

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE BACKWARD STOCHASTIC LORENZ SYSTEM

P. SUNDAR AND HONG YIN

ABSTRACT. The backward stochastic Lorenz system is studied in this paper. Suitable a priori estimates for adapted solutions of the backward stochastic Lorenz system are obtained. The existence and uniqueness of solutions is shown by the use of suitable truncations and approximations. The continuity of the adapted solutions with respect to the terminal data is also established.

1. Introduction

In a celebrated work, Edward N. Lorenz introduced a nonlinear system of ordinary differential equations describing fluid convection of nonperiodic flows (Lorenz [9]). The derivation of these equations is from a model of fluid flow within a region of uniform depth and with higher temperature at the bottom (Rayleigh [14]).

Lorenz introduced three time-dependent variables. The variable X is proportional to the intensity of the convective motion, Y is proportional to the temperature difference between ascending and descending currents, and Z is proportional to distortion of the vertical temperature profile from linearity. The model consists of the following three equations:

$$\begin{cases} \dot{X} = -aX + aY \\ \dot{Y} = -XZ + bX - Y \\ \dot{Z} = XY - cZ \end{cases} \quad (1.1)$$

where a is the *Prandtl number*, b is the temperature difference of the heated layer and c is related to the size of the fluid cell. The numbers a , b , and c are all positive.

In the past 40 years, ranging from physics (Sparrow [17]) to physiology of the human brain (Weiss [19]), Lorenz systems have been widely studied in many areas for a variety of parameter values. Randomness has also been introduced into Lorenz system and some properties of the forward system have been studied (Schmalfuß[16] and Keller [7]).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space on which a 3-dimensional Wiener process $\{W_t\}$ is defined, such that $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration of $\{W_t\}$,

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augmented by all the P -null subsets of \mathcal{F} . Define a matrix A as follows:

$$A = \begin{pmatrix} a & -a & 0 \\ -b & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$$

For any $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $\bar{y} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix}$ in \mathbb{R}^3 , we define an operator B as follows:

$$B(y, \bar{y}) = \begin{pmatrix} 0 \\ y_1 \bar{y}_3 \\ -y_1 \bar{y}_2 \end{pmatrix}$$

Then the *backward stochastic Lorenz system* corresponding to equation (1.1) is given by the following terminal value problem:

$$\begin{cases} dY(t) = -(AY(t) + B(Y(t), Y(t)))dt + Z(t)dW(t) \\ Y(T) = \xi \end{cases} \quad (1.2)$$

for $t \in [0, T]$ and $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^3)$. The integral form of the backward stochastic Lorenz system is as follows:

$$Y(t) = \xi + \int_t^T (AY(s) + B(Y(s), Y(s)))ds - \int_t^T Z(s)dW(s).$$

The problem consists in finding a pair of adapted solutions $\{(Y(t), Z(t))\}_{t \in [0, T]}$.

Definition 1.1. A pair of processes $(Y(t), Z(t)) \in \mathcal{M}[0, T]$ is called an adapted solution of (1.2) if the following holds:

$$Y(t) = \xi + \int_t^T (AY(s) + B(Y(s), Y(s)))ds - \int_t^T Z(s)dW(s) \quad \forall t \in [0, T], \text{ P-a.s.}$$

Here $\mathcal{M}[0, T] = L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^3)) \times L^2_{\mathcal{F}}(\Omega; L^2(0, T; \mathbb{R}^{3 \times 3}))$ and it is equipped with the norm

$$\|Y(\cdot), Z(\cdot)\|_{\mathcal{M}[0, T]} = \{E(\sup_{0 \leq t \leq T} |Y(t)|^2) + E \int_0^T |Z(t)|^2 dt\}^{\frac{1}{2}}.$$

It is worthwhile to emphasize that the solution pair $\{(Y(t), Z(t)) : t \in [0, T]\}$ is required to be adapted to the *forward* filtration $\{\mathcal{F}_t : t \in [0, T]\}$, and $Y(T)$ is specified as an \mathcal{F}_T -measurable random variable where T is the terminal time. The stochastic integrals that appear throughout this article are therefore forward Itô integrals though the time parameter t appears as the lower limit of the integrals. This is in contrast to the backward Itô integrals that are employed in the book by Kunita [8].

