Communications on Stochastic Analysis
Vol. 10, No. 1 (2016) 13-30

Serials Publications
www.serialspublications.com

STOCHASTIC INTEGRAL REPRESENTATIONS OF $\mathcal{F}$-SELFDECOMPOSABLE AND $\mathcal{F}$-SEMI-SELFDECOMPOSABLE DISTRIBUTIONS

NADJIB BOUZAR

ABSTRACT. We study two specific mappings defined on the class of infinitely divisible distributions on $\mathbb{Z}_+$ with finite log-moments. As a consequence, we re-derive some known stochastic integral representations for $\mathcal{F}$-selfdecomposable distributions and obtain some new ones for $\mathcal{F}$-semi-self-decomposable distributions. Stochastic integral representations for discrete distributions in nested subclasses of $\mathcal{F}$-selfdecomposable and $\mathcal{F}$-semi-self-decomposable distributions are established via the iterates of the aforementioned mappings.

1. Introduction

Let $\mathcal{I}(\mathbb{R}^d)$ denote the class of infinitely divisible distributions on $\mathbb{R}^d$. Stochastic integral representations for distributions in sub-classes of $\mathcal{I}(\mathbb{R}^d)$ have been the object of numerous articles over the last three decades. The subclasses $\mathcal{SD}(\mathbb{R}^d)$ of selfdecomposable distributions (see Sato [14], for e.g.) and $\mathcal{SSD}_\alpha(\mathbb{R}^d)$, $\alpha \in (0,1)$, of semi-selfdecomposable distributions (Maejima and Naito [11]) have been paid particular attention. We cite two important characterization results. For that we will need some notation first. We denote by $\mathcal{L}(Y)$ the probability law of a random variable Y and by $\lfloor t \rfloor$ the greatest integer function at $t \in \mathbb{R}$. For a probability law $\mu \in \mathcal{I}(\mathbb{R}^d)$, $\{X_t^{(\mu)}\}$ will designate a Lévy process such that $\mathcal{L}(X_1^{(\mu)}) = \mu$. We define the subclass $\mathcal{I}_{\log}(\mathbb{R}^d)$ by

$$\mathcal{I}_{\log}(\mathbb{R}^d) = \Big\{\mu \in \mathcal{I}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \ln^+ |x|\, \mu(dx) < \infty\Big\},$$

where $\ln^+ |x| = max(\ln |x|, 0)$.

A random variable X on $\mathbb{R}^d$ has a selfdecomposable distribution if and only if there exits a Lévy process $\{X_t^{(\mu)}\}$ such that

$$X \stackrel{d}{=} \int_0^\infty e^{-t}\, dX_t^{(\mu)}, \tag{1.1}$$

in which case $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{R}^d)$.

Received 2015-5-12; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60E07; Secondary 60G51.
Key words and phrases. Lévy processes, stochastic integral, composition semigroups, $\mathcal{F}$-stability, infinite divisibility, probability generating functions, weak convergence.

A random variable X on $\mathbb{R}^d$ has a semi-selfdecomposable distribution with order (span) $\alpha \in (0,1)$ if and only if there exits a Lévy process $\{X_t^{(\mu)}\}$ such that

$$X \stackrel{d}{=} \int_0^\infty \alpha^{\lfloor t \rfloor} \, dX_t^{(\mu)}, \tag{1.2}$$

in which case $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{R}^d)$.

The representation (1.1) was obtained by Wolfe [19] in the case $d = 1$ and generalized to the case $d > 1$ by Sato and Yamazato [15], and to Banach space-valued random variables by Jurek and Vervaat [9]. The result was extended to operator selfdecomposable distributions by Sato and Yamazato [16]. Versions of (1.1) for distributions in nested subclasses of $\mathcal{SD}(\mathbb{R}^d)$ were given in [9], [15], and Barndorff-Nielsen et al. [2].

Maejima and Ueda [12] established the representation (1.2) for distributions in $\mathcal{SSD}_\alpha(\mathbb{R}^d)$ and extended it to distributions in nested subclasses of $\mathcal{SSD}_\alpha(\mathbb{R}^d)$. Maejima and Miura [10] obtained a stochastic integral representation for distributions in $\mathcal{SSD}_\alpha(\mathbb{R}^d)$ similar to (1.1) but where the integration was with respect to a semi-Lévy process.

For more on the stochastic integrals with respect to Lévy processes, additive processes, and semi-Lévy processes, we refer to Rocha-Arteaga and Sato [13] and to references therein.

van Harn et al. [7] proposed discrete analogues of self-decomposability and stability for distributions on $\mathbb{Z}_+ := \{0, 1, 2, \cdots\}$. We recall below some important facts on the topic and refer without further mention to [7] and to Steutel and van Harn [18], Chapter 5, Section 8, for more details.

The $\mathbb{Z}_+$-valued multiple $\alpha \odot_{\mathcal{F}} X$ for a $\mathbb{Z}_+$-valued random variable X and $0 < \alpha < 1$ is defined as follows:

$$\alpha \odot_{\mathcal{F}} X = \sum_{k=1}^{X} Y_k(t) := Z_X(t) \quad (t = -\ln \alpha), \tag{1.3}$$

where $Y_1(\cdot), Y_2(\cdot), \cdots$ are independent copies of a continuous-time Markov branching process, independent of X, such that for every $k \geq 1$, $P(Y_k(0) = 1) = 1$. The processes $(Y_k(\cdot), k \geq 1)$ are driven by a composition semigroup of probability generating functions (pgf's) $\mathcal{F} := (F_t, t \geq 0)$:

$$F_s \circ F_t(z) = F_{s+t}(z) \quad (|z| \leq 1;\ s, t \geq 0). \tag{1.4}$$

The process $Z_X(\cdot)$ of (1.3) is itself a Markov branching process driven by $\mathcal{F}$ and starting with X individuals ($Z_X(0) = X$).

Let $P(z)$ be the pgf of X. Then the pgf $P_{\alpha \odot_{\mathcal{F}} X}(z)$ of $\alpha \odot_{\mathcal{F}} X$ is given by

$$P_{\alpha \odot_{\mathcal{F}} X}(z) = P(F_t(z)) \quad (\ t = -\ln \alpha;\ 0 \leq z \leq 1). \tag{1.5}$$

As an analogue of scalar multiplication, the operation $\odot_{\mathcal{F}}$ must satisfy some minimal conditions. In particular, the following regularity conditions are imposed on the composition semigroup $\mathcal{F}$:

$$\lim_{t \downarrow 0} F_t(z) = F_0(z) = z, \quad \lim_{t \to \infty} F_t(z) = 1. \tag{1.6}$$

The first part of (1.6) implies the continuity of the semigroup $\mathcal{F}$ (by way of (1.4)) and the second part is equivalent to assuming that $m = F_1'(1) \leq 1$, which implies the (sub-)criticality of the continuous-time Markov branching process $Y_k(\cdot)$ in (1.3).

We will restrict ourselves to the subcritical case ($m < 1$) and we will assume without loss of generality that $m = e^{-1}$.

A $\mathbb{Z}_+$-valued random variable X, or its distribution, is said to have an $\mathcal{F}$-self-decomposable distribution if for every $t > 0$,

$$X \stackrel{d}{=} e^{-t} \odot_{\mathcal{F}} X + X_t, \tag{1.7}$$

where X_t is $\mathbb{Z}_+$-valued and X and X_t are independent (see [7]).

Equivalently, by (1.5) and (1.7), a distribution on $\mathbb{Z}_+$ with pgf $P(z)$ is $\mathcal{F}$-self-decomposable if for every $t > 0$, there exists a pgf $P_t(z)$ such that

$$P(z) = P(F_t(z))P_t(z) \quad (0 \leq z \leq 1), \tag{1.8}$$

where P and P_t are the pgf's of X and X_t, respectively, in (1.7).

$\mathcal{F}$-self-decomposable distributions are infinitely divisible. Moreover the distribution of X_t is infinitely divisible for every $t > 0$.

Let $0 < \alpha < 1$. A $\mathbb{Z}_+$-valued random variable X with a nondegenerate distribution is said to be $\mathcal{F}$-semi-selfdecomposable of order α if X satisfies (1.7) for some infinitely divisible $\mathbb{Z}_+$-valued random variable X_t independent of X and for $t = -\ln\alpha$ (Bouzar [4]). $\mathcal{F}$-semi-selfdecomposable distributions are infinitely divisible. $\mathcal{F}$-semistable and $\mathcal{F}$-geometric semistable distributions are $\mathcal{F}$-semi-selfdecomposable (see [4]).

Steutel et al. [17] introduced a stochastic integral with respect to a $\mathbb{Z}_+$-valued Lévy process and used it to give an integral representation of (continuous time) sub-critical branching processes with immigration. They showed that $\mathcal{F}$-selfdecomposable distributions arise as the weak limit for such processes (as $t \to \infty$). Although not explicitly noted by the authors, but implicit in the proof of their main result (Theorem 3.2), is an integral representation for $\mathcal{F}$-selfdecomposable distributions.

