1 and W_2 , respectively,

$$W_1 = \frac{1-A-pB-A\sqrt{1-p^2}}{p(q-p)(A^2+B^2)}, \quad W_2 = \frac{1-A-qB-A\sqrt{1-q^2}}{q(p-q)(A^2+B^2)}.$$

Moreover, with the above values of W_0, W_1, W_2 , it can be easily checked that Equation (6.8) is an identity for all $|z| \leq 1$. Thus ν defined by Equation (6.5) is the unique probability measure satisfying Equation (6.4). \square

Next, we determine the values of W_1 and W_2 in terms of the parameters A and B . We can easily derive the equalities:

$$p-q = \frac{2A\sqrt{B^2+2A-1}}{A^2+B^2}, \\ \sqrt{1-p^2} = \frac{|A(1-A) - B\sqrt{B^2+2A-1}|}{A^2+B^2}, \\ \sqrt{1-q^2} = \frac{|A(1-A) + B\sqrt{B^2+2A-1}|}{A^2+B^2}.$$

Then use them to rewrite W_1 and W_2 as follows:

$$W_1 = \frac{|A(1 - A) - B\sqrt{B^2 + 2A - 1}| - \{A(1 - A) - B\sqrt{B^2 + 2A - 1}\}}{2p(A^2 + B^2)\sqrt{B^2 + 2A - 1}},$$

$$W_2 = \frac{\{A(1 - A) + B\sqrt{B^2 + 2A - 1}\} - |A(1 - A) + B\sqrt{B^2 + 2A - 1}|}{2q(A^2 + B^2)\sqrt{B^2 + 2A - 1}}.$$

Note that when $0 < A \leq 1$, we have $W_2 = 0$. When $0 < A \leq 1$ and $0 \leq B \leq 1 - A$, we have $W_1 = 0$. In the cases: (1) $0 < A \leq 1$ and $B > 1 - A$ and (2) $A > 1$, the number inside the absolute value sign of W_1 is a negative number. On the other hand, when $A > 1$, it is easy to check that the number inside the absolute value sign of W_2 is negative if and only if $B < A - 1$. Therefore, the values of W_1 and W_2 are given by the following chart.

Region	W_1	W_2
$0 < A \leq 1$ $0 \leq B \leq 1 - A$	0	0
$0 < A \leq 1$ $B > 1 - A$	$\frac{B\sqrt{B^2+2A-1}-A(1-A)}{A(B^2+2A-1)+B(1-A)\sqrt{B^2+2A-1}}$	0
$1 < A$ $0 \leq B \leq A - 1$	$\frac{B\sqrt{B^2+2A-1}-A(1-A)}{A(B^2+2A-1)+B(1-A)\sqrt{B^2+2A-1}}$	$\frac{A(A-1)-B\sqrt{B^2+2A-1}}{A(B^2+2A-1)+B(A-1)\sqrt{B^2+2A-1}}$
$1 < A$ $B > A - 1$	$\frac{B\sqrt{B^2+2A-1}-A(1-A)}{A(B^2+2A-1)+B(1-A)\sqrt{B^2+2A-1}}$	0

Here are some examples for the various regions in the above chart.

Example 6.3. $\alpha = \gamma = 1, \beta = 0, a = \frac{4}{9}, b = \frac{1}{3}$.

In this case, $A = \frac{4}{9}, B = \frac{1}{3}$ and $\mu = \nu$. Then we can find the values

$$p = q = \frac{9}{15}, \quad W_0 = \frac{36}{25}, \quad W_1 = W_2 = 0.$$

Therefore, by Theorem 6.1 the probability measure μ is given by

$$d\mu(x) = \frac{36\sqrt{1-x^2}}{\pi(25-30x+9x^2)} 1_{(-1,1)}(x) dx.$$

Example 6.4. $\alpha = \gamma = 1, \beta = 0, a = \frac{1}{8}, b = 1$.

In this case, $A = \frac{1}{8}, B = 1, \mu = \nu$ and we have the values

$$p = \frac{12}{13}, \quad q = \frac{4}{5}, \quad W_0 = \frac{8}{65}, \quad W_1 = \frac{5}{6}, \quad W_2 = 0.$$

Therefore, by Theorem 6.1 the probability measure μ is given by

$$d\mu(x) = \frac{8\sqrt{1-x^2}}{\pi(13-12x)(5-4x)} 1_{(-1,1)}(x) dx + \frac{5}{6} d\delta_{\frac{13}{12}}(x).$$

Example 6.5. $\alpha = \gamma = 1, \beta = 0, a = 8, b = 1$.

