Communications on Stochastic Analysis
Vol. 10, No. 2 (2016) 185-193

Serials Publications
www.serialspublications.com

CONTINUITY OF RANDOM FIELDS ON RIEMANNIAN MANIFOLDS

ANNIKA LANG*, JÜRGEN POTTHOFF, MARTIN SCHLATHER,
AND DIMITRI SCHWAB

Dedicated to the memory of Robert Schrader

Abstract. A theorem of the Kolmogorov–Chentsov type is proved for random fields on a Riemannian manifold.

1. Introduction

One of the key theorems in the theory of stochastic processes is the Kolmogorov–Chentsov theorem (the classical references are [8] and [2]), which states the existence of a continuous modification of a given stochastic process based on tail or moment estimates of its increments.

In the present paper we prove a theorem of the Kolmogorov–Chentsov type for random fields indexed by a finite–dimensional Riemannian manifold. A result of similar kind has been proved in a recent paper, in which also results about differentiablity are shown [1]. Concerning the Kolmogorov–Chentsov theorem the differences between the present and the cited paper are twofold: For one, the continuity statement in [1] is formulated in terms of coordinate mappings, while here we give an intrinsic formulation, i.e., a direct formulation in terms of the Riemannian (topological) metric. Secondly, the methods of proof are quite different: While the Sobolev embedding theorem is employed in [1], we basically use here the classical method via dyadic approximations and the Borel–Cantelli lemma. In fact, our proof of the Kolmogorov–Chentsov theorem for random fields on a Riemannian manifold is based on a localized variant of a theorem in [7], which is combined with a local coordinatization of the underlying Riemannian manifold in terms of the exponential map.

The organization of the paper is as follows. In Section 2 we recall the setup of the general Kolmogorov–Chentsov theorem in [7], and prove the above mentioned localized version of Theorem 2.8 in [7]. In Theorem 3.1 below we state and prove our main assertion, namely our theorem of Kolmogorov–Chentsov type for Riemannian manifolds. Finally, in Section 4 we discuss the existence of locally Hölder

Received 2016-7-20; Communicated by Hui-Hsiung Kuo.
2010 *Mathematics Subject Classification.* Primary 60G60, 60G17; Secondary 53B21, 60G15.
Key words and phrases. Random fields, Riemannian manifolds, Kolmogorov–Chentsov theorem, Gaussian random fields.
* The work of AL was supported in part by the Swedish Research Council under Reg. No. 621-2014-3995 as well as the Knut and Alice Wallenberg foundation.

continuous modifications, provide sufficient conditions in terms of moments of the increments, and consider the special case of Gaussian random fields as examples.

2. A Kolmogorov–Chentsov Theorem for Metric Spaces

In this section we give a variant of the Kolmogorov–Chentsov type theorem in [7], which follows rather directly from it, and in some sense sharpens that result.

Suppose that (M, d) is a separable metric space, that $(\Omega, \mathcal{A}, P)$ is a probability space, and that $\phi = \big(\phi(x),\, x \in M\big)$ is a real-valued random field on this probability space indexed by M.

Assume furthermore that r and q are two strictly increasing functions on an interval $[0, \rho)$, $\rho > 0$, such that $r(0) = q(0) = 0$. Throughout this paper we suppose that for all x, $y \in M$ with $d(x, y) < \rho$, we have the bound

$$P\Big(\big|\phi(x) - \phi(y)\big| > r\big(d(x, y)\big)\Big) \leq q\big(d(x, y)\big). \tag{2.1}$$

Remark 2.1. We take the occasion to correct a minor error in [7]: There, this bound has been formulated and used with a "$\geq$" sign for the event under the probability, which for $x = y$ is obviously absurd. However, an inspection shows that all arguments and results in [7] remain correct, when replacing the condition there with the inequality (2.1). Alternatively, the condition in [7] could be supplied with the additional restriction $x \neq y$.

We make the following assumptions on the metric space (M, d):

Assumptions 2.2.