Linear backward stochastic differential equations were introduced by Bismut in 1973 [1], and the systematic study of general backward stochastic differential equations (BSDEs for short) were put forward first by Pardoux and Peng in 1990 [13]. Since the theory of BSDEs is well connected with nonlinear partial differential equations, nonlinear semigroups and stochastic controls, it has been intensively studied in the past two decades. There are also various applications of BSDEs in

the theory of mathematical finance. For instance, the hedging and pricing of a contingent claim can be described as linear BSDEs.

In the present work, since the coefficient B is nonlinear and unbounded, the existing theory of BSDEs does not apply. To overcome this difficulty, a truncation of the coefficient and an approximation scheme have been used. The Lorenz system shares certain features with the Navier-Stokes equations. Adapted solutions of the two dimensional backward stochastic Navier-Stokes equations have also been studied recently [18].

The organization of the paper is as follows. In section 2, a priori estimates for the solutions of systems with uniformly bounded terminal values are obtained and a truncation of the system is introduced. In section 3, we prove the existence and uniqueness of the solution to the Lorenz system using an approximation scheme. Section 4 is devoted to the continuity of the solutions with respect to terminal data.

2. A Priori Estimates

Let us list two frequently used results. The first one is a simple property of B , and the second result is the Gronwall inequality for backward differential equations.

Proposition 2.1. *If y and $\bar{y} \in \mathbb{R}^3$, then $\langle B(y, \bar{y}), \bar{y} \rangle = 0$ and $|B(y, y) - B(\bar{y}, \bar{y})|^2 \leq |y|^2 + |\bar{y}|^2 |y - \bar{y}|^2$*

Proposition 2.2. *Suppose that $g(t)$, $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are integrable functions, and $\beta(t), \gamma(t) \geq 0$. For $0 \leq t \leq T$, if*

$$g(t) \leq \alpha(t) + \beta(t) \int_t^T \gamma(\rho)g(\rho)d\rho \tag{2.1}$$

then

$$g(t) \leq \alpha(t) + \beta(t) \int_t^T \alpha(\eta)\gamma(\eta)e^{\int_\eta^t \beta(\rho)\gamma(\rho)d\rho} d\eta.$$

In particular, if $\alpha(t) \equiv \alpha$, $\beta(t) \equiv \beta$ and $\gamma(t) \equiv 1$, then

$$g(t) \leq \alpha(2 - e^{-\beta(T-t)})$$

Let $E^{\mathcal{F}_t} X$ to be the conditional expectation $E(X|\mathcal{F}_t)$, and let us list two assumptions:

- 1° $|\xi|^2 \leq K$ for some constant K , P-a.s.
- 2° $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$

Proposition 2.3. *Under Assumption 1°, if $(Y(t), Z(t))$ is an adapted solution for Lorenz system (1.2), then there exists a constant N_0 , such that $|Y(t)| \leq N_0$ for all $t \in [0, T]$, P-a.s.*

Proof. Applying the Itô formula to $|Y(t)|^2$, and by Proposition 2.1,

$$|Y(t)|^2 + \int_t^T \|Z(s)\|^2 ds = |\xi|^2 + \int_t^T 2\langle Y(s), AY(s) \rangle ds - 2 \int_t^T \langle Y(s), Z(s) \rangle dW(s).$$

For all $0 \leq r \leq t \leq T$, we have:

$$\begin{aligned} & E^{\mathcal{F}_r} |Y(t)|^2 + E^{\mathcal{F}_r} \int_t^T \|Z(s)\|^2 ds \\ &= E^{\mathcal{F}_r} |\xi|^2 + 2E^{\mathcal{F}_r} \int_t^T \langle Y(s), AY(s) \rangle ds \\ &\leq E^{\mathcal{F}_r} |\xi|^2 + 2\|A\| \int_t^T E^{\mathcal{F}_r} |Y(s)|^2 ds. \end{aligned}$$

