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SOME CONSIDERATIONS ON THE STRUCTURE OF TRANSITION DENSITIES OF SYMMETRIC LÉVY PROCESSES

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Abstract. For a class of symmetric Lévy processes $(Y_t)_{t\geq 0}$ with characteristic exponent ψ we show that $\nu_t = e^{-\frac{1}{t}\psi(\cdot)}/p_{\frac{1}{t}}(0)$, $t > 0$, gives rise to an additive process $(X_t)_{t\geq 0}$ with t-dependent characteristic exponent $-\frac{\partial}{\partial t}\ln\left(p_{\frac{1}{t}}(\xi)/p_{\frac{1}{t}}(0)\right)$ where $(p_t)_{t>0}$ are the transition densities of $(Y_t)_{t\geq 0}$. We estimate (from above and below) p_t in terms of two metrics $d_{\psi,t}$ and $\delta_{\psi,t}$, $d_{\psi,t}$ controlling $p_t(0)$ and $\delta_{\psi,t}$ the spatial decay, and we prove that the transition density $\pi_{t,0}$ of $P_{X_t-X_0}$ is controlled by $\delta_{\psi,\frac{1}{t}}$ and $d_{\psi,\frac{1}{t}}$ now with $\delta_{\psi,\frac{1}{t}}$ controlling $\pi_{t,0}(0)$ and $d_{\psi,\frac{1}{t}}$ the spatial decay.

1. Introduction

In the mid 1980's it became clear that heat kernels of second order elliptic partial differential operators are best understood in terms of the underlying Riemannian geometry, see [4] and [5] as seminal contributions. Further investigation led to the concept of metric measure spaces associated with (local) Dirichlet forms, and now geometry is used to construct corresponding diffusions and their generators, see [21], and for more recent developments [7, 8, 9].

We bypass the work of the E.M.Stein school on the corresponding sub-elliptic problem which led to the emergence of sub-Riemannian geometry as we want to emphasise the efforts in [3] to extend the programme to non-local generators of Markovian semigroups. The crucial role of the carré du champ operator, see in particular [16], was highlighted, however to our best knowledge so far this programme has not led to the desired results.

In studying concrete transition densities of symmetric Lévy processes as well as the observation that the characteristic exponent of such a process often led to a metric measure space, the second named author suggested to use this metric to study the transition densities. In [15] for the diagonal term a first result could be proved, and in [14] a suggestion of a theory was outlined to interpret the transition density of certain symmetric Lévy processes in terms of two time-dependent metrics $d_{\psi,t}$ and $\delta_{\psi,t}$ where ψ is the characteristic exponent of the Lévy process. Assuming volume doubling for the metric measure space $\left(\mathbb{R}^n, d_{\psi,t}, \lambda^{(n)}\right)$ the result

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reads as,

$$p_t(x-y) \asymp \lambda^{(n)}\big(B^{d_{\psi,t}}(0,1)\big)e^{-\delta^2_{\psi,t}(x,y)}, \tag{1.1}$$

where $B^{d_{\psi,t}}(0,1)$ denotes the unit ball with respect to $d_{\psi,t}$ and $a_t \asymp b_t$ means for two constants $0<\gamma_0<\gamma_1$ that $\gamma_0 a_t \le b_t \le \gamma_1 a_t$. While in [14] the existence of $\delta_{\psi,t}$ could only be proved for some classes, the estimate,

$$p_t(0) \asymp \lambda^{(n)}\big(B^{d_{\psi,t}}(0,1)\big), \tag{1.2}$$

holds always under the doubling condition and we can write,

$$p_t(x-y) = p_t(0)\frac{p_t(x-y)}{p_t(0)} = p_t(0)e^{-\left(-\ln\frac{p_t(x-y)}{p_t(0)}\right)}. \tag{1.3}$$

In the case we can show that,

$$x \mapsto -\ln\frac{p_t(x)}{p_t(0)}, \tag{1.4}$$

is a continuous negative definite function, see [13, Vol I.] for the definition (but note that the continuous negative definite function is just another name for the characteristic exponent of a Lévy process), our conditions will imply that,

$$\delta_{\psi,t}(x,y) = \left(-\ln\frac{p_t(x-y)}{p_t(0)}\right)^{1/2}, \tag{1.5}$$

is a metric. So far we do not know a general result of this type, but plenty of examples, see [2] or [14]. However even for the transition density of the relativistic Hamiltonian process associated with $\psi(\xi) = (|\xi|^2+m)^{1/2}-m$, see [10], the problem is still open. On the other hand, using subordination in the sense of Bochner, see as a general reference [1] or [19], new examples can be constructed. It is helpful to note in this context that for subordination a good functional calculus is available [18], and that certain functional inequalities are stable under subordination, see [20], since functional inequalities are useful tools to handle transition densities. We want to also mention that in [6] the suggested approach was tested for Q-matrices with state space $\mathbb{Z}^n$ relying much on commutative harmonic analysis. It would be of interest to extend these ideas to locally compact groups since they allow a corresponding harmonic analysis and are well studied objects in probability theory, see H. Heyer [11].