The aim of this paper is to obtain the discrete analogues of the representations (1.1) and (1.2) for $\mathcal{F}$-selfdecomposable and $\mathcal{F}$-semi-selfdecomposable distributions and to extend Theorem 3.2 in [17] to $\mathcal{F}$-semi-selfdecomposable distributions. We parallel the treatment for the continuous case, as covered in [12]. In Section 2, we recall some facts on the stochastic integral with respect to a $\mathbb{Z}_+$-valued Lévy process and formally describe the associated improper integral. In Section 3, we define the mapping Ψ_α ($\alpha \in (0,1)$) on the class of infinitely divisible distributions on $\mathbb{Z}_+$ and identify its domain and range. As a corollary, we re-derive a stochastic integral representation for $\mathcal{F}$-selfdecomposable distributions obtained implicitly in [17]. In addition, we describe a fixed-point property for Ψ_α in terms of $\mathcal{F}$-stable distributions. In Section 4, we introduce and study the operator Φ_α and study its properties. We obtain a stochastic integral representation for $\mathcal{F}$ semi-selfdecomposable distributions with order $\alpha \in (0,1)$ and we show that these distributions arise as weak limits to a specific class of $\mathbb{Z}_+$-valued stochastic processes. Finally, in Section 5, stochastic integral representations for distributions in nested subclasses of $\mathcal{F}$-decomposable and $\mathcal{F}$-semi-selfdecomposable distributions are derived via the iterates of Ψ_α and Φ_α, respectively.

In the rest of this section we recall some additional definitions and results about the continuous composition semigroup of pgf's $\mathcal{F} := (F_t, t \geq 0)$.

The following characterization of $\mathcal{F}$-self-decomposable distributions (see [18], Chapter V, Theorem 8.3) plays a key role throughout the paper.

Theorem 1.1. *A function $P(z)$ on $[0,1]$ is the pgf of an $\mathcal{F}$-selfdecomposable on $\mathbb{Z}_+$ if and only if it has the form*

$$\ln P(z) = \int_z^1 \frac{\ln Q(x)}{U(x)}\, dx = \int_0^\infty \ln Q(F_t(z))\, dt \quad (0 \leq z \leq 1), \tag{1.9}$$

where $U(z)$ is the infinitesimal generator of the semigroup $\mathcal{F}$ (see definition below) and $Q(z)$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$ such that

$$\int_0^1 \frac{[-\ln Q(x)]}{U(x)}\, dx < \infty. \tag{1.10}$$

The infinitesimal generator U of the semigroup $\mathcal{F}$ is defined by

$$U(z) = \lim_{t \downarrow 0} (F_t(z) - z)/t \quad (|z| \leq 1), \tag{1.11}$$

and satisfies $U(z) > 0$ for $0 \leq z < 1$. Moreover,

$$U(x) \sim 1 - x \qquad (x \uparrow 1). \tag{1.12}$$

The related A-function is defined by

$$A(z) = \exp\Big\{ -\int_0^z (U(x))^{-1}\, dx \Big\} \quad (0 \leq z < 1). \tag{1.13}$$

The functions $U(z)$ and $A(z)$ satisfy for any $t > 0$,

$$\frac{\partial}{\partial t} F_t(z) = U(F_t(z)) = U(z) F_t'(z) \quad (|z| \leq 1), \tag{1.14a}$$

and

$$A(F_t(z)) = e^{-t} A(z) \quad (0 \leq z < 1). \tag{1.14b}$$

Following standard terminology, we will refer to any $\mathbb{Z}_+$-valued Lévy process as a $\mathbb{Z}_+$-valued subordinator which we denote by $\{X_t^{(\mu)}\}$, with $\mathcal{L}(X_1^{(\mu)}) = \mu$. We will be referring quite frequently to the following sets:

- $\mathcal{I}(\mathbb{Z}_+)$: the set of infinitely divisible distributions on $\mathbb{Z}_+$.
- $\mathcal{I}_{\log}(\mathbb{Z}_+)$: the subset of distributions $\mu = (q_n, n \geq 0)$ in $\mathcal{I}(\mathbb{Z}_+)$ such that $\sum_{n=0}^\infty q_n \ln(n+1) < \infty$.
- $\mathcal{F}$-$\mathcal{SD}(\mathbb{Z}_+)$: the subset of $\mathcal{F}$-self-decomposable distributions in $\mathcal{I}(\mathbb{Z}_+)$.
- $\mathcal{F}$-$\mathcal{SSD}_\alpha(\mathbb{Z}_+)$: the subset of distributions in $\mathcal{I}(\mathbb{Z}_+)$ that are $\mathcal{F}$-semi-selfdecomposable of order $\alpha \in (0,1)$.

2. A Stochastic Integral With Respect to a $\mathbb{Z}_+$-valued Subordinator

Let $(X_t^{(\mu)}, t \geq 0)$ be a $\mathbb{Z}_+$-valued subordinator. We will assume without loss of generality that $X_0 = 0$. Since $\mathbb{Z}_+$-valued subordinators are piecewise constant, it follows from Theorem 21.2, p. 135, in [14], that $\{X_t^{(\mu)}\}$ is a compound Poisson process, i.e., there exists a Poisson process $\{N_t\}$ and a sequence of $\mathbb{Z}_+$-valued random variables $\{C_k\}$ with $C_0 = 0$, independent of $\{N_t\}$, such that

$$X_t^{(\mu)} = \sum_{k=1}^{N_t} C_k,$$

or, equivalently,

$$X_t^{(\mu)} = \sum_{\{k:\ 0<T_k\leq t\}} C_k, \tag{2.1}$$

where $(T_k, k \geq 1)$ is the sequence of jump times of $\{X_t^{(\mu)}\}$.

We recall the definition of the stochastic integral with respect to $\{X_t^{(\mu)}\}$ introduced in [17]. Let $f(s)$ be a (Lebesgue) measurable function on an interval $[t_0, t_1]$ in $[0, \infty)$ and taking values in $(0, 1]$. We define

$$\int_{t_0}^{t_1} f(s) \odot_{\mathcal{F}}\ dX_s^{(\mu)} \stackrel{d}{=} \sum_{\{k:\ t_0<T_k\leq t_1\}} f(T_k) \odot_{\mathcal{F}} C_k. \tag{2.2}$$

As noted in [17], the operation $A \odot_{\mathcal{F}} X$ for a random variable A taking values in $[0, 1]$ has pgf

$$P_{A\odot_{\mathcal{F}}}(z) = \int_0^1 P_{a\odot_{\mathcal{F}}}(z)\, dG_A(a),$$

where G_A is the distribution function of A.

We also recall a useful representation theorem obtained in [17].

Theorem 2.1. *Let $f(s)$ be a measurable function on $[t_0, t_1]$ with range in $(0, 1]$. The stochastic integral $\int_{t_0}^{t_1} f(s) \odot_{\mathcal{F}}\ dX_s^{(\mu)}$ has an infinitely divisible distribution with pgf*

$$P(z) = \exp\Big\{ \int_{t_0}^{t_1} \ln Q_\mu\big(F_{-\ln f(s)}(z)\big)\, ds \Big\}, \tag{2.3}$$

where $Q_\mu(z)$ is the (infinitely divisible) pgf of $X_1^{(\mu)}$.

Next, we describe an improper integral.

Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$-valued subordinator and $f(s)$ a measurable function on $[0, \infty)$ with values in $(0, 1]$. We define the $\mathbb{Z}_+$-valued process $\{Y_t\}$ by

$$Y_t \stackrel{d}{=} \int_0^t f(s) \odot_{\mathcal{F}}\ dX_s^{(\mu)} \stackrel{d}{=} \sum_{\{k:\ 0<T_k\leq t\}} f(T_k) \odot_{\mathcal{F}} C_k \qquad (t \geq 0). \tag{2.4}$$

If $\{Y_t\}$ converges in distribution to a $\mathbb{Z}_+$-valued random variable Y_∞ as $t \to \infty$, then we set

$$\int_0^\infty f(s) \odot_{\mathcal{F}}\ dX_s^{(\mu)} \stackrel{d}{=} Y_\infty. \tag{2.5}$$

Theorem 2.2. *Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$-valued subordinator and $f(t)$ a measurable function on $[0,\infty)$ with values in $(0,1]$. Assume that the process $\{Y_t\}$ of (2.4) converges weakly as $t \to \infty$. Then*
(i) the distribution of $\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ is infinitely divisible and its pgf admits the representation

$$P(z) = \exp\Big\{\int_0^\infty \ln Q_\mu\big(F_{-\ln f(s)}(z)\big)\, ds\Big\}. \tag{2.6}$$

(ii) $\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ admits the following infinite series representation

$$\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{k=1}^\infty f(T_k) \odot_{\mathcal{F}} C_k. \tag{2.7}$$

Proof. By Theorem 2.1, $L(\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)})$ is infinitely divisible as the weak limit of the infinitely divisible distributions $\{\mathcal{L}(Y_t)\}$, when $t \to \infty$. Since the pgf of Y_t is (by (2.3))

$$P_t(z) = \exp\Big\{\int_0^t \ln Q_\mu\big(F_{-\ln f(s)}(z)\big)\, ds\Big\}, \tag{2.8}$$

it ensues that the pgf of $\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ admits the representation (2.6) by letting $t \to \infty$ in (2.8). The infinite series representation (2.7) follows from the second equation in (2.4). □

Remark 2.3. One can arrive at an equivalent definition of the stochastic integral (2.2) that is similar to the one constructed in the continuous case (see for e.g. [13]), by first defining it for step functions and then extending the definition to measurable functions via the standard limit process.

