

## REPRESENTATION AND GAUSSIAN BOUNDS FOR THE DENSITY OF BROWNIAN MOTION WITH RANDOM DRIFT

AZMI MAKHLOUF

ABSTRACT. We study the density, with respect to Lebesgue measure, of the law of multidimensional Brownian motion with random drift. We do not assume any Markovianity, and the drift may be unbounded. We state a representation of this density in terms of the Gaussian one, from which we derive explicit upper and local lower bounds of Gaussian type.

### 1. Introduction

In this paper, we are interested in multidimensional Brownian motion with random drift  $b$ , defined by

$$X_t := W_t + \int_0^t b_s ds, \quad (1.1)$$

where  $(W_t)_{t \geq 0}$  is a  $d$ -dimensional Brownian motion and  $(b_t)_{t \geq 0}$  a stochastic process, adapted to the natural filtration of  $W$ . We aim at studying the existence and bounds of the density of  $X_t$  (with respect to Lebesgue measure).

Several works have dealt with this topic in the particular (Markovian) framework of SDE (stochastic differential equations)

$$\tilde{X}_t = W_t + \int_0^t \tilde{b}(\tilde{X}_s) ds, \quad (1.2)$$

where  $\tilde{b}(\cdot)$  is a deterministic measurable function.

By working on the fundamental solution of the PDE associated to (1.2), Aronson [1] obtained absolute continuity and Gaussian upper and lower bounds when  $\tilde{b}(\cdot)$  is bounded, allowing also for the presence of a uniformly elliptic diffusion coefficient  $\tilde{\sigma}(\cdot)$  (in the  $dW_t$  term).

There is a vast literature concerning the stochastic methods used to derive such bounds. For regular drift  $\tilde{b}(\cdot)$  and diffusion  $\tilde{\sigma}(\cdot)$ , we refer to [8], [2] and [5], where Malliavin calculus is used.

For irregular drift, some results are obtained by stochastic methods. Still in the Markovian setting, sharp Gaussian bounds have been obtained for the density of (1.2) by [11] (for bounded drift  $\tilde{b}(\cdot)$ ) and [12] (for  $\tilde{b}(\cdot)$  of linear growth).

---

Received 2015-12-11; Communicated by editors.

2010 *Mathematics Subject Classification*. Primary 60J65; Secondary 60H10; 60G99.

*Key words and phrases*. Brownian motion, random drift, stochastic equations, density, Gaussian bounds.

In [3], the authors prove the existence of a density for one-dimensional SDE (1.2), with possibly path-dependent coefficients. They use a simple approach consisting in freezing the coefficients at time  $t - \varepsilon$ , then computing the characteristic function (law Fourier transform)  $\hat{\mu}$  of the approximating variable  $\tilde{X}_t^\varepsilon$ , which is Gaussian conditionally on the past up to time  $t - \varepsilon$ . The absolute continuity of  $\tilde{X}_t$  follows by choosing  $\varepsilon$  such that  $\hat{\mu}$  can be shown to be in  $L^2(\mathbb{R})$ . However, this argument cannot be generalized to higher dimensions, and does not lead to any pointwise bound on the density.

The approach of [3] has been extended by [7] and [4] to study the existence and regularity of the density of multidimensional SDE (1.2) with bounded irregular drift  $\tilde{b}(\cdot)$ . In [6], the authors have used a more general approach that enables to state an integration-by-parts formula for functionals of SDE's, from which they have derived Gaussian bounds on the density. In the latter three references, the authors perform a change of measure by Girsanov theorem to get rid of the irregular drift, and expand it as a sum of multiple stochastic integrals, using Itô-Taylor expansion up to a certain order. In each integral, a conditional expectation is taken in order to regularize the drift (relying on the Markovianity of the SDE), so that Malliavin calculus can be finally applied.

Recently, in [10], the authors have combined the techniques of [7] and [6] with a Clark-Ocone representation of the covariance, to state Gaussian estimates for one-dimensional additive functionals of SDE's with irregular drift. Besides, they have applied the mentioned covariance representation to the density estimation of solutions of some one-dimensional stochastic equations, under regularity assumptions on the coefficients (including the drift).

In this paper, we consider  $(X_t)$ , the multidimensional Brownian motion with random drift  $(b_t)$  defined by (1.1), and we do not assume any Markovianity nor boundedness on the drift. In such a general framework, we are not aware of any attempt to establish existence and Gaussian bounds for the density of  $X_t$ .

Our idea is simpler and more direct than those cited above. We express some distance between both laws of the Brownian motion with drift,  $X_t$ , and of the (driftless) Brownian motion  $W_t$ , by plugging the process  $X$  in a test function for the law of  $W$ . More precisely, we apply Itô's lemma to  $\phi(s, X_s)$  for  $0 \leq s \leq t$ , where  $\phi(s, x) := \mathbb{E}[f(W_t) | W_s = x]$  and  $f$  is a test function. Then we take the expectation of the expansion, and we estimate how both laws «diverge» from each other, from time 0 to time  $t$ . This estimation enables us to have existence, a representation, and Gaussian bounds for the density of  $X_t$ .