In this paper we start with an obvious observation. If $(\mu_t)_{t\ge0}$ is a convolution semigroup of probability measures on $\mathbb{R}^n$ with Fourier transform,

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2}e^{-t\psi(\xi)}, \tag{1.6}$$

then by,

$$\rho_t(\mathrm{d}x) := \frac{e^{-t\psi(x)}}{(2\pi)^n p_t(0)}\,\mathrm{d}x, \tag{1.7}$$

a family of probability measures $(\rho_t)_{t\ge0}$ is given. Using a ratio limit result as proved in [14] it turns out that the family,

$$\nu_t := \rho_{\frac{1}{t}},\ \ t>0, \tag{1.8}$$

is for $t\to0$ weakly continuous and it gives rise to a family of strongly continuous convolution operators $(S_t)_{t\ge0}$ which are contractions in either $\big(C_\infty(\mathbb{R}^n),\|\cdot\|_\infty\big)$ or

$\big(\mathcal{L}^2(\mathbb{R}^n), \|\cdot\|_0\big)$ and are either positivity preserving or sub-Markovian. However, in general $(S_t)_{t\geq 0}$ is not a semigroup. We can show that with,

$$q(t,\xi) := -\frac{\partial}{\partial t} \ln \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)}, \tag{1.9}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ we get,

$$\frac{\partial}{\partial t} S_t u + q(t,D) S_t u = 0, \;\; \lim_{t\to 0} S_t u = u, \tag{1.10}$$

where $q(t,D)$ is the pseudo-differential operator with symbol $q(t,\xi)$. We give examples for $\xi \mapsto q(t,\xi)$ being a continuous negative definite function. Under the assumption that $q : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is continuous and $\xi \mapsto q(t,\xi)$ is a negative definite function we prove further that we can associate with $q(t,\xi)$ an additive process $(X_t)_{t\geq 0}$ with,

$$P_{X_t - X_s} = \gamma_{t,s}, \;\; 0 \leq s < t, \tag{1.11}$$

where,

$$\hat{\gamma}_{t,s}(\xi) = (2\pi)^{-n/2} e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau}. \tag{1.12}$$

Studying the density $\pi_{t,0}$ of $P_{X_t - X_0}$ in comparison with the density p_t of $(Y_t)_{t\geq 0}$ we obtain (under some additional assumptions, see Theorem 5.1) our main result:

$$p_t(x-y) \asymp \lambda^{(n)}\big(B^{d_{\psi,t}}(0,1)\big) e^{-\delta^2_{\psi,t}(x,y)}, \tag{1.13}$$

and,

$$\pi_{t,0}(x-y) \asymp \lambda^{(n)}\big(B^{\delta_{\psi,\frac{1}{t}}}(0,1)\big) e^{-d^2_{\psi,\frac{1}{t}}(x,y)}, \tag{1.14}$$

with $d_{\psi,t}(x,y)\sqrt{t\psi(x-y)}$ and $\delta_{\psi,t}(x,y) = \Big(-\ln \frac{p_t(x-y)}{p_t(0)}\Big)^{1/2}$.

We refer to [2], but also [14], where attempts were made to extend the results to processes generated by pseudo-differential operators with symbol $q(t,x,\xi)$ and the x-dependence is subjected to oscillation conditions with respect to a reference function as in [12].

In general, our notations are the ones used in [13].

2. Families of Measures Associated with Convolution Operators

Let $(\mu_t)_{t\geq 0}$ be a symmetric convolution semigroup of probability measures, i.e., each μ_t is a probability measure on $\mathbb{R}^n$, $\mu_0 = \epsilon_0$, $\mu_s * \mu_t = \mu_{s+t}$, and $\mu_t \to \epsilon_0$ vaguely, hence weakly, for $t \to 0$, with Fourier transform,

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2} e^{-t\psi(\xi)}, \tag{2.1}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function. Since all measures μ_t are assumed to be probability measures it follows that $\psi(0) = 0$. We add the following assumptions on ψ:

Ai) $\psi(\xi) = 0$ if and only if $\xi = 0$;
Aii) $\liminf_{|\xi|\to\infty} \psi(\xi) > 0$;
Aiii) $e^{-t\psi}, \psi e^{-t\psi} \in \mathcal{L}^1(\mathbb{R}^n)$ for all $t > 0$.