3. A Stochastic Integral Representation for $\mathcal{F}$-selfdecomposable Distributions

Let $\alpha \in (0,1)$. We define the mapping $\Psi_\alpha(\cdot)$ on $\mathcal{I}(\mathbb{Z}_+)$ by

$$\Psi_\alpha(\mu) = \mathcal{L}\Big(\int_0^\infty \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)}\Big), \tag{3.1}$$

where $\{X_t^{(\mu)}\}$ is a $\mathbb{Z}_+$-valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$.

We identify the domain $\mathcal{D}(\Psi_\alpha)$ and range $\mathcal{R}(\Psi_\alpha)$ of the mapping Ψ_α and as a consequence we obtain an integral representation for $\mathcal{F}$-selfdecomposable distributions on $\mathbb{Z}_+$.

We start out with a useful lemma.

Lemma 3.1. *Assume $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z)$. The following assertions are equivalent.*
(i) $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$.
(ii) (1.10) holds for $Q = Q_\mu(z)$.
(iii) $\int_0^\infty [-\ln Q_\mu(F_s(z))]\, ds < \infty$ for every $z \in [0,1]$.

Proof. (i)⇔(ii): By Proposition 4.2, Appendix A, in [18], $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ if and only if $\int_0^1 \frac{[-\ln Q_\mu(z)]}{1-x}\,dx < \infty$. The conclusion follows by applying (1.12). The statement (ii)⇔(iii) follows essentially from the proof of Theorem 8.3, Chapter V, in [18]. □

Next, we give the main result of the section. Parts of the statement of Theorem 3.2 below can be derived from the proof of Theorem 3.2 in [17]. We provide a somewhat abbreviated proof for the sake of completeness.

Theorem 3.2. *$\mathcal{D}(\Psi_\alpha) = \mathcal{I}_{\log}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha) = \mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$.*

Proof. Assume $\mu \in \mathcal{D}(\Psi_\alpha)$. By applying (2.6) to $f(s) = \alpha^s$, one can show that the pgf $P_{\Psi_\alpha(\mu)}(z)$ of $\Psi_\alpha(\mu)$ admits the representation (1.9) with $Q(z) = Q_\mu^{-1/\ln\alpha}(z)$. Therefore, by Theorem 1.1, $\Psi_\alpha(\mu)$ is $\mathcal{F}$-selfdecomposable and $Q_\mu(z)$ satisfies (1.10). It follows by Lemma 3.1[(ii)⇒(i)] that $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. We have thus shown that $\mathcal{D}(\Psi_\alpha) \subset \mathcal{I}_{\log}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha) \subset \mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$. Assume now that $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. By Lemma 3.1[(i)⇒(iii)] and Theorem 1.1, the function $P(z) = \exp\bigl\{\int_0^\infty \ln Q_\mu(F_{-s\ln\alpha}(z))\,ds\bigr\}$ is the pgf of an $\mathcal{F}$-selfdecomposable distribution. Letting $P_t(z)$ be the pgf of $Y_t = \int_0^t \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)}$ and applying Theorem 2.1, as well as (2.8), to $f(s) = \alpha^s$, we conclude that $\lim_{t\to\infty} P_t(z) = P(z)$. Therefore, $\mathcal{L}(\int_0^\infty \alpha^t \odot_{\mathcal{F}} dX_s^{(\mu)})$ exists as the weak limit of $\{Y_t\}$ as $t \to \infty$, or equivalently, $\mu \in \mathcal{D}(\Psi_\alpha)$. We have thus shown $\mathcal{I}_{\log}(\mathbb{Z}_+) \subset \mathcal{D}(\Psi_\alpha)$. It remains to prove that $\mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+) \subset \mathcal{R}(\Psi_a)$. Let $P(z)$ be the pgf of an $\mathcal{F}$-selfdecomposable distribution on $\mathbb{Z}_+$. Then $P(z)$ admits the canonical representation (1.9) for some infinitely divisible pgf $Q(z)$. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z) = Q^{-\ln\alpha}(z)$. It follows that $P(z)$ satisfies (2.6) with $f(s) = \alpha^s$. Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$-valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and let $\{Y_t\}$ be as above. It is clear that $\{Y_t\}$ converges weakly and therefore $\mathcal{L}(\int_0^\infty \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)})$ is definable and that $P(z)$ is its pgf. □

Corollary 3.3. *Let $\alpha \in (0,1)$. The mapping Ψ_α is one-to-one from $\mathcal{I}_{\log}(\mathbb{Z}_+)$ onto $\mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$.*

Proof. We only need prove Ψ_α is one-to-one. Let $\mu_1,\ \mu_2 \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ such that $\Psi_\alpha(\mu_1) = \Psi_\alpha(\mu_2)$. Denote by $P(z)$ the pgf of $\Psi_\alpha(\mu_1)$ (and thus of $\Psi_\alpha(\mu_2)$). It follows by (2.6) and (1.4) that

$$\ln \frac{P(z)}{P(F_{-t\ln\alpha}(z))} = \int_0^t \ln Q_{\mu_i}(F_{-s\ln\alpha}(z))\,ds \qquad (z \in [0,1];\ i = 1,2).$$

Therefore, $Q_{\mu_1}(F_{-t\ln\alpha}(z)) = Q_{\mu_2}(F_{-t\ln\alpha}(z))$ for every $z \in [0,1]$ and $t \geq 0$. Letting $t \downarrow 0$ and using (1.6) leads to $Q_{\mu_1}(z) = Q_{\mu_2}(z)$, or $\mu_1 = \mu_2$. □

The corollary below follows straightforwardly from Theorem 3.2 and the infinite series representation (2.7).

Corollary 3.4. *A $\mathbb{Z}_+$-valued random variable X has an $\mathcal{F}$-selfdecomposable distribution if and only if for some $\alpha \in (0,1)$, and therefore for every $\alpha \in (0,1)$, there* *exists a $\mathbb{Z}_+$-valued subordinator $\{X_t^{(\mu)}\}$, with $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ (depending on α), such that*

$$X \stackrel{d}{=} \int_0^\infty \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{k=1}^\infty \alpha^{T_k} \odot_{\mathcal{F}} C_k. \tag{3.2}$$

Here the sequences $\{T_k\}$ and $\{C_k\}$ respectively represent the jump times and jump sizes of $\{X_t^{(\mu)}\}$ (see(2.1)).

For the sake of completeness, and as a consequence of Theorem 3.2, we restate the convergence result obtained in [17] (Theorem 3.2, therein).

Corollary 3.5. *Let $\alpha \in (0,1)$. Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$-valued subordinator and $Y_t = \int_0^t \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)}$. Then Y_t converges in distribution to some $\mathbb{Z}_+$-valued random variable Y_∞ (as $t \to \infty$) if and only if $E(\ln(1 + X_1^{(\mu)})) < \infty$, in which case the distribution of Y_∞ is $\mathcal{F}$-selfdecomposable.*

Next, we derive a fixed point property for the mapping Ψ_a that is the analogue of a result obtained in [9] for distributions on Banach and Euclidean spaces.

First, we briefly recall some results on $\mathcal{F}$-stable distributions (see Chapter V, Section 8, in [18]).

A $\mathbb{Z}_+$-valued random variable X, or its distribution, is said to be $\mathcal{F}$-stable with exponent $\gamma \in (0,1]$ if for every $\alpha \in (0,1)$,

$$X \stackrel{d}{=} \alpha \odot_{\mathcal{F}} X + (1-\alpha^\gamma)^{1/\gamma} \odot_{\mathcal{F}} X', \tag{3.3}$$

where $X \stackrel{d}{=} X'$ and X and X' are independent.