(a) There exists an at most countable open cover $(U_n,\, n \in \mathbb{N})$ of M, and for every $n \in \mathbb{N}$, there exists a metric d_n on U_n so that $\alpha_n\, d_n(x, y) \leq d(x, y) \leq d_n(x, y)$ for all x, $y \in U_n$ and some $\alpha_n \in (0, 1]$;

(b) for every $n \in \mathbb{N}$, (U_n, d_n) is well separable in the sense of [7], i.e.:

 i) there exists an increasing sequence $(D_{n,k},\, k \in \mathbb{N})$ of finite subsets of U_n such that $D_n = \bigcup_k D_{n,k}$ is dense in (U_n, d_n), and for $x \in D_{n,k}$, let $C_{n,k}(x) = \{y \in D_{n,k},\, d_n(x, y) \leq \delta_{n,k}\}$, where $\delta_{n,k}$ denotes the minimal distance of distinct points in $D_{n,k}$ with respect to d_n;

 ii) every $z \in U_n$ has a neighborhood $V \subset U_n$ so that for almost all $k \in \mathbb{N}$ and all x, $y \in D_{n,k+1} \cap V$, there exist x', $y' \in D_{n,k} \cap V$ with $x' \in C_{n,k+1}(x)$, $y' \in C_{n,k+1}(y)$, and $d_n(x', y') \leq d_n(x, y)$;

(c) for n, $k \in \mathbb{N}$, let $\pi_{n,k}$ be the set of all unordered pairs $\langle x, y\rangle$, x, $y \in D_{n,k}$ with $d_n(x, y) \leq \delta_{n,k}$, and let $|\pi_{n,k}|$ denote the number of elements in this set, then

$$\sum_k |\pi_{n,k}|\, q(\delta_{n,k}) < +\infty, \tag{2.2}$$

$$\sum_k r(\delta_{n,k}) < +\infty \tag{2.3}$$

hold true.

We remark that due to the assumption on the metrics d and d_n, $n \in \mathbb{N}$, in (a) above, the relative topology on U_n generated by d coincides with the topology generated by d_n.

For x, $y \in U_n$ with $d(x,y) < \alpha_n \rho$, we can estimate as follows

$$\begin{aligned} P\Big(\big|\phi(x) - \phi(y)\big| > r\big(d_n(x,y)\big)\Big) \\ &\le P\Big(\big|\phi(x) - \phi(y)\big| > r\big(d(x,y)\big)\Big) \\ &\le q\big(d(x,y)\big) \\ &\le q\big(d_n(x,y)\big), \end{aligned}$$

because r and q are both increasing. Theorem 2.8 in [7] shows that from this estimate, together with the Assumptions (a), (b), and (c) above, it follows that for every $n \in \mathbb{N}$, the restriction ϕ_n of ϕ to U_n has a locally uniformly continuous modification ψ_n which is such that ϕ_n, ψ_n, and ϕ coincide on D_n. In more detail we have that for every $n \in \mathbb{N}$, there exists a random field ψ_n indexed by U_n such that

(i) for every $\omega \in \Omega$, the mapping $\psi_n(\cdot, \omega) : U_n \to \mathbb{R}$ is locally uniformly continuous;
(ii) for every $x \in U_n$, there exists a P–null set $N_{x,n}$ so that $\psi_n(x,\omega) = \phi(x,\omega)$ for all ω in the complement of $N_{x,n}$, and if $x \in D_n$, $N_{x,n}$ can be chosen as the empty set.

In order to get for $x \in M$ a universal P-null set N_x, we set $N_x = \bigcup_{n'} N_{x,n'}$, where the union is over all $n' \in \mathbb{N}$ such that $x \in U_{n'}$. Since this is a countable union, N_x is indeed a P–null set.