Note that if f is a Bernstein function growing as a power at infinity, e.g. $f(s) = s^\alpha, 0 < \alpha < 1$, then $f(|\xi|^2)$ as well as $f(|\xi|^{\alpha_1} + |\xi_2|^{\alpha_2})$, $\alpha_j \in (0,2]$, will satisfy these assumptions. More examples are given in [14]. From condition Ai) we deduce, compare Lemma 3.6.21 in [13, Vol I.], that by,

$$d_\psi(\xi,\eta) := \psi^{1/2}(\xi - \eta), \tag{2.2}$$

a metric is given on $\mathbb{R}^n$ and Aii) assures, see Lemma 3.2 in [14], that the metric d_ψ generates on $\mathbb{R}^n$ the Euclidean topology. Furthermore, by Aiii) the measures μ_t have a density $p_t \in C_\infty(\mathbb{R}^n) \cap \mathcal{L}^1(\mathbb{R}^n)$ with respect to the Lebesgue measure $\lambda^{(n)}$ given by,

$$p_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)}\,\mathrm{d}\xi, \tag{2.3}$$

and of course we have,

$$\int_{\mathbb{R}^n} p_t(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} 1\,\mathrm{d}\mu_t = 1. \tag{2.4}$$

Furthermore we find that

$$\frac{\partial p_t}{\partial t}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi}\big(-\psi(\xi)\big)e^{-t\psi(\xi)}\,\mathrm{d}\xi,$$

exists. Note that in the case that ψ has at least power growth for $|\xi| \to \infty$ the condition $\psi e^{-t\psi} \in L^1(\mathbb{R}^n)$ is trivial. Thus with ψ (or the convolution semigroup $(\mu_t)_{t\geq 0}$ or the corresponding canonical Lévy process $(X_t)_{t\geq 0}$) we can associate a metric measure space $\big(\mathbb{R}^n, d_\psi, \lambda^{(n)}\big)$, see [14]. In the next section we will employ this metric measure space to study p_t. Observe that,

$$p_t(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t\psi(\xi)}\,\mathrm{d}\xi, \tag{2.5}$$

and therefore it follows that,

$$\rho_t := \frac{e^{-t\psi(\cdot)}}{(2\pi)^n p_t(0)}\,\lambda^{(n)}, \tag{2.6}$$

is for $t > 0$ a symmetric probability measure on $\mathbb{R}^n$ with Fourier transform,

$$\hat{\rho}_t(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} \frac{e^{-t\psi(\xi)}}{(2\pi)^n p_t(0)}\,\mathrm{d}\xi = (2\pi)^{-\frac{n}{2}} \frac{p_t(y)}{p_t(0)}. \tag{2.7}$$

We now introduce the family of measures $(\nu_t)_{t>0}$ by,

$$\nu_t := \rho_{\frac{1}{t}} = \frac{e^{-\frac{1}{t}\psi(\cdot)}}{(2\pi)^n p_{\frac{1}{t}}(0)}\,\lambda^{(n)}. \tag{2.8}$$

First we note that,

$$\begin{aligned}\nu_t(\mathbb{R}^n) &= \int_{\mathbb{R}^n} 1\,\nu_t(\mathrm{d}\xi) = \int_{\mathbb{R}^n} \frac{e^{-\frac{1}{t}(\xi)}}{(2\pi)^n p_{\frac{1}{t}}(0)}\,\lambda^{(n)}(\mathrm{d}\xi)\\ &= \frac{(2\pi)^{-n}\int_{\mathbb{R}^n} e^{-\frac{1}{t}\psi(\xi)}\,\lambda^{(n)}(\mathrm{d}\xi)}{p_{\frac{1}{t}}(0)} = \frac{p_{\frac{1}{t}}(0)}{p_{\frac{1}{t}}(0)} = 1,\end{aligned}$$

i.e., ν_t is a probability measure on $\mathbb{R}^n$. Further we have,

Proposition 2.1. *For $t \to 0$ the family of measures $(\nu_t)_{t>0}$ converges weakly to the Dirac measure ε_0, i.e.,*

$$\lim_{t\to 0} \nu_t = \varepsilon_0 \quad \textit{(weak limit).} \tag{2.9}$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. By Plancherel's theorem we find,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi(x-y)\,\nu_t(\mathrm{d}y) \\
&= \int_{\mathbb{R}^n} \frac{e^{-\frac{1}{t}\psi(y)}}{(2\pi)^n p_{\frac{1}{t}}(0)} \varphi(x-y)\,\mathrm{d}y \\
&= \frac{1}{(2\pi)^{n/2} p_{\frac{1}{t}}(0)} \int_{\mathbb{R}^n} \mathrm{F}^{-1}\big((2\pi)^{-n/2} e^{-\frac{1}{t}\psi(\cdot)}\big)(\xi)\overline{\mathrm{F}^{-1}\big(\varphi(x-\cdot)\big)(\xi)}\,\mathrm{d}\xi,
\end{aligned}$$