The paper is organized as follows. We first introduce, in Section 2, the notations and assumptions used throughout the article. Then we state and comment our main results in Section 3. The proofs are given in Section 4, and a summarizing conclusion in Section 5.

## 2. Notations and Assumptions

For  $x = (x_1, \dots, x_d)^* \in \mathbb{R}^d$ ,  $x^*$  denotes its transpose and  $|x|$  its Euclidian norm:  $|x| = (\sum_{i=1}^d x_i^2)^{\frac{1}{2}}$ . For any bounded function  $f$ ,  $\|f\|_\infty$  denotes its supremum norm.

For a space-differentiable function  $f(t, x) : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\nabla f(t, x)$  stands for its gradient with respect to  $x$ , i.e.  $\nabla f(t, x) = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d})(t, x)$ .

The function  $g(\cdot, \cdot)$  denotes the Gaussian heat kernel: for  $t > 0$ ,  $x \in \mathbb{R}^d$ ,

$$g(t, x) := (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right).$$

For any random variable  $Z$ , and any real number  $p \geq 1$ ,  $\|Z\|_p := (\mathbb{E}[|Z|^p])^{\frac{1}{p}}$ .

Let  $W$  be a  $d$ -dimensional Brownian motion, defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , where  $(\mathcal{F}_t)_{t \geq 0}$  is the natural filtration of  $W$ , augmented with the  $\mathbb{P}$ -null sets.

Let  $\mathbb{E}$  denote the expectation under  $\mathbb{P}$ . Let  $(b_t)_{t \geq 0}$  be a  $\mathbb{R}^d$ -valued,  $(\mathcal{F}_t)$ -adapted stochastic process such that  $\int_0^t |b_s| ds < \infty$  ( $\mathbb{P}$ -a.s.). We define the Brownian motion with random drift  $b$  by

$$X_t := W_t + \int_0^t b_s ds.$$

Throughout the paper, we use the following assumption. Set

$$\mathcal{E}_{b,\alpha}(t) := \mathbb{E}[\exp(\alpha \int_0^t |b_s|^2 ds)];$$

$$\mathcal{L}_{b,q}(t) := \sup_{s \in [0,t]} \mathbb{E}[|b_s|^q]^{\frac{1}{q}} = \sup_{s \in [0,t]} \|b_s\|_q.$$

*Assumption 2.1.* There exist real constants  $\alpha \in (0, \infty)$  and  $q \in (1, \infty)$  such that, for  $t > 0$ ,

$$\mathcal{E}_{b,\alpha}(t) + \mathcal{L}_{b,q}(t) < \infty. \tag{A_{b,\alpha,q}}$$

Notice that no boundedness is assumed for the drift process  $(b_t)$ , and that the condition on  $\mathcal{E}_{b,\alpha}(t)$  is even weaker than the standard Novikov condition when  $\alpha$  is smaller than  $\frac{1}{2}$ . It will be though sufficient to estimate the change of measure coming from the application of Girsanov theorem (see the proof of Lemma 4.1, Section 4.4).

Let us mention here that Novikov's condition ensures (by Girsanov theorem) the existence of a weak solution to SDE (1.2), and that a similar condition has been recently shown in [9] to be sufficient, in some cases, to have a strong solution. One may then apply our results to this particular setting.

### 3. Main Results

**Theorem 3.1** (Existence and representation). *Let  $t > 0$ . Assume that (A\_{b,\alpha,q}) holds for some  $\alpha \in (0, \infty)$  and  $q \in (1, \infty)$ , and set  $p := (1 + \frac{1}{q-1})(1 + \frac{1+\sqrt{1+8\alpha}}{2\alpha})$ . If*

- (i) either the dimension  $d = 1$ ,
- (ii) or,  $d > 1$  and  $p < \frac{d}{d-1}$ ,

then  $X_t$  admits a density  $\gamma(t, \cdot)$  with respect to Lebesgue measure, and for  $x \in \mathbb{R}^d$ ,

$$\gamma(t, x) = g(t, x) + \int_0^t \mathbb{E}[\nabla g(t-s, x - X_s) b_s] ds. \tag{3.1}$$

Notice that, under Assumption  $(A_{b,\alpha,q})$ , the representation (3.1) always holds true in the one-dimensional case. For  $d > 1$ , it still holds if  $\alpha$  and  $q$  are sufficiently large for condition (ii) in Theorem 3.1 to be fulfilled: this is the case if, for example (but not necessarily),  $b$  is uniformly bounded.

Identity (3.1) may be seen as a first-order expansion of the density  $\gamma(t, \cdot)$  of  $X_t$ , whose leading term is the density  $g(t, \cdot)$  of Brownian motion  $W_t$ . One may think about Brownian motion with deterministic drift as a reference process: then we derive the following generalization of Theorem 3.1.