By Gronwall's inequality (2.1),

$$\begin{aligned} & E^{\mathcal{F}_r} |Y(t)|^2 + E^{\mathcal{F}_r} \int_t^T \|Z(s)\|^2 ds \\ &\leq E^{\mathcal{F}_r} |\xi|^2 + 2\|A\| \int_t^T E^{\mathcal{F}_r} |\xi|^2 e^{\int_s^t 2\|A\| dv} ds \\ &= E^{\mathcal{F}_r} |\xi|^2 (2 - e^{-2\|A\|(T-t)}). \end{aligned}$$

Letting r to be t , and by Assumption 1 $^\circ$, $|Y(t)|^2 \leq N_0$ for some constant $N_0 > 0$ which is only related to K . \square

Definition 2.4. Let $b(y) = Ay + B(y, y)$, and for all $N \in \mathbb{N}$ and $y \in \mathbb{R}^3$, we define

$$b^N(y) = \begin{cases} b(y) & \text{if } |y| \leq N \\ b(\frac{y}{|y|}N) & \text{if } |y| > N, \end{cases}$$

and the *truncated Lorenz system* is the following BSDE:

$$\begin{cases} dY^N(t) = -b^N(Y^N(t))dt + Z^N(t)dW(t) \\ Y^N(T) = \xi \end{cases} \quad (2.2)$$

where $W(t)$ is the 3-dimensional Wiener process defined on a complete probability space (Ω, \mathcal{F}, P) and $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^3)$.

Corollary 2.5. Under Assumption 1 $^\circ$, if $(Y^N(t), Z^N(t))$ is an adapted solution for truncated Lorenz system (2.2), then there exists a constant N_0 , such that $|Y^N(t)| \leq N_0$ for all $N \in \mathbb{N}$ and $t \in [0, T]$, P -a.s.

Proof. If $|Y^N(t)| \leq N$, then $\langle Y^N(t), B(Y^N(t), Y^N(t)) \rangle = 0$. If $|Y^N(t)| > N$, then we also have $\langle Y^N(t), B(\frac{Y^N(t)}{|Y^N(t)|}N, \frac{Y^N(t)}{|Y^N(t)|}N) \rangle = 0$. Let

$$a^N(y) = \begin{cases} Ay & \text{if } |y| \leq N \\ A\frac{y}{|y|}N & \text{if } |y| > N. \end{cases}$$

Then $\langle Y^N(t), b^N(Y^N(t)) \rangle = \langle Y^N(t), a^N(Y^N(t)) \rangle$.

An application of Itô formula to $|Y^N(t)|^2$ and the above equality yields

$$\begin{aligned} |Y^N(t)|^2 + \int_t^T \|Z^N(s)\|^2 ds &= |\xi|^2 + \int_t^T 2\langle Y^N(s), a^N(Y^N(s)) \rangle ds \\ &\quad - 2 \int_t^T \langle Y^N(s), Z^N(s) \rangle dW(s). \end{aligned}$$

Similar to the proof of Proposition 2.3, we get $|Y^N(t)|^2 \leq N_0$ for some constant $N_0 > 0$ which is only related to K . \square

3. Existence and Uniqueness of Solutions

Proposition 3.1. *The function b^N is Lipschitz continuous on \mathbb{R}^3 .*

Proof. For any y and $\bar{y} \in \mathbb{R}^3$, let us assume that $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $\bar{y} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix}$.