An $\mathcal{F}$ stable distribution μ on $\mathbb{Z}_+$ is $\mathcal{F}$-selfdecomposable and thus necessarily infinitely divisible. Moreover, its pgf P_μ admits the following canonical representation:

$$P_\mu(z) = \exp[-\lambda A(z)^\gamma] \qquad (0 \le z \le 1), \tag{3.4}$$

where $A(z)$ is the A-function given in (1.13) and λ is a positive constant.

Theorem 3.6. *Let $\alpha \in (0,1)$. A distribution μ on $\mathbb{Z}_+$ is $\mathcal{F}$-stable if and only if $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ and there exists $c > 0$ such that the pgf's P_μ and $P_{\Psi_\alpha(\mu)}$ of μ and $\Psi_\alpha(\mu)$, respectively, satisfy the equation*

$$\ln P_{\Psi_\alpha(\mu)}(z)) = c \ln P_\mu(z), \qquad (0 \le z \le 1), \tag{3.5}$$

in which case $\Psi_\alpha(\mu)$ is also $\mathcal{F}$-stable with the same exponent γ as μ, $\gamma = \frac{-1}{c \ln \alpha} \in (0,1]$.

Proof. The "only if" part: assume that μ is $\mathcal{F}$-stable with exponent $\gamma \in (0,1]$. We have by (3.4) and the fact that $\frac{A'(x)}{A(x)} = -\frac{1}{U(x)}$, $x \in (0,1)$ (see (1.13)),

$$\int_0^1 \frac{[-\ln P_\mu(x)]}{U(x)}\, dx = \lambda \int_0^1 \frac{A(x)^\gamma}{U(x)}\, dx = \lambda \int_0^1 A(x)^{\gamma-1} A'(x)\, dx \quad (\lambda > 0).$$

A simple change of variable argument shows that the third integral in the above equation is finite, and thus $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. By (2.6) and (3.4),

$$\ln P_{\Psi_\alpha(\mu)}(z) = \int_0^\infty \ln P_\mu(F_{-s\ln\alpha}(z))\,ds = -\lambda\int_0^\infty [A(F_{-s\ln\alpha}(z)]^\gamma\,ds,$$

for some $\lambda > 0$ and $\gamma \in (0,1]$. It follows by (1.14b) that

$$\ln P_{\Psi_\alpha(\mu)}(z) = -\lambda A(z)^\gamma \int_0^\infty \alpha^{\gamma s}\,ds = \frac{\lambda}{\gamma\ln\alpha}A(z)^\gamma.$$

Therefore (3.5) holds with $c = -1/\gamma\ln\alpha$ and $\Psi_\alpha(\mu)$ is $\mathcal{F}$-stable with exponent γ (by (3.4)). We now prove the "if part". Assume that $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ and that (3.5) holds. We have via (2.6) and (1.14a)

$$\ln P_{\Psi_\alpha(\mu)}(z) = \frac{-1}{\ln\alpha}\int_0^\infty \ln P_\mu(F_s(z)\,ds = \frac{-1}{\ln\alpha}\int_0^\infty \frac{\ln P_\mu(F_s(z)}{U(F_s(z))}\frac{\partial}{\partial s}F_s(z)\,ds.$$

The change of variable $x = F_t(z)$ and (1.6) yield $\ln P_{\Psi_\alpha(\mu)}(z) = \frac{-1}{\ln\alpha}\int_z^1 \frac{\ln P_\mu(x)}{U(x)}\,dx$, which, combined with (3.5), yields in turn the differential equation

$$c\frac{d}{dx}\ln P_\mu(x) = \frac{1}{\ln\alpha}\frac{\ln P_\mu(x)}{U(x)}.$$

Since $\frac{A'(x)}{A(x)} = -\frac{1}{U(x)}$, one easily deduces that the solution $P_\mu(z)$ takes the form (3.4) with $\lambda = -\ln P_\mu(0)$ and $\gamma = -1/(c\ln\alpha)$. The fact that $\gamma \in (0,1]$ follows from a result in van Harn and Steutel [8] (see the proof of Lemma 4.2 therein). □

4. A Stochastic Integral Representation for $\mathcal{F}$-semi-selfdecomposable Distributions

Let $\alpha \in (0,1)$. We define the mapping $\Phi_\alpha(\cdot)$ on $\mathcal{I}(\mathbb{Z}_+)$ by

$$\Phi_\alpha(\mu) = \mathcal{L}\Big(\int_0^\infty \alpha^{\lfloor s\rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}\Big), \tag{4.1}$$

where $\{X_t^{(\mu)}\}$ is a $\mathbb{Z}_+$-valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and, recall, $\lfloor s\rfloor$ denotes the greatest integer function at s.

The first two results identify the domain $\mathcal{D}(\Phi_\alpha)$ and range $\mathcal{R}(\Phi_\alpha)$ of the mapping Φ_α.

Theorem 4.1. *$\mathcal{D}(\Phi_\alpha) = \mathcal{I}_{\log}(\mathbb{Z}_+)$.*

Proof. Let $f(s) = \alpha^{\lfloor s\rfloor}$ for $s \in [0,t]$, and let $\{s_0, s_1, \cdots, s_{\lfloor t\rfloor}, s_{\lfloor t\rfloor+1}\}$ be the subdivision of $[0,t]$ such that $s_j = j$, $j = 0,1,\cdots,\lfloor t\rfloor$, and $s_{\lfloor t\rfloor+1} = t$. The function $f(s)$ can be rewritten as

$$f(s) = \sum_{j=0}^{\lfloor t\rfloor} \alpha^j I_{[s_j,s_{j+1})}(s), \quad (s \in [0,t]).$$

A straightforward pgf argument, by way of Theorem 2.1 and equation (2.3), leads to

$$\int_0^t \alpha^{\lfloor s\rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)} = \sum_{j=0}^{\lfloor t\rfloor} \alpha^j \odot_F (X_{s_{j+1}}^{(\mu)} - X_{s_j}^{(\mu)}).$$

Moreover, the pgf $P_t(z)$ of $\int_0^t \alpha^{\lfloor s\rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}$ simplifies to

$$P_t(z) = \Big\{\prod_{j=0}^{\lfloor t\rfloor-1} Q_\mu(F_{-j\ln\alpha}(z))\Big\} Q_\mu^{t-\lfloor t\rfloor}(F_{-\lfloor t\rfloor\ln\alpha}(z)). \tag{4.2}$$

Since $0 \le t-\lfloor t\rfloor \le 1$ and $\lim_{t\to\infty} F_{-\lfloor t\rfloor \ln\alpha}(z) = 1$, we have $\lim_{t\to\infty} Q_\mu^{t-\lfloor t\rfloor}(F_{-\lfloor t\rfloor\ln\alpha}(z))$ $= 1$. Therefore, $\lim_{t\to\infty} P_t(z)$ exists if and only if $\lim_{n\to\infty}\prod_{j=0}^{n} Q_\mu(F_{-j\ln\alpha}(z))$ is finite and

$$\lim_{t\to\infty} P_t(z) = \lim_{n\to\infty}\prod_{j=0}^{n} Q_\mu(F_{-j\ln\alpha}(z)). \tag{4.3}$$

Now $\prod_{j=0}^n Q_\mu(F_{-j\ln\alpha}(z))$ is the pgf of the size of the n-th generation of a Galton-Watson branching process with stationary immigration described by the pgf $Q_\mu(z)$ and an offspring distribution with pgf $F_{-\ln\alpha}(z)$ (see Athreya and Ney [1], Chapter 6). By Corollary 2 in Foster and Williamson [5], $\prod_{j=0}^n Q_\mu(F_{-j\ln\alpha}(z))$ converges to a pgf if and only if $\sum_{n=0}^\infty q_n \ln n < \infty$, where $\{q_n\}$ is the distribution with pgf Q_μ, or equivalently (since $\ln(n+1) \sim \ln n$ as $n\to\infty$), if and only if $E(\ln(1+X_1^{(\mu)})) < \infty$. Therefore, $\Phi_\alpha(\mu)$ exists if and only if $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. □

Theorem 4.2. $\mathcal{R}(\Phi_\alpha) = \mathcal{F}\text{-}\mathcal{SSD}_\alpha(\mathbb{Z}_+)$.