In this case, $A = 8, B = 1, \mu = \nu$ and we have the following values

$$p = \frac{5}{13}, \quad q = -\frac{3}{5}, \quad W_0 = \frac{8}{65}, \quad W_1 = \frac{3}{5}, \quad W_2 = \frac{1}{3}.$$

Therefore, by Theorem 6.1 the probability measure μ is given by

$$d\mu(x) = \frac{8\sqrt{1-x^2}}{\pi(13-5x)(5+3x)} 1_{(-1,1)}(x) dx + \frac{3}{5} d\delta_{\frac{13}{5}}(x) + \frac{1}{3} d\delta_{-\frac{5}{3}}(x).$$

Finally, we mention that the case $B < 0$ can be taken care of by replacing $d\nu(y)$ with $d\nu(-y)$ as pointed out just before the statement of Theorem 6.1. As for the general case, we simply use Equation (6.2) to derive μ from ν . Thus μ is obtained from ν in Theorem 6.1 through translation, dilation, and reflection.

Example 6.6. $\alpha = 4, \beta = 3, \gamma = 2, a = 3, b = 7.$

In this case, $A = \frac{3}{2}, B = \sqrt{2}$, and we have the following values

$$p = \frac{4}{6 + \sqrt{2}}, \quad q = \frac{4}{\sqrt{2} - 6}, \quad W_0 = \frac{6}{17}, \quad W_1 = \frac{2 + 3\sqrt{2}}{8}, \quad W_2 = 0.$$

Hence by Theorem 6.1 ν is given by

$$d\nu(y) = \frac{6\sqrt{1-y^2}}{\pi(17 + 4\sqrt{2}y - 8y^2)} 1_{(-1,1)}(y) dy + \frac{2 + 3\sqrt{2}}{8} d\delta_{\frac{6+\sqrt{2}}{4}}(y).$$

Then make the change of variables $x = 2\sqrt{2}y + 3$ in Equation (6.2) to get

$$d\mu(x) = \frac{3\sqrt{6x-1-x^2}}{4\pi(2+8x-x^2)} 1_{(3-2\sqrt{2}, 3+2\sqrt{2})}(x) dx + \frac{2 + 3\sqrt{2}}{8} d\delta_{4+3\sqrt{2}}(x).$$

7. Interacting Fock spaces

Let μ be a probability measure and let $\{P_n(x), \alpha_n, \omega_n\}_{n=0}^\infty$ be the associated orthogonal polynomials and Jacobi–Szegő parameters specified by Equation (1.1). In the interacting Fock space given by μ , we have the preservation operator \mathcal{A}^0 , annihilation operator \mathcal{A}^- , and creation operators \mathcal{A}^+ densely defined by

$$\mathcal{A}^0 P_n = \alpha_n P_n, \quad \mathcal{A}^- P_n = \omega_{n-1} P_{n-1}, \quad \mathcal{A}^+ P_n = P_{n+1}, \quad n \geq 0,$$

where $\omega_{-1} = P_{-1} = 0$ by convention as in Equation (1.1). For more information, see the paper by Accardi and Bożejko [2].

It has been shown in [3] [4] [5] that some properties of a probability measure can be characterized by the preservation operator \mathcal{A}^0 and the commutator $[\mathcal{A}^-, \mathcal{A}^+]$. For the probability measure μ in Theorem 4.1, we have

$$\mathcal{A}^0 P_n = \begin{cases} bP_n, & \text{if } n = 0, \\ \beta P_n, & \text{if } n \geq 1. \end{cases}$$

$$[\mathcal{A}^-, \mathcal{A}^+] P_n = \begin{cases} \frac{a\alpha}{2} P_n, & \text{if } n = 0, \\ \frac{\alpha}{4} (\gamma - 2a) P_n, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

Thus the class of probability measures determined by the parameters α, β, γ, a , and b provides interesting examples for the paper [5].

The special case with $\alpha = \gamma = 1$ and $\beta = b = 0$ has been studied in our previous paper [17]. These symmetric probability measures are vacuum distributions of the

field operators for the interacting Fock spaces related to the Anderson model [6] [14] [18]. Using a different type of generating function, Lu [18] has derived these symmetric probability measures. However, we do not know whether his method can be used to derive our class of probability measures in this paper.