**Corollary 3.2.** *Let  $t > 0$  and  $\beta(\cdot) : [0, t] \rightarrow \mathbb{R}^d$  any deterministic function such that  $(A_{b-\beta,\alpha,q})$  holds for some  $\alpha \in (0, \infty)$  and  $q \in (1, \infty)$ . Set*

$$I_\beta(t) := \int_0^t \beta(s) ds.$$

Then under conditions (i) and (ii) of Theorem 3.1, we have for  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} \gamma(t, x) &= g(t, x - I_\beta(t)) \\ &\quad + \int_0^t \mathbb{E} \left[ \nabla g(t-s, x - I_\beta(t) - (X_s - I_\beta(s))) (b_s - \beta(s)) \right] ds. \end{aligned}$$

*Proof.* Setting  $Y_t := W_t + \int_0^t (b_s - \beta(s)) ds$ , we have  $X_t = Y_t + I_\beta(t)$ . Since  $\beta(\cdot)$  is deterministic,

$$\gamma(t, x) = \gamma_Y(t, x - I_\beta(t)),$$

where  $\gamma_Y(t, \cdot)$  is the density of  $Y_t$ . Applying Theorem 3.1 to  $Y_t$  proves Corollary 3.2.  $\square$

From Theorem 3.1 (and Lemma 4.1), we get the following Gaussian bounds on the density  $\gamma(t, x)$ . The upper bound (Theorem 3.3 and Corollary 3.4) is explicitly given for any  $t > 0$  and  $x \in \mathbb{R}^d$ . The lower bound (Theorem 3.5) is only valid for small time-space  $(t, x)$ .

**Theorem 3.3** (Upper bound). *Under the assumptions of Theorem 3.1, we have for  $t > 0$  and  $x \in \mathbb{R}^d$ ,*

$$\gamma(t, x) \leq g(t, x) + \tilde{C}_{p,d} \mathcal{L}_{b,q}(t) \mathcal{E}_{b,\alpha}(t)^{\frac{1}{2r}} \left( \sqrt{t} + |x| \right) g(pt, x),$$

with  $\tilde{C}_{p,d} := (-\frac{d}{2} + \frac{d}{2p} + \frac{1}{2})^{-1} C_{p,d}$ , and  $p, r$  and  $C_{p,d}$  defined as in Lemma 4.1.

As with Corollary 3.2, we immediately derive a bound which is not only optimal in the class of deterministic drifts, but also sharper than that of Theorem 3.3. For instance, by choosing  $\beta(s) := \mathbb{E}[b_s]$ , the bound involves centered moments  $\mathcal{L}_{b-\beta,q}(t)$  (the standard deviation, if  $q = 2$ ) of the drift  $b$ , instead of the crude moments  $\mathcal{L}_{b,q}(t)$ .

**Corollary 3.4.** *Under the assumptions of Corollary 3.2, we have for  $t > 0$  and  $x \in \mathbb{R}^d$ ,*

$$\begin{aligned} \gamma(t, x) &\leq g(t, x - I_\beta(t)) \\ &\quad + \tilde{C}_{p,d} \mathcal{L}_{b-\beta,q}(t) \mathcal{E}_{b-\beta,\alpha}(t)^{\frac{1}{2r}} \left( \sqrt{t} + |x - I_\beta(t)| \right) g(pt, x - I_\beta(t)). \end{aligned}$$

**Theorem 3.5** (Lower bound). *Assume that the conditions of Theorem 3.1 hold. For  $t_0 > 0$  and  $R > 0$ , set*

$$\mathcal{O}_{t_0,R} := \{(t, x) \in (0, \infty) \times \mathbb{R}^d, \text{ s.t. } 0 < t \leq t_0 \text{ and } \frac{|x|}{\sqrt{t}} \leq R\}.$$

*Then for every  $R > 0$ , there exists  $t_0 > 0$  (depending on  $R, \alpha, q$  and  $d$ ) such that, for  $(t, x) \in \mathcal{O}_{t_0,R}$ ,*

$$\gamma(t, x) \geq \frac{1}{2}g(t, x).$$

*Remark 3.6.* It is not clear (at least from our work) how an explicit lower bound, valid for any time  $t$ , could be. It cannot be of the same type as the bounds given by Theorems 3.3 and 3.5, in the sense that other powers of  $t$  (smaller than  $-\frac{d}{2}$ ) may appear. Here is an example:

$$X_t := W_t + \int_0^t B_s ds,$$

where  $B$  is a  $d$ -dimensional Brownian motion, independent from  $W$ .

Then the random variable  $X_t$  is Gaussian with density

$$\gamma(t, x) = (2\pi t + \frac{2\pi}{3}t^3)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2(t + \frac{1}{3}t^3)}\right),$$

which is equivalent, for large  $t$ , to  $(\frac{2\pi}{3}t^3)^{-\frac{d}{2}}$  (thus negligible compared to  $t^{-\frac{d}{2}}$ ).

Finding a lower bound, valid for any time  $t$ , remains as a challenging problem.

#### 4. Proofs

The following lemma gives a crucial estimate for our proofs. For the convenience of the reader, its proof is postponed to Section 4.4.