Case I: $|y|, |\bar{y}| \leq N$. Then we have

$$\begin{aligned} |b^N(y) - b^N(\bar{y})| &= |b(y) - b(\bar{y})| \\ &= |Ay + B(y, y) - A\bar{y} - B(\bar{y}, \bar{y})| \\ &\leq \|A\| |y - \bar{y}| + \sqrt{(y_1 y_3 - \bar{y}_1 \bar{y}_3)^2 + (y_1 y_2 - \bar{y}_1 \bar{y}_2)^2} \\ &\leq |ac - abc| |y - \bar{y}| + N |y - \bar{y}|. \end{aligned}$$

Let $L_N = |ac - abc| + N$. Thus $|b^N(y) - b^N(\bar{y})| \leq L_N |y - \bar{y}|$.

Case II: $|y| \leq N$, but $|\bar{y}| > N$. Then by Case I, we have

$$|b^N(y) - b^N(\bar{y})| = |b(y) - b(\frac{\bar{y}}{|\bar{y}|} N)| \leq L_N |y - \frac{\bar{y}}{|\bar{y}|} N|.$$

Let us prove that $|y - \frac{\bar{y}}{|\bar{y}|} N| \leq |y - \bar{y}|$.

By carefully choosing a coordinate system, It is possible to make $\bar{y} = (\bar{y}_1, 0, 0)$. Under such coordinate system, we have

$$\begin{aligned} |y - \frac{\bar{y}}{|\bar{y}|} N| &= [(y_1 - \text{sign}(\bar{y}_1) N)^2 + y_2^2 + y_3^2]^{\frac{1}{2}} \quad \text{and} \\ |y - \bar{y}| &= [(y_1 - \bar{y}_1)^2 + y_2^2 + y_3^2]^{\frac{1}{2}}. \end{aligned}$$

Since $|y_1| \leq N < |\bar{y}_1|$, it is clear that $|y_1 - \text{sign}(\bar{y}_1) N| \leq |y_1 - \bar{y}_1|$. So $|y - \frac{\bar{y}}{|\bar{y}|} N| \leq |y - \bar{y}|$ and thus $|b^N(y) - b^N(\bar{y})| \leq L_N |y - \bar{y}|$.

Case III: $|y| > N$ and $|\bar{y}| > N$. Then by Case I, we have

$$|b^N(y) - b^N(\bar{y})| = |b(\frac{y}{|y|} N) - b(\frac{\bar{y}}{|\bar{y}|} N)| \leq L_N |\frac{y}{|y|} N - \frac{\bar{y}}{|\bar{y}|} N|.$$

Without lose of generality, let us assume that $|y| \leq |\bar{y}|$. Consider $|y|$ as N in Case II. It is clear that

$$|\frac{y}{|y|} N - \frac{\bar{y}}{|\bar{y}|} N| = \frac{N}{|y|} |y - \frac{|y|}{|\bar{y}|} \bar{y}| \leq \frac{N}{y} |y - \bar{y}|,$$

Thus we have shown that

$$|b^N(y) - b^N(\bar{y})| \leq L_N |y - \bar{y}| \quad \text{for Case III}$$

and the proof is complete. \square

Theorem 3.2. *Under Assumption 1°, the Lorenz system (1.2) has a unique solution.*

Proof. First let us prove the existence of the solution of Lorenz system (1.2). By Proposition 3.1, b^N is Lipschitz. Thus there exists a unique solution $(Y^N(t), Z^N(t))$ of truncated Lorenz system (2.2) with such b^N for each $N \in \mathbb{N}$ (see Yong and Zhou [20]).

Because of Assumption 1^o, by Corollary 2.5, there exists a natural number N_0 , such that $|Y^N(t)| \leq N_0$ for all $N \in \mathbb{N}$. By taking $N = N_0$, it follows that

$$|Y^{N_0}(t)| \leq N_0 \implies b^{N_0}(Y^{N_0}(t)) = b(Y^{N_0}(t))$$

by the definition of $b^N(y)$. Thus for N_0 , truncated Lorenz system (2.2) is the same as Lorenz system (1.2). Hence $(Y^{N_0}(t), Z^{N_0}(t))$ is also solution of Lorenz system (1.2).