Proof. Assume that $\Phi_\alpha(\mu)$ exists for some $\mu \in \mathcal{I}(\mathbb{Z}_+)$. Let $P_t(z)$ be the pgf of $\int_0^t \alpha^{\lfloor s\rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}$. Then by (4.3), the pgf $P_{\Phi_\alpha(\mu)}(z)$ of $\Phi_\alpha(\mu)$ can be written as

$$P_{\Phi_\alpha(\mu)}(z) = \lim_{n\to\infty}\prod_{j=0}^{n} Q_\mu(F_{-j\ln\alpha}(z)) = \Big[\lim_{n\to\infty}\prod_{j=1}^{n} Q_\mu(F_{-j\ln\alpha}(z))\Big] Q_\mu(z), \tag{4.4}$$

or,

$$\begin{aligned} P_{\Phi_\alpha(\mu)}(z) &= \Big[\lim_{n\to\infty}\prod_{j=1}^{n} Q_\mu(F_{-(j-1)\ln\alpha}(F_{-\ln\alpha}(z))\Big] Q_\mu(z) \\ &= P(F_{-\ln\alpha}(z))Q_\mu(z). \end{aligned} \tag{4.5}$$

Since $Q_\mu(z)$ is infinitely divisible, it follows by (1.8) (applied at $t = -\ln\alpha$) that $\Phi_\alpha(\mu) \in \mathcal{F}\text{-}\mathcal{SSD}_\alpha(\mathbb{Z}_+)$ and thus $\mathcal{R}(\Phi_\alpha) \subset \mathcal{F}\text{-}\mathcal{SSD}_\alpha(\mathbb{Z}_+)$. Assume now that $P(z)$ is the pgf of an $\mathcal{F}$-semi-selfdecomposable distribution on $\mathbb{Z}_+$ with order α. We have by (1.8) that $P(z) = P(F_{-\ln\alpha}(z))Q(z)$ for some infinitely divisible pgf $Q(z)$. It follows by equation (2.2) in [4] that

$$P(z) = \lim_{n\to\infty}\prod_{j=0}^{n} Q(F_{-j\ln\alpha}(z)). \tag{4.6}$$

Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q(z)(= Q_\mu(z))$ and let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$-valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$. We conclude by (4.3) and (4.6) that $\mathcal{L}(\int_0^\infty \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)})$ is definable and that $P(z)$ is its pgf. We have thus shown $\mathcal{F}\text{-}\mathcal{SSD}_\alpha(\mathbb{Z}_+) \subset \mathcal{R}(\Phi_\alpha)$. □

Corollaries 4.3 and 4.4 below are straightforward consequences of Theorems 4.1 and 4.2 and the infinite series representation (2.7).

Corollary 4.3. *A $\mathbb{Z}_+$-valued random variable X with pgf $P(z)$ has an $\mathcal{F}$-semi-selfdecomposable distribution with order $\alpha \in (0,1)$ if and only if there exists a $\mathbb{Z}_+$-valued subordinator $\{X_t^{(\mu)}\}$ such that*

$$X \stackrel{d}{=} \int_0^\infty \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{k=1}^\infty \alpha^{\lfloor T_k \rfloor} \odot_{\mathcal{F}} C_k, \tag{4.7}$$

in which case $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. Here the sequences $\{T_k\}$ and $\{C_k\}$ respectively represent the jump times and jump sizes of $\{X_t^{(\mu)}\}$ (see(2.1)).

Corollary 4.4. *Let $\alpha \in (0,1)$. Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$-valued subordinator and $Y_t = \int_0^t \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}$. Then Y_t converges in distribution to some $\mathbb{Z}_+$-valued random variable Y_∞ (as $t \to \infty$) if and only if $E(\ln(1 + X_1^{(\mu)})) < \infty$, in which case the distribution of Y_∞ is $\mathcal{F}$-semi-selfdecomposable with order α.*

The proof of the next result is similar to that of Corollary 3.3 (by way of (4.5)).

Corollary 4.5. *Let $\alpha \in (0,1)$. The mapping Φ_α is one-to-one from $\mathcal{I}_{\log}(\mathbb{Z}_+)$ onto $\mathcal{F}\text{-}\mathcal{SSD}_\alpha(\mathbb{Z}_+)$.*

5. Integral Representations for Elements in Nested Subclasses of $\mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$ and $\mathcal{F}\text{-}\mathcal{SSD}_\alpha(\mathbb{Z}_+)$

We define nested subclasses of $\mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$ and $\mathcal{F}\text{-}\mathcal{SSD}_\alpha(\mathbb{Z}_+)$ and study the actions of the mappings Ψ_α and Φ_a, respectively, on their elements.

First, we formulate an equivalent definition of $\mathcal{F}$-selfdecomposability and $\mathcal{F}$-semi-selfdecomposability (see (1.7) and (1.8)). A distribution μ on $\mathbb{Z}_+$ belongs to $\mathcal{F}\text{-}\mathcal{SD}(\mathbb{Z}_+)$ (resp. $\mathcal{F}\text{-}\mathcal{SSD}_\alpha(\mathbb{Z}_+)$ for some $\alpha \in (0,1)$) if and only if for every $t > 0$ (resp. for $t = -\ln \alpha$),

$$\mu = (\mu \vee \nu_t) * \mu_t, \tag{5.1}$$

where ν_t is the distribution with pgf $F_t(z)$ and $\mu_t \in \mathcal{I}(\mathbb{Z}_+)$. The binary operation $\vee$ is the compounding operation and $*$ is the convolution operation.

We define $\mathcal{F}\text{-}\mathcal{SD}_0(\mathbb{Z}_+) = \mathcal{I}(\mathbb{Z}_+)$ and for $m \geq 0$, $\mathcal{F}\text{-}\mathcal{SD}_{m+1}(\mathbb{Z}_+)$ as the subset of distributions $\mu \in \mathcal{I}(\mathbb{Z}_+)$ such that every $t > 0$, there exists $\mu_t \in \mathcal{F}\text{-}\mathcal{SD}_m(\mathbb{Z}_+)$ such that (5.1) holds.

Likewise, we define for $\alpha \in (0,1)$, $\mathcal{F}\text{-}\mathcal{SSD}_{\alpha,0}(\mathbb{Z}_+) = \mathcal{I}(\mathbb{Z}_+)$ and for $m \geq 0$, $\mathcal{F}\text{-}\mathcal{SSD}_{\alpha,m+1}(\mathbb{Z}_+)$ as the subset of distributions $\mu \in \mathcal{I}(\mathbb{Z}_+)$ that satisfy (5.1) for $t = -\ln \alpha$ and some $\mu_t \in \mathcal{F}\text{-}\mathcal{SSD}_{\alpha,m}(\mathbb{Z}_+)$.

For $m \geq 1$, we define

$$\mathcal{I}_{\log^m}(\mathbb{Z}_+) = \{\mu = (q_n, n \geq 0) \in \mathcal{I}(\mathbb{Z}_+) : \sum_{n=0}^{\infty} q_n (\ln(n+1))^m < \infty\}.$$

The sequences $\{\mathcal{I}_{\log^m}(\mathbb{Z}_+)\}$, $\{\mathcal{F}\text{-}\mathcal{SD}_m(\mathbb{Z}_+)\}$, and $\{\mathcal{F}\text{-}\mathcal{SSD}_{\alpha,m}(\mathbb{Z}_+)\}$ are easily checked to be decreasing in $\mathcal{I}(\mathbb{Z}_+)$ by an induction argument.

Hansen [6] studied extensively the subclasses $\{\mathcal{F}\text{-}\mathcal{SD}_m(\mathbb{Z}_+)\}$ (see also Berg and Forst [3]).

We start out by recalling the discrete versions of two important results obtained in [6].

Theorem 5.1. *(Theorem 7.4 in [6]) Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $P(z)$ and $m \geq 0$. Then $\mu \in \mathcal{F}\text{-}\mathcal{SD}_{m+1}(\mathbb{Z}_+)$ if and only if*

$$\ln P(z) = \int_0^{\infty} \ln Q(F_s(z)) s^m \, ds,$$

where $Q(z)$ is the pgf of a distribution in $\mathcal{I}(\mathbb{Z}_+)$.

Theorem 5.2. *(Theorem 7.5 in [6]) Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q(z)$ and $m \geq 0$. The following assertions are equivalent.*

(i) $\int_0^{\infty} [-\ln Q(F_s(z))] s^m \, ds < \infty \qquad (0 \leq z \leq 1)$

(ii) The canonical sequence $\{r_k\}$ of μ defined by $\frac{Q'(z)}{Q(z)} = \sum_{k=0}^{\infty} r_k z^k$ satisfies

$$\sum_{k=0}^{\infty} (\ln(k+1))^{m+1} \frac{r_k}{k+1} < \infty \qquad (r_k \geq 0).$$

Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$-valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu \in \mathcal{I}(\mathbb{Z}_+)$. For $\alpha \in (0,1)$, we define the sequences of mappings $\{\Psi_\alpha^m(\mu)\}$ and $\{\Phi_\alpha^m(\mu)\}$ recursively as follows: $\Psi_\alpha^0(\mu) = \Phi_\alpha^0(\mu) = \mu$ and for every $m \geq 0$, $\Psi_\alpha^{m+1}(\mu) = \Psi_\alpha(\Psi_\alpha^m(\mu))$ and $\Phi_\alpha^{m+1}(\mu) = \Phi_\alpha(\Phi_\alpha^m(\mu))$ where Ψ_α and Φ_α are defined by (3.1) and (4.1), respectively. The aim of this section is to identify the domains and ranges of Ψ_α^m and Φ_α^m.