From the modifications ψ_n of ϕ_n, $n \in \mathbb{N}$, we construct a locally uniformly continuous modification ψ of ϕ. We show

Lemma 2.3. *There exists a P–null set $N \in \mathcal{A}$ so that for all n, $n' \in \mathbb{N}$, $n \neq n'$, with $U_n \cap U_{n'} \neq \emptyset$ and every $x \in U_n \cap U_{n'}$ the equality $\psi_n(x,\omega) = \psi_{n'}(x,\omega)$ holds true for all ω in the complement of N.*

Proof. Assume that $x \in U_n \cap U_{n'}$. Since ψ_n and $\psi_{n'}$ are modifications of ϕ when all these random fields are restricted to $U_n \cap U_{n'}$, we get $\psi_n(x) = \psi_{n'}(x)$ on the complement of the P–null set N_x. Since (M,d) is separable so is $(U_n \cap U_{n'}, d)$, and letting x range over a countable dense subset $E_{n,n'}$ and taking the union of all associated P–null sets, we get the existence of a P–null set $N_{n,n'}$ such that for all $x \in E_{n,n'}$, we have $\psi_n(x) = \psi_{n'}(x)$ on the complement of $N_{n,n'}$. ψ_n and $\psi_{n'}$ are continuous on $U_n \cap U_{n'}$, and hence we obtain for all $x \in U_n \cap U_{n'}$ the equality $\psi_n(x) = \psi_{n'}(x)$ on the complement of $N_{n,n'}$. Finally, we set $N = \bigcup_{n,n'} N_{n,n'}$ so that we find for all n, n', and all $x \in U_n \cap U_{n'}$ the equality $\psi_n(x) = \psi_{n'}(x)$ on the complement of the P–null set N. □

On the exceptional set N of the last lemma we define $\psi(x) = 0$ for all $x \in M$. On its complement we set $\psi(x) = \psi_n(x)$ whenever $x \in U_n$, and the last lemma shows that this makes ψ well-defined. For $x \in M$, define the P–null set $N'_x = N_x \cup N$ where N_x and N are the P–null sets defined above. We have $x \in U_n$ for some

$n \in \mathbb{N}$, and for all ω in the complement of N'_x, we find that $\psi(x,\omega) = \psi_n(x,\omega) = \phi(x,\omega)$. Thus ψ is a modification of ϕ. We have proved the following

Theorem 2.4. *Under Condition* (2.1) *on the random field ϕ and under the above Assumptions 2.2 on (M,d), r and q, ϕ has a locally uniformly continuous modification.*

3. A Kolmogorov–Chentsov Theorem for Riemannian Manifolds

We begin this section by recalling the necessary terminology of Riemannian geometry, setting up our notation at the same time. For further background the interested reader is referred to the standard literature, e.g. [3, 4, 6].

Assume that $m \in \mathbb{N}$ and that (M,g) is an m–dimensional Riemannian manifold as defined in [3]. That is, M is a connected, m–dimensional C^∞–manifold together with a symmetric, strictly positive definite tensor field g of type $(0,2)$. For each $x \in M$, the Riemannian metric g determines an inner product $g_x(\cdot,\cdot)$ on the tangent space T_xM at x:

$$\begin{aligned} g_x : T_xM \times T_xM &\to \mathbb{R} \\ (X,Y) &\mapsto g_x(X,Y). \end{aligned}$$

The corresponding norm on T_xM is given by

$$\|X\| = g_x(X,X)^{1/2}, \qquad X \in T_xM.$$

Let $c : [a,b] \to M$ be a smooth curve in M. Then its derivative $c'(t)$ at $t \in (a,b)$ belongs to $T_{c(t)}M$, and the length of c is given by

$$L(c) = \int_a^b \|c'(t)\|\, dt.$$

The Riemannian distance $d(x,y)$ of two points x, $y \in M$ is defined as the infimum of the lengths of curve segments joining x and y. Indeed, d is a metric on M and it can be shown that under the given assumptions on M the metric space (M,d) is separable, locally compact and connected [3, Proposition I.9.6]. Furthermore, the original topology and the topology defined by d coincide [3, Corollary I.9.5].

We denote the open ball of radius $R > 0$ centered at $x \in M$ relative to the metric d by $B^d_R(x)$, while the ball of radius R in T_xM with center at $X \in T_xM$ with respect to the norm $\|\cdot\|$ is denoted by $B_R(X)$.