Let $(Y(t), Z(t))$ and $(\bar{Y}(t), \bar{Z}(t))$ be two pairs of solutions of Lorenz system (1.2). By Proposition 2.3, there exists a natural number N_0 , such that $|Y(t)| \leq N_0$ and $|\bar{Y}(t)| \leq N_0$. Since Lorenz system (1.2) and truncated Lorenz system (2.2) for $N = N_0$ are the same, $(Y(t), Z(t))$ and $(\bar{Y}(t), \bar{Z}(t))$ are also solutions of truncated Lorenz system (2.2) for $N = N_0$. Since the truncated Lorenz system (2.2) has a unique solution for $N = N_0$, we know that $(Y(t), Z(t)) = (\bar{Y}(t), \bar{Z}(t))$ P-a.s. Thus the uniqueness of the solution has been shown. \square

Definition 3.3. For any ξ satisfies Assumption 2^o and $n \in \mathbb{N}$, we define $\xi^n = \xi \vee (-n) \wedge n$, and the n -Lorenz system is the following BSDE:

$$\begin{cases} dY^n(t) = -b(Y^n(t))dt + Z^n(t)dW(t) \\ Y^n(T) = \xi^n \end{cases} \tag{3.1}$$

where $W(t)$ is the 3-dimensional Wiener process defined on a complete probability space (Ω, \mathcal{F}, P) .

Proposition 3.4. Under Assumption 2^o, the solutions of n -Lorenz systems are Cauchy in $\mathcal{M}[0, T]$.

Proof. Since ξ^n is bounded by n , the existence and uniqueness of the solution of n -Lorenz system is guaranteed by Proposition 2.3.

For all n and $m \in \mathbb{N}$, let $(Y^n(t), Z^n(t))$ and $(Y^m(t), Z^m(t))$ be the unique solutions of n -Lorenz system and m -Lorenz system, respectively. Define $\tilde{Y}(t) = Y^n(t) - Y^m(t)$, $\tilde{Z}(t) = Z^n(t) - Z^m(t)$ and $\tilde{\xi} = \xi^n - \xi^m$. Apply Itô formula to $|\tilde{Y}(t)|^2$ to get

$$\begin{aligned} & |\tilde{Y}(t)|^2 + \int_t^T \|\tilde{Z}(s)\|^2 ds \leq |\tilde{\xi}|^2 + \int_t^T 2\|A\| |\tilde{Y}(s)|^2 ds \\ & + 2 \int_t^T \langle \tilde{Y}(s), B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s)) \rangle ds \\ & - 2 \int_t^T \langle \tilde{Y}(s), \tilde{Z}(s) \rangle dW(s) \end{aligned} \tag{3.2}$$

It is easy to show that

$$\begin{aligned}
 & \langle \tilde{Y}(s), B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s)) \rangle \\
 &= \langle Y^n(s) - Y^m(s), B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s)) \rangle \\
 &= -\langle Y^n(s), B(Y^m(s), Y^m(s)) \rangle - \langle Y^m(s), B(Y^n(s), Y^n(s)) \rangle \\
 &= \langle Y^m(s), B(Y^m(s), Y^n(s)) \rangle - \langle Y^m(s), B(Y^n(s), Y^n(s)) \rangle \\
 &= \langle Y^m(s), B(Y^m(s) - Y^n(s), Y^n(s)) \rangle \\
 &= \langle Y^m(s) - Y^n(s), B(Y^m(s) - Y^n(s), Y^n(s)) \rangle \\
 &= \langle \tilde{Y}(s), B(\tilde{Y}(s), Y^n(s)) \rangle \\
 &\leq |\tilde{Y}(s)|^2 |Y^n(s)|
 \end{aligned}$$

It follows from Proposition 2.3 that

$$|Y^n(t)|^2 \leq (2 - e^{-2\|A\|(T-t)}) E^{\mathcal{F}_t} |\xi^n|^2 \leq 2n^2.$$

So $|Y^n(t)| \leq \sqrt{2n} \forall t$.