Next, we derive a useful lemma.

Lemma 5.3. *Let $\alpha \in (0,1)$ and m a positive integer. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q(z)$. Then the following assertions are equivalent.*

(i) $J_1(z) = \int_0^{\infty} [-\ln Q(F_{-\lfloor s \rfloor \ln \alpha}(z))] \binom{\lfloor s \rfloor + m}{m} ds < \infty \qquad (0 \leq z \leq 1).$

(ii) $J_2(z) = \int_0^{\infty} [-\ln Q(F_s(z))] s^m \, ds < \infty \qquad (0 \leq z \leq 1).$

(iii) $\mu \in \mathcal{I}_{log^{m+1}}(\mathbb{Z}_+)$

Proof. First, we note that for every $s > 0$, $\lfloor s \rfloor \leq s \leq \lfloor s \rfloor + 1$ and for every $z \in [0,1]$, $F_s(z)$ is increasing as a function of $s \in (0,\infty)$ (see 1.14a). Moreover,

$\binom{\lfloor s\rfloor+m}{m} \sim \frac{s^m}{m!}$ as $s\uparrow\infty$. Rewriting $J_2(z)$ as

$$J_2(z) = t_0^{m+1}\int_0^\infty [-\ln Q(F_{st_0}(z))]s^m\,ds,$$

where $t_0 = -\ln\alpha$, and using a standard integration argument, one can easily show that (i)⇔(ii). The equivalence (ii)⇔(iii) follows by Theorem 5.2, combined with the discrete versions of Theorem 25.3 and Proposition 25.4 in [14]. □

The next result identifies the domain $\mathcal{D}(\Psi_\alpha^m)$ and the range $\mathcal{R}(\Psi_\alpha^m)$ of the mapping Ψ_α^m, $m\ge 1$.

Theorem 5.4. *For every positive integer m, $\mathcal{D}(\Psi_\alpha^m) = \mathcal{I}_{log^m}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha^m) = \mathcal{F}\text{-}\mathcal{SD}_m(\mathbb{Z}_+)$. Moreover, if $\mu\in\mathcal{I}_{log^m}(\mathbb{Z}_+)$, then*

$$\ln P_{\Psi_\alpha^m(\mu)}(z) = \int_0^\infty \frac{s^{m-1}}{(m-1)!}\ln Q_\mu(F_{-s\ln\alpha}(z))\,ds. \tag{5.2}$$

Proof. We prove the theorem by induction. The case $m=1$ follows from Theorem 3.2 and equation (2.6) (applied to $f(s)=\alpha^s$). Assume that Theorem 5.4 holds for $m\ge 1$. Suppose that $\mu\in\mathcal{D}(\Psi_\alpha^{m+1})$. Let $Q_\mu(z)$ be the pgf of $X_1^{(\mu)}$. Then by (2.6), (5.2), and (1.4),

$$\begin{aligned}\ln P_{\Psi_\alpha^{m+1}(\mu)}(z) &= \int_0^\infty \ln P_{\Psi_\alpha^m(\mu)}(F_{-t\ln\alpha}(z))\,dt\\ &= \int_0^\infty\Big(\int_0^\infty \frac{s^{m-1}}{(m-1)!}\ln Q_\mu(F_{-(s+t)\ln\alpha}(z))\,ds\Big)\,dt,\end{aligned}$$

which implies

$$\ln P_{\Psi_\alpha^{m+1}(\mu)}(z) = \int_0^\infty\Big(\int_0^s \frac{(s-t)^{m-1}}{(m-1)!}\,dt\Big)\ln Q_\mu(F_{-s\ln\alpha}(z))\,ds,$$

which in turn implies that (5.2) holds for $m+1$. By making the change of variable $t=-s\ln\alpha$ in (5.2), with $m+1$ in place of m, we obtain

$$\ln P_{\Psi_\alpha^{m+1}(\mu)}(z) = B_m\int_0^\infty t^m\ln Q_\mu(F_t(z))\,dt, \tag{5.3}$$

where

$$B_m = \frac{(-\ln\alpha)^{-m-1}}{m!}. \tag{5.4}$$

Since the integral in (5.3) converges, we conclude by Lemma 5.3[(ii)⇒(iii)] that $\mu\in\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$. Moreover, noting that $Q_\mu^{B_m}(z)$ is an infinitely divisible pgf, the representation (5.3) implies that $\Psi_\alpha^{m+1}(\mu)\in\mathcal{F}\text{-}\mathcal{SD}_{m+1}(\mathbb{Z}_+)$ by Theorem 5.1. We have thus shown that $\mathcal{D}(\Psi_\alpha^{m+1})\subset\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha^{m+1})\subset\mathcal{F}\text{-}\mathcal{SD}_{m+1}(\mathbb{Z}_+)$. Assume now that $\mu\in\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$. By Lemma 5.3[(iii)⇒(ii)], the integral $J_2(z)$ converges for every $z\in[0,1]$. Since $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)\subset\mathcal{I}_{\log^m}(\mathbb{Z}_+)$, $\Psi_\alpha^m(\mu)$ exists by the induction hypothesis. Using the same argument as above, we deduce

$$\int_0^\infty[-\ln P_{\Psi_\alpha^m(\mu)}(F_{-t\ln\alpha}(z))]\,dt = B_mJ_2(z)<\infty \qquad (z\in[0,1]),$$

where B_m is given by (5.4). It ensues by Lemma 3.1[(iii)⇒(i)], along with an obvious change of variable argument, that $\Psi_\alpha^m(\mu) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. Therefore $\Psi_\alpha^{m+1}(\mu) = \psi_\alpha(\Psi_\alpha^m(\mu))$ exists by Theorem 3.2 and thus $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{D}(\Psi_\alpha^{m+1})$. What remains to be shown is $\mathcal{F}$-$\mathcal{SD}_{m+1}(\mathbb{Z}_+) \subset \mathcal{R}(\Psi_\alpha^{m+1})$. Let $P(z)$ be the pgf of a distribution in $\mathcal{F}$-$\mathcal{SD}_{m+1}(\mathbb{Z}_+)$. By Theorem 5.1, $P(z)$ admits the representation

$$\ln P(z) = \int_0^\infty t^m \ln Q(F_t(z))\, dt \tag{5.5}$$

for some infinitely divisible pgf $Q(z)$. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z) = Q^{1/B_m}(z)$, where B_m is as in (5.4). We have by (5.5) that the integral $J_2(z)$ of Lemma 5.3 converges for every $0 \le z \le 1$. By Lemma 5.3[(ii)⇒(iii)], $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and thus, as shown above, $\Psi_\alpha^{m+1}(\mu)$ exists. It is easily seen that

$$\ln P(z) = \int_0^\infty \frac{s^m}{m!} \ln Q_\mu(F_{-s\ln\alpha}(z))\, ds,$$

or, $P(z) = P_{\Psi_\alpha^{m+1}(\mu)}(z)$. □

The following corollary gives an integral representation of the iterate $\Psi_\alpha^m(\mu)$. We will need the following sequence of functions $\{f_m\}$ defined on $[0,\infty)$ as follows:

$$f_m(s) = ((m+1)!s)^{1/(m+1)} \qquad (m \ge 0). \tag{5.6}$$

Corollary 5.5. *Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$-valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and m a positive integer. If $\mu \in \mathcal{I}_{log^m}(\mathbb{Z}_+)$, then*

$$\Psi_\alpha^m(\mu) = \mathcal{L}\Big(\int_0^\infty \alpha^{f_{m-1}(s)} \odot_{\mathcal{F}} dX_s^{(\mu)}\Big). \tag{5.7}$$

Proof. Assume $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ $(m \ge 1)$. We define the process $\{Y_t\}$ by

$$Y_t = \int_0^t \alpha^{f_{m-1}(s)} \odot_{\mathcal{F}} dX_s^{(\mu)}. \tag{5.8}$$

We have by Theorem 2.1 and (2.3) that the pgf $P_t(z)$ of Y_t is

$$P_t(z) = \exp\Big\{\int_0^t \ln Q_\mu\big(F_{-f_{m-1}(s)\ln\alpha}(z)\big)\, ds\Big\}. \tag{5.9}$$

Making the change of variable $u = f_{m-1}(s)$ in (5.9) and letting $t \to \infty$ leads to

$$\lim_{t\to\infty} P_t(z) = \exp\Big\{\int_0^\infty \frac{u^{m-1}}{(m-1)!} \ln Q_\mu(F_{-u\ln\alpha}(z)\, du\Big\},$$

or, by Theorem 5.3 and (5.2), $\lim_{t\to\infty} P_t(z) = P_{\Psi_\alpha^m(\mu)}(z)$. It follows that the process $\{Y_t\}$ converges weakly and therefore, by Theorem 2.2, $\mathcal{L}\Big(\int_0^\infty \alpha^{f_{m-1}(s)} \odot_{\mathcal{F}} dX_s^{(\mu)}\Big)$ is definable and has pgf $P_{\Psi_\alpha^m(\mu)}(z)$. □

We now turn to the problem of characterizing the domain and range of the iterates $\{\Phi_\alpha^m(\mu)\}$. We denote by $\mathcal{D}(\Phi_\alpha^m)$ and $\mathcal{R}(\Phi_\alpha^m)$, the domain and range of Φ_α^m, respectively.