8. Appendix: integral formulas

In Section 5 we have used the following integral formulas. Here ξ, λ , and η are real numbers.

1. $\xi^2 - \lambda\eta < 0$:

$$\begin{aligned} \int_0^\infty \frac{1}{\lambda u^2 - 2\xi u + \eta} du &= \frac{1}{\sqrt{\lambda\eta - \xi^2}} \left(\frac{\pi}{2} + \tan^{-1} \frac{\xi}{\sqrt{\lambda\eta - \xi^2}} \right) \\ \int_1^\infty \frac{1}{\lambda u^2 - 2\xi u + \eta} du &= \frac{1}{\sqrt{\lambda\eta - \xi^2}} \left(\frac{\pi}{2} - \tan^{-1} \frac{\lambda - \xi}{\sqrt{\lambda\eta - \xi^2}} \right) \\ \int_{|u|>1} \frac{1}{\lambda u^2 - 2\xi u + \eta} du \\ &= \frac{1}{\sqrt{\lambda\eta - \xi^2}} \left(\pi - \tan^{-1} \frac{\lambda - \xi}{\sqrt{\lambda\eta - \xi^2}} - \tan^{-1} \frac{\lambda + \xi}{\sqrt{\lambda\eta - \xi^2}} \right) \\ \text{p.v.} \int_{|u|>1} \frac{u}{\lambda u^2 - 2\xi u + \eta} du &= \frac{1}{2\lambda} \ln \left| \frac{\lambda + 2\xi + \eta}{\lambda - 2\xi + \eta} \right| \\ &+ \frac{\xi}{\lambda\sqrt{\lambda\eta - \xi^2}} \left(\pi - \tan^{-1} \frac{\lambda - \xi}{\sqrt{\lambda\eta - \xi^2}} - \tan^{-1} \frac{\lambda + \xi}{\sqrt{\lambda\eta - \xi^2}} \right) \end{aligned}$$

2. $\xi^2 - \lambda\eta > 0$:

$$\begin{aligned} \text{p.v.} \int_0^\infty \frac{1}{\lambda u^2 - 2\xi u + \eta} du &= \frac{1}{2\sqrt{\xi^2 - \lambda\eta}} \ln \left| \frac{\xi + \sqrt{\xi^2 - \lambda\eta}}{\xi - \sqrt{\xi^2 - \lambda\eta}} \right| \\ \text{p.v.} \int_1^\infty \frac{1}{\lambda u^2 - 2\xi u + \eta} du &= \frac{1}{2\sqrt{\xi^2 - \lambda\eta}} \ln \left| \frac{\lambda - \xi + \sqrt{\xi^2 - \lambda\eta}}{\lambda - \xi - \sqrt{\xi^2 - \lambda\eta}} \right| \\ \text{p.v.} \int_{|u|>1} \frac{1}{\lambda u^2 - 2\xi u + \eta} du &= \frac{1}{2\sqrt{\xi^2 - \lambda\eta}} \ln \left| \frac{\lambda - \eta + 2\sqrt{\xi^2 - \lambda\eta}}{\lambda - \eta - 2\sqrt{\xi^2 - \lambda\eta}} \right| \\ \text{p.v.} \int_{|u|>1} \frac{u}{\lambda u^2 - 2\xi u + \eta} du &= \frac{1}{2\lambda} \ln \left| \frac{\lambda + 2\xi + \eta}{\lambda - 2\xi + \eta} \right| \\ &+ \frac{\xi}{2\lambda\sqrt{\xi^2 - \lambda\eta}} \ln \left| \frac{\lambda - \eta + 2\sqrt{\xi^2 - \lambda\eta}}{\lambda - \eta - 2\sqrt{\xi^2 - \lambda\eta}} \right| \end{aligned}$$

3. $\xi^2 - \lambda\eta = 0$

$$\begin{aligned} \int_0^\infty \frac{1}{\lambda u^2 - 2\xi u + \eta} du &= -\frac{1}{\xi}, \quad \text{if } \lambda\xi < 0 \\ \int_1^\infty \frac{1}{\lambda u^2 - 2\xi u + \eta} du &= \frac{1}{\lambda - \xi}, \quad \text{if } \frac{\xi}{\lambda} < 1 \end{aligned}$$

$$\int_{|u|>1} \frac{1}{\lambda u^2 - 2\xi u + \eta} du = \frac{2\lambda}{\lambda^2 - \xi^2}, \quad \text{if } \xi^2 < \lambda^2$$

$$\text{p.v.} \int_{|u|>1} \frac{u}{\lambda u^2 - 2\xi u + \eta} du = \frac{1}{\lambda} \ln \left| \frac{\lambda + \xi}{\lambda - \xi} \right| + \frac{2\xi}{\lambda^2 - \xi^2}$$