**Lemma 4.1.** *Let  $0 < s < t$ . Assume that  $(A_{b,\alpha,q})$  holds for some  $\alpha \in (0, \infty)$  and  $q \in (1, \infty)$ , and set  $p := (1 + \frac{1}{q-1})(1 + \frac{1+\sqrt{1+8\alpha}}{2\alpha})$  and  $r := (1 - \frac{1}{p} - \frac{1}{q})^{-1}$ . Then for  $x \in \mathbb{R}^d$ ,*

$$\begin{aligned} & \mathbb{E}[|\nabla g(t-s, x - X_s)b_s|] \\ & \leq C_{p,d} \mathcal{L}_{b,q}(t) \mathcal{E}_{b,\alpha}(t)^{\frac{1}{2r}} \left(\frac{t-s}{t}\right)^{-\frac{d}{2} + \frac{d}{2p}} \left((t-s)^{-\frac{1}{2}} + \frac{|x|}{t}\right) g(pt, x), \end{aligned}$$

where  $C_{p,d}$  is a constant depending only on  $p$  and  $d$ , and explicitly given by (4.6)-(4.5).

**4.1. Proof of Theorem 3.1 (Existence and representation).** Roughly speaking, the idea is the following: since we want to compare the law of  $X$  to that of  $W$ , we apply a test function characterizing the law of  $W$ , here  $\mathbb{E}[f(W_t)|W_s = \cdot]$  ( $0 \leq s \leq t$ ), to the process  $X$ . We then expand it using Itô's lemma; we only need to be careful with the possible blow up of the derivative when  $s$  is close to  $t$ . But this is controlled by Lemma 4.1.

*Proof of Theorem 3.1.* Applying the usual characterization of the probability density function of a random variable, we have to show that, for every bounded and continuous test function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$\mathbb{E}[f(X_t)] = \int_{\mathbb{R}^d} f(x)g(t, x)dx + \int_{\mathbb{R}^d} f(x) \int_0^t \mathbb{E}[\nabla g(t-s, x - X_s)b_s]dsdx. \quad (4.1)$$

For fixed  $t > 0$ , and for all  $s \in [0, t]$  and  $y \in \mathbb{R}^d$ , we set

$$\phi(s, y; t) := \mathbb{E}[f(W_t)|W_s = y] = \mathbb{E}[f(y + W_t - W_s)].$$

Note that  $\phi(0, 0; t) = \mathbb{E}[f(W_t)]$  and  $\phi(t, y; t) = f(y)$ , and that  $\phi(\cdot, \cdot; t)$  is continuous on  $[0, t] \times \mathbb{R}^d$ .

When  $0 \leq s < t$ , we have

$$\phi(s, y; t) = \int_{\mathbb{R}^d} f(x)g(t-s, x-y)dx,$$

and

$$\nabla \phi(s, y; t) = \int_{\mathbb{R}^d} f(x)\nabla g(t-s, x-y)dx = \int_{\mathbb{R}^d} f(x) \frac{(y-x)^*}{t-s} g(t-s, x-y)dx.$$

Moreover, the following estimates hold:

$$|\phi(s, y; t)| \leq \|f\|_\infty, \quad (4.2)$$

and (by the change of variable  $z = \frac{x-y}{\sqrt{t-s}}$ )

$$|\nabla \phi(s, y; t)| \leq \frac{\|f\|_\infty}{\sqrt{t-s}}. \quad (4.3)$$

Clearly, for every  $\varepsilon \in (0, t)$ , the function  $\phi(\cdot, \cdot; t)$  is of class  $\mathcal{C}^{1,2}$  on  $[0, t-\varepsilon] \times \mathbb{R}^d$ . By Itô's lemma (and the fact that  $\phi(\cdot, \cdot; t)$  is solution to the backward heat equation),

$$\phi(t-\varepsilon, X_{t-\varepsilon}; t) = \phi(0, 0; t) + \int_0^{t-\varepsilon} \nabla \phi(s, X_s; t)dW_s + \int_0^{t-\varepsilon} \nabla \phi(s, X_s; t)b_s ds.$$

Estimates (4.2) and (4.3) allow to take expectations in the above equality to get

$$\begin{aligned} & \mathbb{E}[\phi(t-\varepsilon, X_{t-\varepsilon}; t)] \\ &= \mathbb{E}[f(W_t)] + \int_0^{t-\varepsilon} \mathbb{E}[\nabla \phi(s, X_s; t)b_s]ds \\ &= \int_{\mathbb{R}^d} f(x)g(t, x)dx + \int_0^{t-\varepsilon} \mathbb{E}\left[\int_{\mathbb{R}^d} f(x)\nabla g(t-s, x - X_s)b_s dx\right]ds. \end{aligned}$$

We have

$$\mathbb{E}[|f(x)\nabla g(t-s, x - X_s)b_s|] \leq \|f\|_\infty \mathbb{E}[|\nabla g(t-s, x - X_s)b_s|].$$

Now, Lemma 4.1 clearly shows that, if  $-\frac{d}{2} + \frac{d}{2p} - \frac{1}{2} > -1$  (which is equivalent to either condition (i) or (ii) of Theorem 3.1), then  $\mathbb{E}[|\nabla g(t-s, x - X_s)b_s|]$  is  $(ds \otimes dx)$ -integrable on  $[0, t] \times \mathbb{R}^d$ .

Therefore, we apply Fubini's theorem in the last identity, and the dominated convergence theorem when  $\varepsilon \rightarrow 0$ , to obtain (4.1) and prove Theorem 3.1.  $\square$

Let us point out here that our method would not immediately apply if we added to our framework a random diffusion coefficient in the Brownian term of (1.1), since non-integrable second-order terms (involving the second derivative of the Gaussian density  $g$ ) would then appear.