and since $\overline{\mathrm{F}^{-1}\big(\varphi(x-\cdot)\big)(\xi)} = e^{-ix\cdot\xi}\mathrm{F}^{-1}\varphi(\xi)$ we obtain,

$$\int_{\mathbb{R}^n} \varphi(x-y)\,\nu_t(\mathrm{d}y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)} e^{-ix\cdot\xi}\big(\mathrm{F}^{-1}\varphi\big)(\xi)\,\mathrm{d}\xi. \tag{2.10}$$

By the ratio limit theorem, Theorem 5.7 in [15], it holds for the transition density π_t of a Lévy process on $\mathbb{R}^n$ that $\lim_{t\to\infty} \frac{\pi_t(x)}{\pi_t(0)} = 1$ for all $x \in \mathbb{R}^n$. Passing in (2.10) to the limit $t \to 0$ we get,

$$\begin{aligned}
\lim_{t\to 0} \int_{\mathbb{R}^n} \varphi(x-y)\,\nu_t(\mathrm{d}y) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\left(\lim_{t\to 0} \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)}\right)\big(\mathrm{F}^{-1}\varphi\big)(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\left(\lim_{t\to \infty} \frac{p_t(\xi)}{p_t(0)}\right)\big(\mathrm{F}^{-1}\varphi\big)(\xi)\,\mathrm{d}\xi \\
&= \varphi(x).
\end{aligned}$$

The density of $C_0^\infty(\mathbb{R}^n)$ in $C_\infty(\mathbb{R}^n)$ implies $\lim_{t\to 0} \nu_t = \varepsilon_0$ vaguely and since $\lim_{t\to 0} \nu_t(\mathbb{R}^n) = 1$ it follows that (2.9) holds. □

Hence the family $(\nu_t)_{t>0}$ is a family of probability measures converging weakly to ε_0. An open question is when they form a projective family or when we can associate with $(\nu_t)_{t>0}$ a stochastic process. To investigate the situation further we want to switch from $(\nu_t)_{t>0}$ to the corresponding family of operators $(S_t)_{t>0}$ defined on $C_b(\mathbb{R}^n)$ by,

$$S_t u(x) := \big(u * \nu_t\big)(x) = \int_{\mathbb{R}^n} u(x-y)\,\nu_t(\mathrm{d}y). \tag{2.11}$$

For $u \in \mathcal{S}(\mathbb{R}^n)$ we find by the convolution theorem when noting that $S_t u = F^{-1}(F(u * \nu_t))$ that,

$$\begin{aligned}
S_t u(x) &= \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\nu}_t(\xi)\hat{u}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)} \hat{u}(\xi)\,\mathrm{d}\xi,
\end{aligned}$$

or with,

$$\sigma_t(\xi) := \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)}, \tag{2.12}$$

we have on $\mathcal{S}(\mathbb{R}^n)$,

$$S_t u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma_t(\xi) \hat{u}(\xi)\, \mathrm{d}\xi, \tag{2.13}$$

i.e., S_t is a pseudo-differential operator with symbol $\sigma_t(\xi)$. Of course $(S_t)_{t>0}$ is in general not a semigroup of linear operators. However from Proposition 2.1 and (2.11) we deduce that $(S_t)_{t>0}$ is a family of contractions on $C_\infty(\mathbb{R}^n)$, i.e.,

$$\|S_t u\|_\infty \leq \|u\|_\infty, \tag{2.14}$$

which is strongly continuous, i.e.,

$$\lim_{t\to 0} \|S_t u - u\|_\infty = 0, \tag{2.15}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ and hence for $u \in C_\infty(\mathbb{R}^n)$. Furthermore,

$$u \geq 0 \text{ implies } S_t u \geq 0, \tag{2.16}$$

i.e., $(S_t)_{t>0}$ is on $C_\infty(\mathbb{R}^n)$ a strongly continuous family of positivity preserving contractions. Moreover, using the density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{L}^2(\mathbb{R}^n)$, Plancherel's theorem and $0 \leq \sigma_t(\xi) \leq 1$ we deduce for $u \in \mathcal{L}^2(\mathbb{R}^n)$,

$$\|S_t u\|_0 \leq \|u\|_0, \tag{2.17}$$

$$\lim_{t\to 0} \|S_t u - u\|_0 = 0, \tag{2.18}$$

and,

$$0 \leq u \leq 1 \text{ a.e. implies } 0 \leq S_t u \leq 1 \text{ a.e.}, \tag{2.19}$$

which means that $(S_t)_{t>0}$ is a strongly continuous family of sub-Markovian contraction on $\mathcal{L}^2(\mathbb{R}^n)$.