4. $t \in \mathbb{R}$ with $|t| \leq 1$ or $t \in \mathbb{C}$ with $\text{Im}(t) \neq 0$

$$\int_{-1}^1 \frac{\sqrt{1-x^2}}{1-tx} dx = \frac{\pi}{t^2} (1 - \sqrt{1-t^2}),$$

where \sqrt{z} is taken to be the branch with $-\frac{\pi}{2} < \arg \sqrt{z} \leq \frac{\pi}{2}$.

References

1. Accardi, L.: Meixner classes and the square of white noise, in: *Finite and Infinite Dimensional Analysis in Honor of Leonard Gross, Contemporary Mathematics* **317** (2003) 1–13, Amer. Math. Soc.
2. Accardi, L. and Bożejko, M.: Interacting Fock space and Gaussianization of probability measures, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **1** (1998) 663–670.
3. Accardi, L., Kuo, H.-H., and Stan, A.: Characterization of probability measures through the canonically associated interacting Fock spaces, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **7** (2004) 485–505.
4. Accardi, L., Kuo, H.-H., and Stan, A.: Probability measures in terms of creation, annihilation, and neutral operators, in: *Quantum Probability and Infinite Dimensional Analysis*, M. Schürmann and U. Franz (eds.) (2005) 1–11, World Scientific.
5. Accardi, L., Kuo, H.-H., and Stan, A.: Moments and commutators of probability measures, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* (to appear).
6. Accardi, L., Lu, Y. G., and Mastropietro, V.: The semi-circle diagrams in the stochastic limit of the Anderson model, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **1** (1998) 467–483.
7. Anshelevich, M.: Orthogonal polynomials with a resolvent-type generating function, *Transactions of the American Mathematical Society* (2006)
8. Asai, N., Kubo, I., and Kuo, H.-H.: Multiplicative renormalization and generating functions I, *Taiwanese Journal of Mathematics* **7** (2003) 89–101.
9. Asai, N., Kubo, I., and Kuo, H.-H.: Generating functions of orthogonal polynomials and Szegő-Jacobi parameters, *Probability and Math. Stat.* **23** (2003) 273–291.
10. Asai, N., Kubo, I., and Kuo, H.-H.: Multiplicative renormalization and generating functions II, *Taiwanese Journal of Mathematics* **8** (2004) 593–628.
11. Asai, N., Kubo, I., and Kuo, H.-H.: Generating function method for orthogonal polynomials and Jacobi-Szegő parameters, in: *Quantum Probability and Infinite Dimensional Analysis*, M. Schürmann and U. Franz (eds.) (2005) 42–55, World Scientific.
12. Asai, N., Kubo, I., and Kuo, H.-H.: Renormalization, orthogonalization, and generating functions, in: *Quantum Information V*, T. Hida and K. Saitô (eds.) (2006) 49–56, World Scientific.
13. Bożejko, M. and Bryc, W.: On a class of free Lévy laws related to a regression problem, *J. Functional Analysis* **236** (2006) 59–77.
14. Bożejko, M., Leinert, M., and Speicher, R.: Convolution and limit theorem for conditionally free random variables, *Pacific J. Math.* **175** (1996) 357–388.
15. Chihara, T. S.: *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.
16. Kubo, I.: Generating functions of exponential type for orthogonal polynomials, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **7** (2004) 155–159.

17. Kubo, I., Kuo, H.-H., and Namli, S.: Interpolation of Chebyshev polynomials and interacting Fock spaces, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **9** (2006) 361–371.
18. Lu, Y. G.: Interacting Fock spaces related to the Anderson model, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **1** (1998) 247–287.
19. Meixner, J.: Orthogonale polynomsysteme mit einer besonderen gestalt der erzeugenden funktion, *J. Lond. Math. Soc.* **9** (1934) 6–13.
20. Shohat, J. and Tamarkin, J.: *The Problem of Moments*, Math. Surveys **1**, Amer. Math. Soc., 1943.
21. Szegő, M.: *Orthogonal Polynomials*, Coll. Publ. **23**, Amer. Math. Soc., 1975.

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