**4.2. Proof of Theorem 3.3 (Upper bound).** It is a direct consequence of Theorem 3.1 and Lemma 4.1.

*Proof of Theorem 3.3.* Conditions (i) and (ii) of Theorem 3.1 ensure that  $-\frac{d}{2} + \frac{d}{2p} - \frac{1}{2} > -1$ , and thus the integrability of the bound given by Lemma 4.1 (with respect to  $s$ ). Then from these two results,

$$\begin{aligned} \gamma(t, x) &= g(t, x) + \int_0^t \mathbb{E}[\nabla g(t-s, x - X_s) b_s] ds \\ &\leq g(t, x) + C_{p,d} \mathcal{L}_{b,q}(t) \mathcal{E}_{b,\alpha}(t)^{\frac{1}{2r}} \int_0^t \left(\frac{t-s}{t}\right)^{-\frac{d}{2} + \frac{d}{2p}} \left((t-s)^{-\frac{1}{2}} + \frac{|x|}{t}\right) ds g(pt, x) \\ &\leq g(t, x) + \left(-\frac{d}{2} + \frac{d}{2p} + \frac{1}{2}\right)^{-1} C_{p,d} \mathcal{L}_{b,q}(t) \mathcal{E}_{b,\alpha}(t)^{\frac{1}{2r}} \left(\sqrt{t} + |x|\right) g(pt, x). \end{aligned}$$

□

**4.3. Proof of Theorem 3.5 (Lower bound).** We rely on the fact that the integral term in the representation (3.1) of the density becomes small for small  $t$  and  $x$  (Notice the multiplicative term  $(\sqrt{t} + |x|)$  that appears in the bound of Theorem 3.3. It is actually the same reason that makes our lower bound only valid around the time-space origin).

*Proof of Theorem 3.5.* From Theorem 3.1, we have (using the triangle inequality)

$$\gamma(t, x) \geq g(t, x) - \int_0^t \mathbb{E}[|\nabla g(t-s, x - X_s) b_s|] ds.$$

From Lemma 4.1 (see also the proof of Theorem 3.3), we get

$$\gamma(t, x) \geq g(t, x) - \tilde{C}_{p,d} \mathcal{L}_{b,q}(t) \mathcal{E}_{b,\alpha}(t)^{\frac{1}{2r}} \left(\sqrt{t} + |x|\right) g(pt, x).$$

Let  $R > 0$  and  $(t, x)$  such that  $\frac{|x|}{\sqrt{t}} \leq R$ . Then

$$\begin{aligned} \gamma(t, x) &\geq (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right) - \tilde{C}_{p,d} \mathcal{L}_{b,q}(t) \mathcal{E}_{b,\alpha}(t)^{\frac{1}{2r}} \sqrt{t} (1+R) (2\pi pt)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2pt}\right) \\ &\geq (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right) \left(1 - \epsilon(t)\right), \end{aligned}$$

where

$$\epsilon(t) := \tilde{C}_{p,d} \mathcal{L}_{b,q}(t) \mathcal{E}_{b,\alpha}(t)^{\frac{1}{2r}} (1+R) p^{-\frac{d}{2}} \exp\left(\frac{R^2}{2} - \frac{R^2}{2p}\right) \sqrt{t}.$$

Clearly,  $\epsilon(t) \xrightarrow[t \rightarrow 0]{} 0$  (as  $C_R\sqrt{t}$ ). Thus, there exists  $t_0 > 0$  (depending on  $R, \alpha, q$  and  $d$ ) such that, for  $t \leq t_0$ ,  $\epsilon(t) \leq \frac{1}{2}$ . Therefore,

$$\gamma(t, x) \geq \frac{1}{2}(2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right).$$

□

**4.4. Proof of Lemma 4.1.** The idea is to apply Girsanov theorem to make a change of measure under which  $X$  is a Brownian motion. Since  $\alpha$  may be smaller than  $\frac{1}{2}$ , the standard Novikov condition is not fulfilled. We proceed in three steps: first we establish an estimate for driftless Brownian motion, then for Brownian motion with bounded drift  $b$ , and finally with unbounded  $b$  by approximation.

*Proof of Lemma 4.1.*

**First step.**

Let us show that, for any real number  $p \in (1, \infty)$ , there exists a constant  $C_{p,d}$  such that, for  $x \in \mathbb{R}^d$ ,

$$\|\nabla g(t-s, x - W_s)\|_p \leq C_{p,d} \left(\frac{t-s}{t}\right)^{-\frac{d}{2} + \frac{d}{2p}} \left((t-s)^{-\frac{1}{2}} + \frac{|x|}{t}\right) g(pt, x). \quad (4.4)$$

We have

$$\begin{aligned} \mathbb{E}[|\nabla g(t-s, x - W_s)|^p] &= \int_{\mathbb{R}^d} |\nabla g(t-s, x - y)|^p g(s, y) dy \\ &= \int_{\mathbb{R}^d} \left(\frac{|y-x|}{t-s}\right)^p (2\pi(t-s))^{-\frac{pd}{2}} \exp\left(-p\frac{|y-x|^2}{2(t-s)}\right) (2\pi s)^{-\frac{d}{2}} \exp\left(-\frac{|y|^2}{2s}\right) dy. \end{aligned}$$