With the Riemannian metric g there is canonically associated — via the notions of parallel transport and geodesics — the exponential map $(\mathrm{Exp}_x,\, x \in M)$, which for each $x \in M$ is a mapping from T_xM into a neighborhood of x in M. It can be shown that for each $x \in M$, there exists a radius $R(x) > 0$ such that Exp_x maps $B_{R(x)}(0)$ diffeomorphically onto $B^d_{R(x)}(x)$ [3, Theorem I.9.9, Proposition I.9.4]. Moreover, for all Y, $Z \in B_{R(x)}(0)$ such that $\mathrm{Exp}_x(Y) = y$, $\mathrm{Exp}_x(Z) = z$, the quotient $\|Y - Z\|/d(y,z)$ converges to 1 as $(y,z) \to (x,x)$ [3, Proposition I.9.10].

In view of Theorem 2.4 in Section 2, we construct a countable cover $(U_n,\, n \in \mathbb{N})$ of M as follows. The separability of M (see above) allows us to fix a countable dense subset $\{x_n,\, n \in \mathbb{N}\}$ of M. For every $n \in \mathbb{N}$, choose $R_n \in \big(0, 1/(2\sqrt{m})\big]$ in such a way that:

1. the exponential map Exp_{x_n} is a diffeomorphism from $B_{R_n}(0) \subset T_{x_n}M$ onto $B^d_{R_n}(x_n) \subset M$,
2. for all X, $Y \in B_{R_n}(0)$ such that $\mathrm{Exp}_{x_n}(X) = x$, $\mathrm{Exp}_{x_n}(Y) = y$, x, $y \in B^d_{R_n}(x_n)$,

$$2^{-1}\|X - Y\| \le d(x, y) \le 2\|X - Y\|. \tag{3.1}$$

The existence of a strictly positive R_n for each $n \in \mathbb{N}$ with these properties follows from the facts mentioned before.

The idea is now to use the exponential map in order to define a convenient coordinatization of U_n and to use inequality (3.1) for the definition of a suitable metric d_n on U_n. To this end, we fix an orthonormal basis $(X_{n,1}, \dots, X_{n,m})$ of $(T_{x_n}M, g_{x_n})$ so that every $X \in T_{x_n}M$ can be written in a unique way as

$$X = \sum_{i=1}^{m} a_i X_{n,i}$$

with $a = (a_1, \dots, a_m) \in \mathbb{R}^m$. Let us denote the so defined linear mapping from $\mathbb{R}^m$ onto $T_{x_n}M$ by L_{x_n}. The orthonormality of $(X_{n,1}, \dots, X_{n,m})$ entails that L_{x_n} is an isometric isomorphism if $\mathbb{R}^m$ is equipped with the standard euclidean metric. In particular, the ball $B_{R_n}(0)$ is under L_{x_n} in one-to-one correspondence with the euclidean ball $B^m_{R_n}(0)$ in $\mathbb{R}^m$. Define

$$\varphi_n(x) = L^{-1}_{x_n} \circ \mathrm{Exp}^{-1}_{x_n}(x), \qquad x \in B^d_{R_n}(x_n),$$

then φ_n is a C^∞–coordinatization of $B^d_{R_n}(x_n)$ which maps this ball onto $B^m_{R_n}(0) \subset \mathbb{R}^m$.

For x, $y \in B^d_{R_n}(x_n)$, define

$$d_n(x, y) = 2\sqrt{m} \max_{i=1,\dots,m} \left|\varphi^i_n(x) - \varphi^i_n(y)\right|, \tag{3.2}$$

where $\varphi^i_n(x)$ denotes the i–th Cartesian coordinate of $\varphi_n(x)$. Set $\alpha_n = 1/(4\sqrt{m})$. If $\|\cdot\|_2$ denotes the usual euclidean norm on $\mathbb{R}^m$, we obtain from (3.1)