Thus it has been shown that

$$\begin{aligned}
 & |\langle \tilde{Y}(s), B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s)) \rangle| \\
 &\leq |\tilde{Y}(s)|^2 |Y^n(s)| \leq \sqrt{2n} |\tilde{Y}(s)|^2
 \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), one gets

$$\begin{aligned}
 |\tilde{Y}(t)|^2 + \int_t^T \|\tilde{Z}(s)\|^2 ds &\leq |\tilde{\xi}|^2 + (2\|A\| + 2\sqrt{2n}) \int_t^T |\tilde{Y}(s)|^2 ds \\
 &\quad - 2 \int_t^T \langle \tilde{Y}(s), \tilde{Z}(s) \rangle dW(s).
 \end{aligned} \tag{3.4}$$

Taking expectation on both sides of (3.4), it follows that

$$E|\tilde{Y}(t)|^2 + E \int_t^T \|\tilde{Z}(s)\|^2 ds \leq E|\tilde{\xi}|^2 + (2\|A\| + 2\sqrt{2n}) \int_t^T E|\tilde{Y}(s)|^2 ds. \tag{3.5}$$

Hence by Gronwall's inequality, we get

$$\begin{aligned}
 & E|\tilde{Y}(t)|^2 + E \int_t^T \|\tilde{Z}(s)\|^2 ds \\
 &\leq E|\tilde{\xi}|^2 + (2\|A\| + 2\sqrt{2n}) \int_t^T E|\tilde{\xi}|^2 e^{(2\|A\| + 2\sqrt{2n})(t-s)} ds \\
 &\leq 2E|\tilde{\xi}|^2.
 \end{aligned} \tag{3.6}$$

Since $E|\xi|^2 < \infty$ and $|\tilde{\xi}|^2 \leq 2|\xi|^2$, an application of the Lebesgue dominated convergence theorem yields

$$\lim_{m,n \rightarrow \infty} E|\tilde{\xi}|^2 = E \lim_{m,n \rightarrow \infty} |\tilde{\xi}|^2 = 0 \tag{3.7}$$

So it follows from (3.6) that

$$\lim_{m,n \rightarrow \infty} E \int_0^T \|\tilde{Z}(s)\|^2 ds = 0 \text{ and } \lim_{m,n \rightarrow \infty} E|\tilde{Y}(t)|^2 = 0 \tag{3.8}$$

On the other hand, by means of the Burkholder-Davis-Gundy inequality, one gets

$$\begin{aligned}
& E\left\{ \sup_{t \leq \rho \leq T} \left| \int_{\rho}^T \langle \tilde{Y}(s), \tilde{Z}(s) \rangle dW(s) \right| \right\} \\
& \leq 2E \sup_{t \leq \rho \leq T} \left| \int_t^{\rho} \langle \tilde{Y}(s), \tilde{Z}(s) \rangle dW(s) \right| \\
& \leq 4\sqrt{2}E \left\{ \int_t^T |\tilde{Y}(s)|^2 \|\tilde{Z}(s)\|^2 ds \right\}^{\frac{1}{2}} \\
& \leq \frac{1}{4}E \left(\sup_{t \leq \rho \leq T} |\tilde{Y}(\rho)|^2 \right) + 32E \int_t^T \|\tilde{Z}(s)\|^2 ds
\end{aligned} \tag{3.9}$$

Thus from (3.2),(3.3) and (3.9), one gets

$$\begin{aligned}
& E \left(\sup_{t \leq \rho \leq T} |\tilde{Y}(\rho)|^2 \right) + E \int_t^T \|\tilde{Z}(s)\|^2 ds \\
& \leq E|\tilde{\xi}|^2 + E \int_t^T (2\|A\| + 2|Y^n(s)|) |\tilde{Y}(s)|^2 ds \\
& \quad + \frac{1}{2}E \left(\sup_{t \leq \rho \leq T} |\tilde{Y}(\rho)|^2 \right) + 64E \int_t^T \|\tilde{Z}(s)\|^2 ds
\end{aligned}$$