By the change of variable  $z = \frac{y-x}{\sqrt{t-s}}$ ,

$$\begin{aligned} \mathbb{E}[|\nabla g(t-s, x - W_s)|^p] &= (2\pi)^{-\frac{pd}{2}} (t-s)^{-\frac{p}{2} - \frac{pd}{2} + \frac{d}{2}} (2\pi s)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |z|^p \exp\left(-p\frac{|z|^2}{2} - \frac{|z\sqrt{t-s} + x|^2}{2s}\right) dz. \end{aligned}$$

By straightforward computation, we have, setting  $\xi := \frac{\sqrt{t-s}}{t-s+ps}x$ ,

$$\begin{aligned} -p\frac{|z|^2}{2} - \frac{|z\sqrt{t-s} + x|^2}{2s} &= -\frac{(t-s+ps)}{2s} |z + \xi|^2 - \frac{p}{2(t-s+ps)} |x|^2 \\ &\leq -\frac{t}{2s} |z + \xi|^2 - \frac{1}{2t} |x|^2. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}[|\nabla g(t-s, x - W_s)|^p] &\leq (2\pi)^{-\frac{pd}{2}} (t-s)^{-\frac{p}{2} - \frac{pd}{2} + \frac{d}{2}} (2\pi s)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right) \int_{\mathbb{R}^d} |z|^p \exp\left(-\frac{t|z + \xi|^2}{2s}\right) dz \\ &= (2\pi)^{-\frac{pd}{2}} (t-s)^{-\frac{p}{2} - \frac{pd}{2} + \frac{d}{2}} (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right) \int_{\mathbb{R}^d} \left|\sqrt{\frac{s}{t}}\zeta - \xi\right|^p \exp\left(-\frac{|\zeta|^2}{2}\right) d\zeta, \end{aligned}$$



where we have made the change of variable  $\zeta := \sqrt{\frac{t}{s}}(z + \xi)$ . From

$$\left| \sqrt{\frac{s}{t}}\zeta - \xi \right|^p \leq 2^{p-1} \left( \left| \sqrt{\frac{s}{t}}\zeta \right|^p + |\xi|^p \right) \leq 2^{p-1} (|\zeta|^p + \frac{(t-s)^{\frac{d}{2}}}{t^p} |x|^p),$$

we get

$$\begin{aligned} & \mathbb{E}[|\nabla g(t-s, x - W_s)|^p] \\ & \leq 2^{p-1} (2\pi)^{-\frac{pd}{2} + \frac{d}{2}} (t-s)^{-\frac{p}{2} - \frac{pd}{2} + \frac{d}{2}} (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right) \mu_{p,d}^p \\ & \quad + 2^{p-1} (2\pi)^{-\frac{pd}{2} + \frac{d}{2}} (t-s)^{-\frac{p}{2} + \frac{d}{2}} |x|^p t^{-p} (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right) \\ & \leq 2^{p-1} (2\pi)^{-\frac{pd}{2} + \frac{d}{2}} \mu_{p,d}^p (t-s)^{-\frac{pd}{2} + \frac{d}{2}} \left( (t-s)^{-\frac{p}{2}} + |x|^p t^{-p} \right) (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2t}\right), \end{aligned}$$

where

$$\mu_{p,d} := \left( \int_{\mathbb{R}^d} |\zeta|^p (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{|\zeta|^2}{2}\right) d\zeta \right)^{\frac{1}{p}} = \|\mathcal{N}(0, I_d)\|_p. \quad (4.5)$$

Therefore (using  $(|a| + |b|)^{\frac{1}{p}} \leq |a|^{\frac{1}{p}} + |b|^{\frac{1}{p}}$ ),

$$\begin{aligned} & (\mathbb{E}[|\nabla g(t-s, x - W_s)|^p])^{\frac{1}{p}} \\ & \leq 2^{1-\frac{1}{p}} (2\pi)^{-\frac{d}{2} + \frac{d}{2p}} \mu_{p,d} (t-s)^{-\frac{d}{2} + \frac{d}{2p}} \left( (t-s)^{-\frac{1}{2}} + |x| t^{-1} \right) (2\pi t)^{-\frac{d}{2p}} \exp\left(-\frac{|x|^2}{2pt}\right) \\ & = C_{p,d} (t-s)^{-\frac{d}{2} + \frac{d}{2p}} \left( (t-s)^{-\frac{1}{2}} + |x| t^{-1} \right) t^{\frac{d}{2} - \frac{d}{2p}} (2\pi pt)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2pt}\right), \end{aligned}$$

where

$$C_{p,d} := 2^{1-\frac{1}{p}} p^{\frac{d}{2}} \mu_{p,d}. \quad (4.6)$$

We have proved estimate (4.4).