$$\begin{aligned}
\alpha_n\, d_n(x, y) &= 2^{-1} \max_i \left|\varphi^i_n(x) - \varphi^i_n(y)\right| \\
&\le 2^{-1}\|\varphi_n(x) - \varphi_n(y)\|_2 \\
&= 2^{-1}\|\mathrm{Exp}^{-1}_{x_n}(x) - \mathrm{Exp}^{-1}_{x_n}(y)\| \\
&\le d(x, y) \\
&\le 2\|\mathrm{Exp}^{-1}_{x_n}(x) - \mathrm{Exp}^{-1}_{x_n}(y)\| \\
&= 2\|\varphi_n(x) - \varphi_n(y)\|_2 \\
&\le d_n(x, y).
\end{aligned}$$

Consider the open hypercube $H^m_{R_n}(0)$

$$H^m_{R_n}(0) = \left\{x \in \mathbb{R}^m,\ \max_{i=1,\dots,m} |x_i| < m^{-1/2} R_n\right\}$$

in $\mathbb{R}^m$ of sidelength $2m^{-1/2}R_n$ centered at the origin. Clearly we have $H^m_{R_n}(0) \subset B^m_{R_n}(0)$. Set

$$U_n = \varphi_n^{-1}\big(H^m_{R_n}(0)\big)$$

so that $(U_n,\, n \in \mathbb{N})$ is an open cover of M.

For each $k \in \mathbb{N}$, define the following subset $G_{n,k}$ of the hypercube $H^m_{R_n}(0)$

$$G_{n,k} = \Big\{a \in \mathbb{R}^m,\, a = -\frac{R_n}{\sqrt{m}} + \frac{lR_n}{2^k\sqrt{m}},\, l \in \big\{1, \dots, 2^{k+1}-1\big\}^m\Big\}.$$

By construction, for each $n \in \mathbb{N}$, $(G_{n,k},\, k \in \mathbb{N})$ is an increasing sequence of finite subsets of $H^m_{R_n}(0)$, and the union of these sets is dense in $H^m_{R_n}(0)$. Next set $D_{n,k} = \varphi_n^{-1}(G_{n,k})$. Then for each $n \in \mathbb{N}$, $(D_{n,k},\, k \in \mathbb{N})$ is an increasing sequence of subsets of U_n, its limit being dense in U_n. Moreover, it is easy to see that Condition (b.ii) of Assumption 2.2 holds true for the sequence $(D_{n,k},\, k \in \mathbb{N})$, where for every $z \in U_n$, we may choose the neighborhood V in this condition as U_n itself. (For an explicit argument, see also [7].)

By construction we have (in terms of the notation of Section 2)

$$\delta_{n,k} = \min\big\{d_n(x,y),\, x, y \in D_{n,k},\, x \neq y\big\} = 2^{-k+1}R_n.$$

The number $|\pi_{n,k}|$ of unordered pairs in $\pi_{n,k}$ (cf. Assumption 2.2.(c)) can be bounded from above by $K_m 2^{mk}$ for some constant K_m.

Now let $\rho \in (0,1]$, and make the usual choices of the functions r, and q:

$$r(h) = \log_2(h^{-1})^{-\beta}, \tag{3.3}$$

$$q(h) = K\, \log_2(h^{-1})^{-\alpha}\, h^m, \tag{3.4}$$

for $h \in (0,\rho)$, and $r(0) = q(0) = 0$. Here $K > 0$ is an arbitrary constant, and α, $\beta > 1$. Then it is straightforward to check that the inequalities (2.2), (2.3) hold true.

Thus we can apply Theorem 2.4 and obtain

Theorem 3.1. *Suppose that ϕ is a random field defined on an m–dimensional Riemannian manifold (M,g) with topological metric d such that for all x, $y \in M$ with $d(x,y) < \rho$,*

$$P\Big(\big|\phi(x) - \phi(y)\big| > r\big(d(x,y)\big)\Big) \leq q\big(d(x,y)\big)$$

holds true, where the functions r, q are defined as in (3.3), (3.4) *for some constants $K > 0$, α, $\beta > 1$. Then ϕ has a locally uniformly continuous modification.*

4. Hölder Continuity and Moment Conditions

While a locally uniformly continuous modification was constructed in the previous section, the goal here is to show higher regularity in terms of orders of Hölder continuity under additional assumptions.