Proposition 2.2. *Let*

$$q(t,\xi) = -\frac{\partial}{\partial t} \ln \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)}. \tag{2.20}$$

Then it follows for $u \in \mathcal{S}(\mathbb{R}^n)$ that,

$$\frac{\partial}{\partial t} S_t u(x) + q(t,D) S_t u(x) = 0, \tag{2.21}$$

and,

$$\lim_{t\to 0} S_t u = u, \tag{2.22}$$

where $q(t,D)$ is the pseudo-differential operator with the time-dependent symbol $q(t,\xi)$, and the limit in (2.22) *can be taken in $C_\infty(\mathbb{R}^n)$, hence also pointwise for $u \in C_\infty(\mathbb{R}^n)$, or in $\mathcal{L}^2(\mathbb{R}^n)$.*

Proof. It remains to prove (2.21) which follows for $u \in \mathcal{S}(\mathbb{R}^n)$ by differentiating (2.13). Note that by (2.3) $t \mapsto p_t(\xi)$ is differentiable for $t > 0$ and every $\xi \in \mathbb{R}^n$. □

Here we encounter a further open problem: As the Fourier transform of the measure σ_t is for every $t > 0$ a continuous positive definite function, we are searching for conditions implying that $q(t,\cdot)$ is a continuous negative definite function. Note that formally we expect $\sigma_t(\xi) = ce^{-\int_0^t q(\tau,\xi)\,d\tau}$ to hold, hence for $q(\tau,\cdot)$ negative definite we would obtain $\sigma_t(\cdot)$ positive definite and $\int_0^t q(\tau,\cdot)\,d\tau$ would be a type of characteristic exponent.

Example 2.3.

A. For Brownian motion in $\mathbb{R}^n$ we have $\psi_B(\xi) = \frac{1}{2}|\xi|^2$ with

$$p_t^B(x) = (2\pi t)^{-n/2} e^{-\frac{|x|^2}{2t}}$$

which yields $q_B(t,\xi) = \frac{1}{2}|\xi|^2$.

B. For the Cauchy process in $\mathbb{R}^n$ we have $\psi_C(\xi) = |\xi|$ with

$$p_t^C(x) = \pi^{-\frac{n+1}{2}} \Gamma(\frac{n+1}{2}) \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}$$

which yields,

$$\begin{aligned} q_C(t,\xi) &= -\frac{\partial}{\partial t} \ln \frac{p_{\frac{1}{t}}^C(\xi)}{p_{\frac{1}{t}}^C(0)} = \frac{\partial}{\partial t} \ln\left(1 + t^2|\xi|^2\right)^{\frac{n+1}{2}} \\ &= \frac{n+1}{t} \frac{|\xi|^2}{|\xi|^2 + \frac{1}{t^2}} = \frac{n+1}{t} f_{\frac{1}{t^2}}\left(|\xi|^2\right), \end{aligned}$$

where $f_r(s) = \frac{s}{s+r}$ is a Bernstein function, hence $q_C(t,\cdot)$ is a continuous negative definite function.

Example 2.4. (See [2]) The symmetric Meixner process on $\mathbb{R}$ has the symbol $\psi_M(\xi) = \ln\cosh\xi$ and the transition density $p_t^M(x) = \frac{2^{t-1}}{\pi\Gamma(t)}\left|\Gamma\left(\frac{t+ix}{2}\right)\right|^2$ and we find, see [14],

$$\frac{p_t^M(\xi)}{p_t^M(0)} = \left|\frac{\Gamma\left(\frac{t+ix}{2}\right)}{\Gamma\left(\frac{t}{2}\right)}\right|^2 = \prod_{j=0}^{\infty}\left(1 + \frac{\xi^2}{(\frac{1}{t}+2j)^2}\right),$$

which implies,

$$-\ln \frac{p_{\frac{1}{t}}^M(\xi)}{p_{\frac{1}{t}}^M(0)} = \sum_{j=0}^{\infty} \ln\left(1 + \frac{\xi^2}{(\frac{1}{t}+2j)^2}\right),$$

and eventually,

$$q_M(t,\xi) = -\frac{\partial}{\partial t} \ln \frac{p_{\frac{1}{t}}^M(\xi)}{p_{\frac{1}{t}}^M(0)} = \sum_{j=0}^{\infty} \frac{2}{t^2(\frac{1}{t}+2j)} \frac{\xi^2}{(\frac{1}{t}+2j)^2 + \xi^2}.$$

This series converges for $t > 0$ locally uniformly with respect to ξ and since $\frac{\xi^2}{(\frac{1}{t}+2j)^2+\xi^2} = f_{(\frac{1}{t}+2j)^2}(\xi^2)$ with the Bernstein function as in Example 2.3.B, we conclude that $q_M(t,\cdot)$ is a continuous negative definite function.