### Second step.

Suppose that the drift process  $b$  is uniformly bounded. By Girsanov theorem,  $(X_s)$  is a Brownian motion under the equivalent probability measure  $\mathbb{Q}$  defined by

$$\Gamma_s := \frac{d\mathbb{P}}{d\mathbb{Q}} \Big|_{\mathcal{F}_s} = \exp\left(\int_0^s b_u^* dW_u + \frac{1}{2} \int_0^s |b_u|^2 du\right).$$

We denote the expectation under  $\mathbb{Q}$  by  $\mathbb{E}_{\mathbb{Q}}$ .

By Hölder's inequality (notice that  $p, q, r > 1$  and  $p^{-1} + q^{-1} + r^{-1} = 1$ ),

$$\begin{aligned} & \mathbb{E}[|\nabla g(t-s, x - X_s) b_s|] \\ & = \mathbb{E}_{\mathbb{Q}}[|\nabla g(t-s, x - X_s) b_s| \Gamma_s^{\frac{1}{q}} \Gamma_s^{1-\frac{1}{q}}] \\ & \leq (\mathbb{E}_{\mathbb{Q}}[|\nabla g(t-s, x - X_s)|^p])^{\frac{1}{p}} (\mathbb{E}_{\mathbb{Q}}[|b_s|^q \Gamma_s])^{\frac{1}{q}} (\mathbb{E}_{\mathbb{Q}}[\Gamma_s^{r(1-\frac{1}{q})}])^{\frac{1}{r}} \\ & = (\mathbb{E}[|\nabla g(t-s, x - W_s)|^p])^{\frac{1}{p}} (\mathbb{E}[|b_s|^q])^{\frac{1}{q}} (\mathbb{E}_{\mathbb{Q}}[\Gamma_s^{\frac{r}{p}+1}])^{\frac{1}{r}}. \end{aligned}$$

The bound on the first term is given by estimate (4.4). The second one is bounded by  $\mathcal{L}_{b,q}(t)$ . For the last term, setting  $r' := \frac{r}{p}$ , we have

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}}[\Gamma_s^{r'+1}] \\
&= \mathbb{E}[\Gamma_s^{r'}] \\
&= \mathbb{E}[\exp(\int_0^s r' b_u^* dW_u + \frac{1}{2} \int_0^s r' |b_u|^2 du)] \\
&= \mathbb{E}[\exp(\frac{2r'^2 + r'}{2} \int_0^s |b_u|^2 du) \exp(\int_0^s r' b_u^* dW_u - \int_0^s |r' b_u|^2 du)] \\
&\leq \mathbb{E}[\exp((2r'^2 + r') \int_0^s |b_u|^2 du)]^{\frac{1}{2}} \mathbb{E}[\exp(\int_0^s 2r' b_u^* dW_u - \frac{1}{2} \int_0^s |2r' b_u|^2 du)]^{\frac{1}{2}} \\
&= \mathbb{E}[\exp((2r'^2 + r') \int_0^s |b_u|^2 du)]^{\frac{1}{2}} \\
&= \mathbb{E}[\exp(\frac{r}{p} (\frac{2r}{p} + 1) \int_0^s |b_u|^2 du)]^{\frac{1}{2}}.
\end{aligned}$$

Now

$$\frac{r}{p} = p^{-1} (1 - \frac{1}{p} - \frac{1}{q})^{-1} = (p(1 - \frac{1}{q}) - 1)^{-1} = \frac{2\alpha}{1 + \sqrt{1 + 8\alpha}},$$

and

$$\frac{r}{p} (\frac{2r}{p} + 1) = \frac{2\alpha}{1 + \sqrt{1 + 8\alpha}} \frac{4\alpha + 1 + \sqrt{1 + 8\alpha}}{1 + \sqrt{1 + 8\alpha}} = \frac{\alpha(1 + \sqrt{1 + 8\alpha})^2}{(1 + \sqrt{1 + 8\alpha})^2} = \alpha.$$

Therefore,

$$(\mathbb{E}_{\mathbb{Q}}[\Gamma_s^{\frac{r}{p}+1}])^{\frac{1}{r}} \leq \mathbb{E}[\exp(\alpha \int_0^s |b_u|^2 du)]^{\frac{1}{2r}} \leq \mathcal{E}_{b,\alpha}(t)^{\frac{1}{2r}}.$$

We have proved the lemma for uniformly bounded  $b$ .

### Third step.

We now examine the general case of unbounded  $b$ . We approximate  $b_s$  by the sequence  $b_s^n := b_s \mathbb{1}_{|b_s| \leq n}$  ( $n \geq 0$ ), and we set

$$X_s^n := W_s + \int_0^s b_u^n du.$$

Clearly,  $b_s^n \xrightarrow[n \rightarrow \infty]{} b_s$  and  $X_s^n \xrightarrow[n \rightarrow \infty]{} X_s$ ,  $\mathbb{P}$ -a.s.