Therefore, let (M,d) be a metric space and ϕ be as in Section 2, Assumption 2.2. For the sequences of minimal distances $(\delta_{n,k}, k \in \mathbb{N})$ of distinct points in the grids $(D_{n,k}, k \in \mathbb{N})$ and the function r, we make the stronger assumptions

Assumptions 4.1.

(a) For every $n \in \mathbb{N}$, there exist constants $\eta_n \in (0,1)$, $C_n > 0$ such that for almost all $k \in \mathbb{N}$,

$$\frac{1}{C_n}\,\eta_n^k \le \delta_{n,k} \le C_n\,\eta_n^k \tag{4.1}$$

holds true;

(b) there exist constants $\gamma \in (0,1)$, $K_\gamma > 0$ so that for all $h \in [0,\rho)$, the inequality

$$r(h) \le K_\gamma h^\gamma \tag{4.2}$$

is valid.

(Note that Assumption (b) on r above is stronger than the requirement of inequality (2.3).) Then, on every (U_n, d_n), we are in the situation of Theorem 2.9 in [7] and get the existence of a modification ψ_n which is locally Hölder continuous of order γ, i.e., for every $\omega \in \Omega$ and every $z \in U_n$, there exists a neighborhood $V(\omega)$ of z in (U_n, d_n) and a constant $\alpha_{\gamma,n}$ such that

$$\sup_{x,y\in V(\omega),\, x\neq y}\left|\frac{\psi_n(x,\omega)-\psi_n(y,\omega)}{d_n(x,y)^\gamma}\right| \le \alpha_{\gamma,n}.$$

Actually, the constant $\alpha_{\gamma,n}$ was explicitly calculated in [7] and is given by

$$\alpha_{\gamma,n} = 2K_\gamma\,\frac{C_n^{2\gamma}}{\eta_n^\gamma(1-\eta_n^\gamma)}.$$

Again we can glue these modifications together to get a modification ψ of ϕ on (M,d) which is locally Hölder continuous of order γ.

Corollary 4.2. *Assume that Condition* (2.1) *on the random field ϕ holds true. Suppose furthermore that the Assumptions 2.2 are valid, together with the additional stronger properties given in Assumptions 4.1. Then ϕ has a modification which is locally Hölder continuous of order γ.*

We return to the case where M is an m–dimensional Riemannian manifold with topological metric d. Let the open cover $((U_n, d_n), n \in \mathbb{N})$, and the sequences of grids $((D_{n,k}, \delta_{n,k}), k \in \mathbb{N})$, $n \in \mathbb{N}$, be defined as in Section 3. Recall that $\delta_{n,k} = 2^{-k+1}R_n$ and $R_n \in (0, 1/2\sqrt{m})]$, n, $k \in \mathbb{N}$. Set $\eta_n = 1/2$, and choose $C_n \ge 1/(2R_n)$. Then Condition (4.1) is fulfilled. As before let ρ be in $(0,1]$. Define q as in (3.4), and

$$r(h) = h^\gamma \tag{4.3}$$

for some $\gamma \in (0,1)$. Then Condition (4.2) is valid as well, and so we arrive at

Corollary 4.3. *Let ϕ be a random field defined on an m–dimensional Riemannian manifold (M,g), $m \in \mathbb{N}$, with topological metric d, such that for all x, $y \in M$ with $d(x,y) < \rho$,*

$$P\Big(\big|\phi(x)-\phi(y)\big| > r\big(d(x,y)\big)\Big) \le q\big(d(x,y)\big)$$

holds true, where the functions r, q are defined as in (4.3), (3.4) *for some constants $K > 0$, $\alpha > 1$, $\gamma \in (0,1)$. Then ϕ has a locally Hölder continuous modification of order γ.*

The standard application of Chebychev's inequality yields sufficient conditions in terms of moments:

Corollary 4.4. *Suppose that ϕ is a random field defined on an m–dimensional Riemannian manifold M, $m \in \mathbb{N}$, with topological metric d.*

(a) *If there exist $\rho \in (0,1]$, $l \geq 1$, $\kappa \geq m$, $\nu > l+1$, and $K > 0$ such that*
$$\mathbb{E}\big(|\phi(x) - \phi(y)|^l\big) \leq K \log_2\big(d(x,y)^{-1}\big)^{-\nu} d(x,y)^\kappa$$
for all x, $y \in M$ with $d(x,y) < \rho$, then ϕ has a modification which is locally uniformly continuous.

(b) *If there are $\rho \in (0,1]$, $l \geq 1$, $\gamma \in (0,1)$, and $\alpha > 1$ such that*
$$\mathbb{E}\big(|\phi(x) - \phi(y)|^l\big) \leq K \log_2\big(d(x,y)^{-1}\big)^{-\alpha} d(x,y)^{m+l\gamma}$$
for all x, $y \in M$ with $d(x,y) < \rho$, the modification can be chosen to have locally Hölder continuous sample paths of order γ.

In case of a Gaussian random field, Corollary 4.4 leads to a condition which can be formulated in terms of the variogram of the random field:

Corollary 4.5. *Assume that ϕ is a Gaussian random field defined on an m–dimensional Riemannian manifold M, $m \in \mathbb{N}$, with topological metric d and variogram $\sigma(x,y)^2 = \mathbb{E}\big((\phi(x) - \phi(y))^2\big)$. If there exist $\rho \in (0,1]$, $\eta \in (0,1)$, and $C > 0$ such that*
$$\sigma(x,y)^2 \leq C\, d(x,y)^\eta \tag{4.4}$$
for all x, $y \in M$ with $d(x,y) < \rho$, then ϕ has a modification which is locally Hölder continuous of order γ for all $\gamma < \eta/2$.

Applied to the specific case of the m–dimensional unit sphere embedded in $\mathbb{R}^{m+1}$, this corollary recovers the results from [5], where isotropic Gaussian random fields on spheres are considered.

Acknowledgment. We are grateful to Martin Schmidt for fruitful discussions.

References

1. Andreev, R. and Lang, A.: Kolmogorov-Chentsov theorem and differentiability of random fields on manifolds, *Potential Analysis* **41** (2014) 761–769.
2. Chentsov, N. N.: Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the "heuristic" approach to the Kolmogorov-Smirnov tests, *Theory Probab. Appl.* **1** (1956) 140–144.
3. Helgason, S.: *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.
4. Jost, J.: *Riemannian Geometry and Geometric Analysis*, Springer, Berlin, Heidelberg, New York, 2011.
5. Lang, A. and Schwab, C.: Isotropic Gaussian random fields on the sphere: regularity, fast simulation and stochastic partial differential equations, *Ann. Appl. Probab.* **25** (2015) 3047–3094.
6. Petersen, P.: *Riemannian Geometry*, Springer, New York, 2006.
7. Potthoff, J.: Sample properties of random fields II: Continuity, *Commun. Stoch. Anal.* **3** (2009) 331–348.

8. Slutsky, E.: Qualche proposizione relative alla teoria delle funzioni aleatorie, *Giorn. Ist. Attuari* **8** (1937) 183–199.

Annika Lang: Department of Mathematical Sciences, Chalmers University of Technology & University of Gothenburg, SE–41296 Göteborg, Sweden
E-mail address: annika.lang@chalmers.se

Jürgen Potthoff: Institut für Mathematik, Universität Mannheim, D–68131 Mannheim, Germany
E-mail address: potthoff@uni-mannheim.de

Martin Schlather: Institut für Mathematik, Universität Mannheim, D–68131 Mannheim, Germany
E-mail address: schlather@uni-mannheim.de

Dimitri Schwab: Institut für Mathematik, Universität Mannheim, D–68131 Mannheim, Germany
E-mail address: dschwab@mail.uni-mannheim.de