Example 2.5. For $\psi_H(\xi) = (m^2+\xi^2)^{1/2} - m$, $m > 0$, it is known, see [10], that $p_t^H(\xi) = \frac{mte^{mt}}{\pi}\frac{K_1\left(m\sqrt{t^2+\xi^2}\right)}{\sqrt{t^2+\xi^2}}$ where K_1 is the modified Bessel function of the second kind with index 1. It is an open problem whether $q_H(t,\cdot)$ is a continuous negative definite function.

In order to clarify the situation further, we need to introduce additive processes.

3. Additive Processes and Fundamental Solutions

Let $q : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $q(t,0) = 0$ and $q(t,\cdot) : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function. For $0 < s < t$ it follows that $\xi \mapsto \int_s^t q(\tau,\xi)\,\mathrm{d}\tau$ is again a continuous negative definite function. The continuity follows directly from our assumptions and since the the pointwise limit of negative definite functions is again negative definite, approximating the integral by a sequence of Riemann sums will yield the negative definiteness. Consequently the function $\xi \mapsto e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau}$ is a continuous positive definite function and hence for $0 < s < t$ we can define a family of bounded measures $(\gamma_{t,s})_{t>s>0}$ by,

$$\hat{\gamma}_{t,s}(\xi) = (2\pi)^{-n/2} e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau}. \tag{3.1}$$

From $q(t,0) = 0$ we deduce that $\gamma_{t,s}$ is a probability measure. Moreover, using results for the Fourier transform of measures, we find that,

Mi) $\gamma_{s,s} = \varepsilon_0$ for $0 \le s$;
Mii) $\gamma_{t,r} * \gamma_{r,s} = \gamma_{t,s}$ for $0 < s \le r \le t < \infty$;
Miii) $\gamma_{t,s} \to \varepsilon_0$ weakly for $s \to t$, $s < t$;
Miv) $\gamma_{t,s} \to \varepsilon_0$ weakly for $t \to s$, $s < t$.

According to K. Sato [17, Theorem 9.7], we can associate with $(\gamma_{t,s})_{0<s<t<\infty}$ a canonical additive process in law $(X_t)_{t\ge o}$ with state space $\mathbb{R}^n$, i.e., $P_{X_t - X_s} = \gamma_{t,s}$, $t > s$.

Theorem 3.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function satisfying Ai) - Aiii). Denote by p_t the density of μ_t where $(\mu_t)_{t\ge0}$ is the convolution semigroup associated with ψ. If $q(t,\xi) := -\frac{\partial}{\partial t}\ln\left(p_{\frac{1}{t}}(\xi)/p_{\frac{1}{t}}(0)\right)$ is with respect to ξ a continuous negative definite function, then we can associate with q a canonical additive process in law $(X_t)_{t\ge0}$ by the relation $P_{X_t-X_s} = \gamma_{t,s}$ where $t > s > 0$ and $\gamma_{t,s}$ is defined by* (3.1).

With the help of the probability measures $\gamma_{t,s}$, $0 < s < t < \infty$, we can define on $C_\infty(\mathbb{R}^n)$ or $\mathcal{L}^2(\mathbb{R}^n)$ the operators,

$$V(t,s)u(x) = \int_{\mathbb{R}^n} u(x-y)\,\gamma_{t,s}(\mathrm{d}y), \tag{3.2}$$

and from Mi) - Miv) we deduce (for either u in $C_\infty(\mathbb{R}^n)$ or in $\mathcal{L}^2(\mathbb{R}^n)$ and convergence is meant accordingly),

- $V(s,s)u = u$;
- $\big(V(t,r)\circ V(r,s)\big)u = V(t,s)u = V(t,s)u$, $r < s < t$;
- $V(t,s)u \to u$ as $s \to t$, $s < t$;
- $V(t,s)u \to u$ as $t \to s$, $s < t$.