First, we give a useful characterization of the distributions in $\mathcal{F}$-$\mathcal{SSD}_{\alpha,m}(\mathbb{Z}_+)$.

Theorem 5.6. *Let $\alpha \in (0,1)$ and m a positive integer. A distribution on $\mathbb{Z}_+$ belongs to $\mathcal{F}$-$\mathcal{SSD}_{\alpha,m}(\mathbb{Z}_+)$ if and only if its pgf $P(z)$ admits the representation*

$$\ln P(z) = \int_0^\infty \ln Q(F_{-\lfloor s\rfloor \ln\alpha}(z)) \binom{\lfloor s\rfloor + m - 1}{m-1} ds, \tag{5.10}$$

where $Q(z)$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$.

Proof. The proof is by induction. First, we note that

$$\int_0^\infty \ln Q(F_{-\lfloor s\rfloor t_0}(z)) \binom{\lfloor s\rfloor + m - 1}{m-1} ds = \sum_{j=0}^\infty \ln Q(F_{-jt_0}(z)) \binom{j+m-1}{m-1}, \tag{5.11}$$

where $t_0 = -\ln\alpha$. Assume $m = 1$. The "only if" part follows from (4.6) and (5.11). For the "if part" we note that if $P(z)$ satisfies (4.6), then $P(F_{t_0}(z)) = \prod_{k=0}^\infty Q(F_{(k+1)t_0}(z))$, and thus (1.8) holds with $P_{t_0}(z) = Q(z)$. This implies that $P(z)$ is the pgf of an $\mathcal{F}$-semi-selfdecomposable distribution. Assume now that the theorem holds for a positive integer m. If $P(z)$ is the pgf of a distribution in $\mathcal{F}$-$\mathcal{SSD}_{\alpha,m+1}(\mathbb{Z}_+)$, then $Q_1(z) = P(z)/P(F_{t_0}(z))$ is the pgf of a distribution in $\mathcal{F}$-$\mathcal{SSD}_{\alpha,m}(\mathbb{Z}_+)$ and

$$\ln P(z) = \sum_{j=0}^\infty \ln Q_1(F_{jt_0}(z)). \tag{5.12}$$

We have by the induction hypothesis that $Q_1(z)$ satisfies (5.10), and thus (5.11), for some infinitely divisible pgf $Q(z)$. It follows by (5.12) and (1.4) that

$$\begin{aligned}\ln P(z) &= \sum_{j=0}^\infty \sum_{k=0}^\infty \ln Q(F_{(k+j)t_0}(z)) \binom{j+m-1}{m-1} \\ &= \sum_{k=0}^\infty \left(\sum_{j=0}^k \binom{k-j+m-1}{m-1} \right) \ln Q(F_{kt_0}(z)).\end{aligned}$$

Since

$$\sum_{j=0}^k \binom{k-j+m-1}{m-1} = \binom{k+m}{m}, \tag{5.13}$$

we conclude that $\ln P(z)$ satisfies (5.11), and thus (5.10), with m in place of $m-1$. Assume now that $P(z)$ is a pgf that admits the representation (5.10), with m in place of $m-1$, for some infinitely divisible pgf $Q(z)$. It follows by (5.11), (1.4), and the identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, that

$$\ln P(z) = \ln P(F_{t_0}(z)) + \sum_{k=0}^\infty \ln Q(F_{kt_0}(z)) \binom{k+m-1}{m-1}. \tag{5.14}$$

Define the function $Q_1(z)$ by the equation

$$\ln Q_1(z) = \sum_{k=0}^\infty \ln Q(F_{kt_0}(z)) \binom{k+m-1}{m-1}.$$

It is clear that $Q_1(0) > 0$. Let $l_k = \binom{k+m-1}{m-1}$. The pgf $Q^{l_k}(z)$ is infinitely divisible. Therefore, by Theorem 4.2, Chapter II, in [18] and the properties of the semigroup $\mathcal{F}$, we have for each $k \geq 0$,

$$-\ln Q^{l_k}(F_{kt_0}(z)) = \int_{F_{kt_0}(z)}^1 R_k(x)\,dx = \int_z^1 R_k(F_{kt_0}(x))F'_{kt_0}(x)\,dx \quad (z \in [0,1],$$

for some absolutely monotone function $R_k(x)$ on $[0,1)$. It follows that $-\ln Q_1(z) = \int_z^1 R(x)\,dx$, where $R(x) = \sum_{k=0}^{\infty} R_k(F_{kt_0}(x))F'_{kt_0}(x)$, $x \in [0,1)$. Since $R(x)$ is absolutely monotone on $[0,1)$ (as the limit of sums of absolutely monotone functions on $[0,1)$), we have, again by Theorem 4.2, Chapter II, in [18], that $Q_1(z)$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$. By the induction hypothesis and (5.11), applied at $m-1$, we can conclude that $Q_1(z)$ is the pgf of a distribution in $\mathcal{F}$-$\mathcal{SSD}_{\alpha,m}(\mathbb{Z}_+)$. By (5.14), $P(z) = P(F_{t_0}(z))Q_1(z)$. Therefore, $P(z)$ is the pgf of a distribution in $\mathcal{F}$-$\mathcal{SSD}_{\alpha,m+1}(\mathbb{Z}_+)$. □

Theorem 5.7. *For every positive integer m, $\mathcal{D}(\Phi_\alpha^m) = \mathcal{I}_{log^m}(\mathbb{Z}_+)$ and $\mathcal{R}(\Phi_\alpha^m) = \mathcal{F}$-$\mathcal{SSD}_{\alpha,m}(\mathbb{Z}_+)$. Moreover,*

$$\ln P_{\Phi_\alpha^m(\mu)}(z) = \int_0^\infty \ln Q_\mu(F_{-\lfloor s\rfloor \ln\alpha}(z)) \binom{\lfloor s\rfloor + m-1}{m-1} ds. \tag{5.15}$$

Proof. The proof is by induction. The case $m=1$ follows from Theorems 4.1 and 4.2 and equation (4.4). Let's assume that Theorem 5.7 holds for $m \geq 1$. Suppose that $\mu \in \mathcal{D}(\Phi_\alpha^{m+1})$. Let $Q_\mu(z)$ be the pgf of $X_1^{(\mu)}$. Since $\Phi_\alpha^{m+1}(\mu) = \Phi_\alpha(\Phi_\alpha^m(\mu))$, we have by the induction hypothesis (or, (5.15)), (4.4), and (1.4) that

$$\begin{aligned}\ln P_{\Phi_\alpha^{m+1}(\mu)}(z) &= \int_0^\infty \ln P_{\Phi_\alpha^m(\mu)}(F_{-\lfloor t\rfloor \ln\alpha}(z))\,dt \\ &= \int_0^\infty \Big(\int_0^\infty \ln Q_\mu(F_{-(\lfloor s\rfloor+\lfloor t\rfloor)\ln\alpha}(z)) \binom{\lfloor s\rfloor + m-1}{m-1} ds\Big)\,dt.\end{aligned}$$

By making the change of variable $u = s + \lfloor t\rfloor$ in the inner integral above, and noting that $\lfloor u\rfloor = \lfloor s\rfloor + \lfloor t\rfloor$, we have

$$\ln P_{\Phi_\alpha^{m+1}(\mu)}(z) = \int_0^\infty \Big(\int_{\lfloor t\rfloor}^\infty \ln Q_\mu(F_{-\lfloor u\rfloor \ln\alpha}(z)) \binom{\lfloor u\rfloor - \lfloor t\rfloor + m-1}{m-1} du\Big)\,dt.$$

Since $u \geq \lfloor t\rfloor$ if and only if $t < \lfloor u\rfloor + 1$, it follows that

$$\ln P_{\Phi_\alpha^{m+1}(\mu)}(z) = \int_0^\infty \ln Q_\mu(F_{-\lfloor u\rfloor \ln\alpha}(z)) \Big(\int_0^{\lfloor u\rfloor+1} \binom{\lfloor u\rfloor - \lfloor t\rfloor + m-1}{m-1} dt\Big)\,du.$$