Hence it follows from (3.6) and the above inequality that

$$\frac{1}{2}E \left(\sup_{t \leq \rho \leq T} |\tilde{Y}(\rho)|^2 \right) \leq 127E|\tilde{\xi}|^2 + E \int_t^T (2\|A\| + 2|Y^n(s)|) |\tilde{Y}(s)|^2 ds \tag{3.10}$$

From Proposition 2.3, one has

$$|Y^n(s)|^2 \leq E^{\mathcal{F}_s} |\xi^n|^2 (2 - e^{-2\|A\|(T-t)}) \leq 2E^{\mathcal{F}_s} |\xi|^2 \tag{3.11}$$

Clearly $\{E^{\mathcal{F}_s} |\xi|^2\}_{s \in [0, T]}$ is a \mathcal{F} -adapted martingale. By Doob's submartingale inequality, for any $\lambda > 0$,

$$P\left\{ \sup_{0 \leq s \leq T} E^{\mathcal{F}_s} |\xi|^2 \geq \lambda \right\} \leq \frac{1}{\lambda} EE^{\mathcal{F}_T} |\xi|^2 = \frac{1}{\lambda} E|\xi|^2 \rightarrow 0 \text{ as } \lambda \rightarrow \infty$$

Let $\tau_R = \inf\{t : E^{\mathcal{F}_t} |\xi|^2 > R\} \wedge T$ for $R > 0$. It is easy to show that $\tau_R \rightarrow T$ a.s. as $R \rightarrow \infty$. From (3.10) and Gronwall's inequality, one gets

$$\begin{aligned}
& E \sup_{t \leq \rho \leq T} |\tilde{Y}(\rho \wedge \tau_R)|^2 \\
& \leq 254E|\tilde{\xi}|^2 + 2E \int_t^T (2\|A\| + 2|Y^n(s \wedge \tau_R)|) |\tilde{Y}(s \wedge \tau_R)|^2 ds \\
& \leq 254E|\tilde{\xi}|^2 + 4(\|A\| + R) \int_t^T E \sup_{s \leq \rho \leq T} |\tilde{Y}(\rho \wedge \tau_R)|^2 ds \\
& \leq 508E|\tilde{\xi}|^2 \text{ for all } t \in [0, T]
\end{aligned}$$

An application of monotone convergence theorem yields

$$\begin{aligned}
 & \lim_{m,n \rightarrow \infty} E(\sup_{0 \leq \rho \leq T} |\tilde{Y}(\rho)|^2) \\
 &= \lim_{m,n \rightarrow \infty} E(\lim_{R \rightarrow \infty} \sup_{0 \leq \rho \leq T} |\tilde{Y}(\rho \wedge \tau_R)|^2) \\
 &= \lim_{m,n \rightarrow \infty} \lim_{R \rightarrow \infty} E(\sup_{0 \leq \rho \leq T} |\tilde{Y}(\rho \wedge \tau_R)|^2) \\
 &\leq \lim_{m,n \rightarrow \infty} 508E[\tilde{\xi}]^2 = 0
 \end{aligned} \tag{3.12}$$

From (3.8), (3.12), and the definition of the norm of $\mathcal{M}[0, T]$, we know that the solutions of n-Lorenz systems are Cauchy. \square

Since $\mathcal{M}[0, T]$ is a Banach space, we know that there exists $(Y, Z) \in \mathcal{M}[0, T]$, such that

$$\lim_{n \rightarrow \infty} \{ E(\sup_{0 \leq t \leq T} |Y^n(t) - Y(t)|^2) + E \int_0^T |Z^n(t) - Z(t)|^2 dt \} = 0$$

We want to show that (Y, Z) is actually the solution of Lorenz system (1.2).