Since $\gamma_{t,s}$ can be viewed as an element in $\mathcal{S}'(\mathbb{R}^n)$ the convolution theorem yields for $u \in \mathcal{S}(\mathbb{R}^n)$ that,

$$\big(V(t,s)u\big)^\wedge(\xi) = e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau}\hat{u}(\xi), \tag{3.3}$$

which gives,

$$\frac{\partial}{\partial t}\big(V(t,s)u\big)^\wedge(\xi) = -q(t,\xi)e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau}\hat{u}(\xi),$$

and,

$$\frac{\partial}{\partial s}\big(V(t,s)u\big)^\wedge(\xi) = q(s,\xi)e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau}\hat{u}(\xi).$$

Therefore we deduce (at least as equations in $\mathcal{S}'(\mathbb{R}^n)$, given $u \in \mathcal{S}(\mathbb{R}^n)$),

$$\frac{\partial}{\partial t}V(t,s)u + q(t,D)V(t,s)u = 0, \tag{3.4}$$

and,

$$\frac{\partial}{\partial s}V(t,s)u - q(s,D)V(t,s)u = 0. \tag{3.5}$$

Depending on properties of $q(t,\cdot)$ we can identify $V(t,s)$, $t > s$, as a fundamental solution in the form of [22] for the initial value problem,

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - A(t)u(t,x) = f(t,x), \\ u(0,x) = u_0(x), \end{cases}$$

in $\mathcal{L}^2\big([0,T];\mathcal{L}^2(\mathbb{R}^n)\big)$ or $C_b\big([0,T];C_\infty(\mathbb{R}^n)\big)$, we refer to [2] or [23] for more details. For the purposes of this note we do not need the details.

Since $q(t,\xi) = -\frac{\partial}{\partial t}\ln\big(p_{\frac{1}{t}}(\xi)/p_{\frac{1}{t}}(0)\big)$ we observe that,

$$\int_s^t q(\tau,\xi)\,\mathrm{d}\tau = -\ln\frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)} + \ln\frac{p_{\frac{1}{s}}(\xi)}{p_{\frac{1}{s}}(0)},$$

or,

$$e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau} = \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)}\frac{p_{\frac{1}{s}}(0)}{p_{\frac{1}{s}}(\xi)},$$

i.e., using the definition of S_t we arrive at,

$$V(t,0) = S_t, \tag{3.6}$$

a relation which even holds when $\xi \mapsto q(t,\xi)$ is not a continuous negative definite function but just given by (2.20). In addition we see that with (2.12),

$$P_{X_t - X_0} = \gamma_{t,0}, \quad \hat{\gamma}_{t,0} = (2\pi)^{-n/2}\sigma_t. \tag{3.7}$$

Assuming that the additive process $(X_t)_{t>s>0}$ associated with $q(t,\xi)$ given by (2.20) exists and denoting the Lévy process associated with ψ by $(Y_t)_{t\,0}$ we find,

$$P_{Y_t - Y_0} = \mu_t = \mathrm{F}^{-1}\big(e^{-t\psi}\big)(\cdot)\,\lambda^{(n)} = p_t(\cdot)\,\lambda^{(n)}, \tag{3.8}$$

and,

$$P_{X_t - X_0} = \gamma_{t,0} = \mathrm{F}^{-1}\left(\frac{p_{\frac{1}{t}}}{p_{\frac{1}{t}}(0)}\right)(\cdot)\,\lambda^{(n)} = \frac{e^{-\frac{1}{t}\psi(\cdot)}}{(2\pi)^n p_{\frac{1}{t}}(0)}\,\lambda^{(n)}. \tag{3.9}$$

In the next section we will use (3.8) and (3.9) to obtain a geometric interpretation of p_t. For later purposes we define,

$$\pi_{t,0}(x) := \frac{e^{-\frac{1}{t}\psi(x)}}{(2\pi)^n p_{\frac{1}{t}}(0)}. \tag{3.10}$$

4. Transition Functions and Geometry I. The Diagonal Term

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function satisfying Ai) - Aiii) with associated metric $d_\psi(\xi,\eta) := \sqrt{\psi(\xi-\eta)}$, and denote the corresponding metric measure space by $\left(\mathbb{R}^n, d_\psi, \lambda^{(n)}\right)$. In this case p_t, the transition density as defined by (2.3), belongs to $\mathcal{L}^1(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n)$ and,

$$p_t(x) \le p_t(0). \tag{4.1}$$

For later purposes it is helpful to note that $p_t(x) < p_t(0)$ for all $x \in \mathbb{R}^n$. Indeed,

$$\begin{aligned} p_t(0) - p_t(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} (e^{i0\cdot\xi} - e^{ix\cdot\xi}) e^{-t\psi(\xi)}\,\mathrm{d}\xi \\ &= 2(2\pi)^{-n} \int_{\mathbb{R}^n} (1 - \cos x\cdot\xi) e^{-t\psi(\xi)}\,\mathrm{d}\xi, \end{aligned}$$