It is easily seen that

$$\int_0^{\lfloor u\rfloor+1} \binom{\lfloor u\rfloor - \lfloor t\rfloor + m-1}{m-1} dt = \sum_{j=0}^{\lfloor u\rfloor} \binom{\lfloor u\rfloor - j + m-1}{m-1} = \binom{\lfloor u\rfloor + m}{m},$$

where the second equation follows from (5.13). We have thus shown that (5.15) holds for $m+1$. Since the integral in (5.15), with $m+1$ in place of m, converges, we conclude that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ (by Lemma 5.3[(i)⇒(iii)]) and that $\Phi_\alpha^{m+1}(\mu) \in \mathcal{F}$-$\mathcal{SSD}_{\alpha,m+1}(\mathbb{Z}_+)$ (by Theorem 5.6). We have thus shown that

$\mathcal{D}(\Phi_\alpha^{m+1}) \subset \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and that $\mathcal{R}(\Phi_\alpha^{m+1}) \subset \mathcal{F}\text{-}\mathcal{SSD}_{\alpha,m+1}(\mathbb{Z}_+)$. Assume now that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$. By Lemma 5.3[(iii)⇒(i)], the integral $J_1(z)$ converges for every $z \in [0,1]$. Since $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, $\Phi_\alpha^m(\mu)$ exists by the induction hypothesis. Using the same argument as above, we deduce

$$\int_0^\infty [-\ln P_{\Phi_\alpha^m(\mu)}(F_{-t\ln\alpha}(z))]\,dt = J_1(z) < \infty \qquad (z \in [0,1]).$$

It follows by Lemma 3.1[(iii)⇒(i)], and an obvious change of variable argument, that $\Phi_\alpha^m(\mu) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. Therefore $\Phi_\alpha^{m+1}(\mu) = \Phi_\alpha(\Phi_\alpha^m(\mu))$ exists by Theorem 4.1 and thus $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{D}(\Phi_\alpha^{m+1})$. We complete the proof by showing that $\mathcal{F}\text{-}\mathcal{SSD}_{\alpha,m+1}(\mathbb{Z}_+) \subset \mathcal{R}(\Phi_\alpha^{m+1})$. Let $P(z)$ be the pgf of a distribution in $\mathcal{F}\text{-}\mathcal{SSD}_{\alpha,m+1}(\mathbb{Z}_+)$. Then by Theorem 5.6, $P(z)$ admits the representation

$$\ln P(z) = \int_0^\infty \ln Q(F_{-\lfloor s\rfloor\ln\alpha}(z))\binom{\lfloor s\rfloor + m}{m}\,ds \tag{5.16}$$

for some infinitely divisible pgf $Q(z)$. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z) = Q(z)$. We have by (5.16) that the integral $J_1(z)$ of Lemma 5.3 converges for every $0 \le z \le 1$. By Lemma 5.3[(i)⇒(iii)], $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and thus, as shown above, $\Phi_\alpha^{m+1}(\mu)$ exists. It is clearly seen that $P(z) = P_{\Phi_\alpha^{m+1}(\mu)}(z)$. □

We now deduce an integral representation of the iterate $\Phi_\alpha^m(\mu)$. We will need the sequence of functions $\{g_m\}$ defined by

$$g_m(s) = \int_0^s \binom{\lfloor u\rfloor + m}{m}\,du \qquad (m \ge 0). \tag{5.17}$$

We denote by $\tilde{g}_m$ the inverse function of g_m.

Corollary 5.8. *Let $\{X_t^{(\mu)}\}$ be a $\mathbb{Z}_+$-valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and m a positive integer. If $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, then*

$$\Phi_\alpha^m(\mu) = \mathcal{L}\Big(\int_0^\infty \alpha^{\lfloor \tilde{g}_{m-1}(s)\rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}\Big). \tag{5.18}$$

Proof. Assume $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ $(m \ge 1)$. We define the process $\{Y_t\}$ by

$$Y_t = \int_0^t \alpha^{\lfloor \tilde{g}_{m-1}(s)\rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}.$$

We have by Theorem 2.1 and (2.3) that the pgf $P_t(z)$ of Y_t is

$$P_t(z) = \exp\Big\{\int_0^t \ln Q_\mu\big(F_{-\lfloor \tilde{g}_{m-1}(s)\rfloor \ln\alpha}(z)\big)\,ds\Big\}. \tag{5.19}$$

Making the change of variable $u = \tilde{g}_{m-1}(s)$ in (5.19) and letting $t \to \infty$ leads to

$$\lim_{t\to\infty} P_t(z) = \exp\Big\{\int_0^\infty \ln Q_\mu(F_{-\lfloor u\rfloor\ln\alpha}(z))\binom{\lfloor u\rfloor + m - 1}{m-1}\,du\Big\}.$$

or, by Theorem 5.7 and (5.15), $\lim_{t\to\infty} P_t(z) = P_{\Phi_\alpha^m(\mu)}(z)$. It follows that the process $\{Y_t\}$ converges weakly and therefore, by Theorem 2.2, $\mathcal{L}\big(\int_0^\infty \alpha^{\lfloor \tilde{g}_{m-1}(s)\rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}\big)$ is definable and has pgf $P_{\Phi_\alpha^m(\mu)}(z)$. □

Acknowledgment. The author is grateful to a referee for very valuable comments and suggestions.

References

1. Athreya, K. B. and Ney, P.E.:*Branching Processes*. Springer-Verlag, Berlin-Heidelberg, 1972.
2. Barndorff-Nielsen, O. E., Maejima, M., and Sato, K.: Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations. Bernoulli **12** (2006), 1–33.
3. Berg, C. and Forst, G.: Multiply self-decomposable probability measures on $\mathbb{R}_+$ and $\mathbb{Z}_+$. Z. Wahrscheinlichkeitstheor. Verw. Gebiete **62,** (1983), 147–163.
4. Bouzar, N.: Semi-self-decomposability induced by semigroups. Electron. J. Probab. **16** (2011), 1117–1133.
5. Foster, J. H. and Williamson, J. A.: Limit theorems for the Galton-Watson process with time-dependent immigration. Z. Wahrscheinlichkeitstheor. Verw. Gebiete **20** (1971), 227–235.
6. Hansen, B. G.: *Self-decomposable distributions and branching processes*. Memoir COSOR 89-06, Eindhoven Univ. of Technology (Eindhoven, Netherlands.), 1989.
7. van Harn, K., Steutel, F. W., and Vervaat, W.: Self-decomposable discrete distributions and branching processes. Z. Wahrscheinlichkeitstheor. Verw. Gebiete **61** (1982), 97–118.
8. van Harn, K. and Steutel, F. W.: Stability equations for processes with stationary independent increments using branching processes and Poisson mixtures. Stochastic Process. Appl. **45** (1993), 209–230.
9. Jurek, Z. J. and Vervaat, W.: An integral representation for self-decomposable Banach space valued random variables. Z. Wahrscheinlichkeitstheor. Verw. Gebiete **62** (1983), 247–262.
10. Maejima, M. and Miura, M.: A characterization of subclasses of semi-self-decomposable distributions by stochastic integral representations. Statist. Probab. Lett. **77** (2007), 838–842.
11. Maejima, M. and Naito, Y.: Semi-selfdecomposable distributions and a new class of limit theorems. Probab. Theory Relat. Fields **112** (1998), 13–31.
12. Maejima M. and Ueda, Y.: Stochastic integral characterizations of semiselfdecomposable distributions and related Ornstein-Uhlenbeck type processes. Commun. Stoch. Anal. **3** (2009), 349–367.
13. Rocha-Arteaga, A. and Sato K.: *Topics in Infinitely Divisible Distributions and Lévy Processes*. Aportaciones Matematicás, Investigación 17, Sociedad Matemática Mexicana, 2003.
14. Sato, K.: *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999.
15. Sato, K. and Yamazato, M.: Stationary processes of Ornstein-Uhlenbeck type. *Probability Theory and Mathematical Statistics, 4-th USSR-Japan, Symp., Proceedings of 1982*, Lecture Notes in Mathematics, vol. 1021, Springer, Berlin, 541–551, 1983.
16. Sato, K. and Yamazato, M.: Operator-selfdecomposable distributions as limit distributions of processes of Ornstein-Uhlenbeck type. Stoch. Proc. Appl. **17**(1984), 73–100.
17. Steutel, F. W., Vervaat, W., and Wolfe, S. J.: Integer-valued branching processes with immigration. Adv. Appl. Probab. **15** (1983), 713–725.
18. Steutel, F. W. and van Harn, K.: *Infinite Divisibility of Probability Distributions on the Real Line*. Marcel Dekker, Inc., New York-Basel, 2004.
19. Wolfe, S. J.: On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$. Stoch. Proc. Appl. **12** (1982), 301–312.

Nadjib Bouzar: Department of Mathematics and Computer Science, University of Indianapolis, Indianapolis, IN 46227, USA

E-mail address: nbouzar@uindy.edu