Theorem 3.5. *Under Assumption A2, Lorenz system (1.2) has a unique solution.*

Proof. By Proposition 3.4, we know that $Y^n(t)$ converge to $Y(t)$ uniformly. Since $Y^n(t)$ and $Y(t)$ are all bounded, it is easy to see that

$$\lim_{n \rightarrow \infty} E \int_t^T |b(Y^n(s)) - b(Y(s))|^2 ds = 0$$

By Itô isometry, we have

$$\lim_{n \rightarrow \infty} E(\int_t^T (Z^n(s) - Z(s)) dW(s))^2 = \lim_{n \rightarrow \infty} E \int_0^T |Z^n(s) - Z(s)|^2 ds = 0$$

Thus we have shown that $(Y(t), Z(t))$ satisfies Lorenz system (1.2), i.e. $(Y(t), Z(t))$ is a solution of Lorenz system (1.2).

Now assume that $(Y(t), Z(t))$ and $(Y'(t), Z'(t))$ are two solutions of Lorenz system (1.2). Similar to the proof of Proposition 3.4, we can get

$$E(\sup_{0 \leq t \leq T} |Y(t) - Y'(t)|^2) = 0 \text{ and } E \int_0^T |Z(t) - Z'(t)|^2 dt = 0,$$

That is, $\|(Y(t), Z(t)) - (Y'(t), Z'(t))\|_{\mathcal{M}[0, T]} = 0$. Thus we have proved the uniqueness of the solution. \square

4. Continuity with respect to Terminal Data

Theorem 4.1. *Assume that $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$. Then the solution of (1.2) is continuous with respect to the terminal data.*

Proof. For any $\xi, \zeta \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, let $(Y(t), Z(t))$ and $(X(t), V(t))$ be solutions of (1.2) under terminal values ξ and ζ , respectively. Let $(Y^n(t), Z^n(t))$ and

$(X^n(t), V^n(t))$ be solutions of corresponding n-Lorenz system. By Proposition 3.4, we know that

$$\lim_{n \rightarrow \infty} E \sup_{0 \leq t \leq T} |Y(t) - Y^n(t)|^2 = \lim_{n \rightarrow \infty} E \int_0^T \|Z(t) - Z^n(t)\|^2 dt = 0 \quad \text{and}$$

$$\lim_{n \rightarrow \infty} E \sup_{0 \leq t \leq T} |X(t) - X^n(t)|^2 = \lim_{n \rightarrow \infty} E \int_0^T \|V(t) - V^n(t)\|^2 dt = 0$$

Similar to the proof of Proposition 3.4, one can show that

$$E \int_0^T \|Z^n(t) - V^n(t)\|^2 dt \leq 2E|\xi^n - \zeta^n|^2$$

and

$$E \sup_{0 \leq t \leq T} |Y^n(t) - X^n(t)|^2 \leq 508E|\xi^n - \zeta^n|^2$$

Hence

$$E \sup_{0 \leq t \leq T} |Y^n(t) - X^n(t)|^2 + E \int_0^T \|Z^n(t) - V^n(t)\|^2 dt \leq 510E|\xi^n - \zeta^n|^2$$

Thus

$$\begin{aligned} & E \sup_{0 \leq t \leq T} |Y(t) - X(t)|^2 + E \int_0^T \|Z(t) - V(t)\|^2 dt \\ & \leq \lim_{n \rightarrow \infty} 3E \sup_{0 \leq t \leq T} (|Y(t) - Y^n(t)|^2 + |Y^n(t) - X^n(t)|^2 + |X(t) - X^n(t)|^2) \\ & \quad + \lim_{n \rightarrow \infty} 3E \int_0^T (\|Z(t) - Z^n(t)\|^2 + \|Z^n(t) - V^n(t)\|^2 + \|V(t) - V^n(t)\|^2) dt \\ & \leq \lim_{n \rightarrow \infty} 1530E|\xi^n - \zeta^n|^2 = 1530E|\xi - \zeta|^2 \end{aligned}$$

□

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P. SUNDAR: DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA

E-mail address: `sundar@math.lsu.edu`

HONG YIN: DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA

E-mail address: `yinhong@math.lsu.edu`