From the previous step, we have

$$\begin{aligned}
& \mathbb{E}[|\nabla g(t-s, x - X_s^n) b_s^n|] \\
&\leq C_{p,d} \|b_s^n\|_q \mathbb{E}[\exp(\alpha \int_0^s |b_u^n|^2 du)]^{\frac{1}{2r}} \left(\frac{t-s}{t}\right)^{-\frac{d}{2} + \frac{d}{2p}} \left((t-s)^{-\frac{1}{2}} + \frac{|x|}{t}\right) g(pt, x).
\end{aligned} \tag{4.7}$$

For all  $r > 0$  and  $y \in \mathbb{R}^d$ , we have

$$\nabla g(r, y) = -\frac{y^*}{r} (2\pi r)^{-\frac{d}{2}} \exp(-\frac{|y|^2}{2r}).$$

$\nabla g(r, \cdot)$  is continuous, and

$$\begin{aligned} |\nabla g(r, y)| &= \frac{1}{\sqrt{r}} (2\pi r)^{-\frac{d}{2}} \frac{|y|}{\sqrt{r}} \exp\left(-\frac{|y|^2}{2r}\right) \\ &\leq \frac{C}{\sqrt{r}} (2\pi r)^{-\frac{d}{2}}, \end{aligned}$$

where  $C = \sup_{z \in \mathbb{R}} \{z \exp(-\frac{z^2}{2})\} = \exp(-\frac{1}{2})$ .

Then

$$|\nabla g(t-s, x - X_s^n) b_s^n| \leq \exp\left(-\frac{1}{2}\right) (t-s)^{-\frac{1}{2}} (2\pi(t-s))^{-\frac{d}{2}} |b_s|.$$

Therefore, by the dominated convergence theorem,

$$\begin{aligned} \mathbb{E}[|\nabla g(t-s, x - X_s^n) b_s^n|] &\xrightarrow{n \rightarrow \infty} \mathbb{E}[|\nabla g(t-s, x - X_s) b_s|]; \\ \|b_s^n\|_q &\xrightarrow{n \rightarrow \infty} \|b_s\|_q; \\ \mathbb{E}[\exp(\alpha \int_0^s |b_u^n|^2 du)] &\xrightarrow{n \rightarrow \infty} \mathbb{E}[\exp(\alpha \int_0^s |b_u|^2 du)]. \end{aligned}$$

Taking the limit when  $n \rightarrow \infty$  in (4.7) proves Lemma 4.1.  $\square$

## 5. Conclusion

We represent the density of Brownian motion with random drift in terms of the Gaussian one. This representation enables us to measure the distance between both densities, leading to explicit Gaussian upper and (local) lower bounds. In particular, this work extends to a non-Markovian setting similar results known for stochastic differential equations with irregular drift, using simpler and more general arguments.

## References

1. Aronson, D. G.: Bounds for the fundamental solution of a parabolic equation, *Bull. Amer. Math. Soc.*, **73** (1967) 890–896.
2. Bally, V.: Lower bounds for the density of locally elliptic Itô processes, *Annals of Probability*, **34** (2006) 2406–2440.
3. Fournier, N. and Printems, J.: Absolute continuity for some one-dimensional processes, *Bernoulli*, **16**(2) (2010) 343–360.
4. Hayashi, M., Kohatsu-Higa, A., and Yūki, G.: Hölder continuity property of the densities of SDEs with singular drift coefficients, *Electron. J. Probab.*, **19**(77) (2014) 1–22.
5. Kohatsu-Higa, A.: Lower bounds for densities of uniformly elliptic non-homogeneous diffusions, *Proceedings of the Stochastic Inequalities Conference in Barcelona. Progress in Probability*, **56** (2003) 323–338.
6. Kohatsu-Higa, A. and Makhlof, A.: Estimates for the density of functionals of SDEs with irregular drift, *Stochastic Processes and their Applications*, **123**(5) (2013) 1716–1728.
7. Kohatsu-Higa, A. and Tanaka, A.: A Malliavin Calculus method to study densities of additive functionals of SDE's with irregular drifts, *Ann. Inst. H. Poincaré Probab. Statist.*, **48**(3) (2012) 871–883.
8. Kusuoka, S. and Stroock, D.: Applications of Malliavin calculus, part III, *J. Faculty Sci. Univ. Tokyo Sect. 1A Math.*, **34** (1987) 391–442.
9. Lanconelli, A.: A comparison theorem for stochastic differential equations under the Novikov condition, *Potential Analysis*, **41**(4) (2014) 1065–1077.

10. Nguyen, T. D., Privault, N., and Torrisi, G. L.: Gaussian estimates for solutions of some one-dimensional stochastic equations. To appear in *Potential Analysis*. Preprint available at [http://www.ntu.edu.sg/home/nprivault/papers/stochastic\\_equations.pdf](http://www.ntu.edu.sg/home/nprivault/papers/stochastic_equations.pdf), 2014.
11. Qian, Z., Russo, F., and Zheng, W.: Comparison theorem and estimates for transition probability densities of diffusion processes, *Probability Theory and Related Fields*, **127**(3) (2003) 388–406.
12. Qian, Z. and Zheng, W.: A representation formula for transition probability densities of diffusions and applications, *Stochastic Processes and their Applications*, **111**(1) (2004) 57–76.

AZMI MAKHLOUF: NATIONAL ENGINEERING SCHOOL OF TUNIS (ENIT) AND LABORATORY OF MATHEMATICAL AND NUMERICAL MODELLING IN ENGINEERING SCIENCES (LAMSIN), UNIVERSITY OF TUNIS EL MANAR, TUNIS 1002, TUNISIA  
*E-mail address:* `azmi.makhlouf@gmail.com`