and the function $\xi \mapsto (1-\cos x\cdot\xi)e^{-t\psi(\xi)}$ is for every $x \in \mathbb{R}^n$ non-negative and continuous. Hence $p_t(0) = p_t(x_0)$ for some $x_0 \in \mathbb{R}^n \setminus \{0\}$ is impossible. From (2.5) we obtain immediately, see [14] or [15], that,

$$p_t(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t\psi(\xi)}\,\mathrm{d}\xi = (2\pi)^{-n} \int_0^\infty \lambda^{(n)}\left(B^{d_\psi}(0, \sqrt{r/t})\right) e^{-r}\,\mathrm{d}r, \tag{4.2}$$

where $B^{d_\psi}(x_0, r) = \{x \in \mathbb{R}^n : d_\psi(x, x_0) < r\}$. We assume further that the metric measure space $\left(\mathbb{R}^n, d_\psi, \lambda^{(n)}\right)$ has the doubling property, i.e., for all $x \in \mathbb{R}^n$ and all $r > 0$ it holds,

$$\lambda^{(n)}\left(B^{d_\psi}(x, 2r)\right) \le c\lambda^{(n)}\left(B^{d_\psi}(x, r)\right). \tag{4.3}$$

In this case, as shown in [14, Theorem 4.1], it follows that,

$$p_t(0) \asymp \lambda^{(n)}\left(B^{d_\psi}(0, 1/\sqrt{t})\right), \tag{4.4}$$

recall that $a_t \asymp b_t$ means that $\gamma_0 a_t \le b_t \le \gamma_1 a_t$ holds with constants $0 < \gamma_0 \le \gamma_1$ independent of t. Switching to the t-dependent metric $d_{\psi,t}(\xi,\eta) = \sqrt{t\psi(\xi-\eta)}$ we can re-write (4.4) as,

$$p_t(0) \asymp \lambda^{(n)}\left(B^{d_{\psi,t}}(0, 1)\right). \tag{4.5}$$

Hence $p_t(0)$, the "diagonal term" of the (translation invariant) transition function $\tilde{p}_t(x,y) = p_t(x-y)$ is controlled by the volume of the unit ball in the (volume doubling) metric measure space $\left(\mathbb{R}^n, d_{\psi,t}, \lambda^{(n)}\right)$.

Our first observation is that such a result carries over to additive processes. For ease of notation we set,

$$Q_{t,s}(\xi) := \int_s^t q(\tau,\xi)\,\mathrm{d}\tau = -\ln \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)} + \ln \frac{p_{\frac{1}{s}}(\xi)}{p_{\frac{1}{s}}(0)}, \quad t > s > 0, \tag{4.6}$$

where q is as in Theorem 3.1. Since $Q_{t,s}(\cdot)$ is a real-valued continuous negative definite function and $Q_{t,s}(\xi) = 0$ implies $q(\tau,\xi) = 0$ for all $\tau \in [s,t]$, note that $q(\tau,\xi) \geq 0$ and by assumption $\tau \mapsto q(\tau,\xi)$ is continuous, we conclude that $Q_{t,s}(\xi) = 0$ if and only if $\xi = 0$, hence by,

$$d_{Q_{t,s}}(\xi,\eta) := Q_{t,s}^{1/2}(\xi-\eta), \tag{4.7}$$

a metric is defined on $\mathbb{R}^n$. Further, we know that if,

$$d_{q(\tau,\cdot)}(\xi,\eta) = q^{1/2}(\tau,\xi-\eta), \tag{4.8}$$

generates the Euclidean topology, then $\liminf_{|\xi|\to\infty} q(\tau,\xi) > 0$, see [14, Lemma 3.2]. Now Fatou's lemma yields,

$$\liminf_{|\xi|\to\infty} Q_{t,s}(\xi) \geq \int_s^t \Big(\liminf_{|\xi|\to\infty} q(\tau,\xi)\Big)\,\mathrm{d}\tau > 0, \tag{4.9}$$

i.e., $d_{Q_{t,s}}$ generates the Euclidean topology too.

In the case that $e^{-\int_s^t q(\tau,\cdot)\,\mathrm{d}\tau} \in \mathcal{L}^1(\mathbb{R}^n)$ we denote the density of the measure $\gamma_{t,s}$ by $\pi_{t,s}$, i.e.,

$$\pi_{t,s}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau}\,\mathrm{d}\xi. \tag{4.10}$$

Theorem 4.1. *Assume that for every $\tau > 0$ the metric* (4.8) *generates the Euclidean topology and that $e^{-\int_s^t q(\tau,\cdot)\,\mathrm{d}\tau} \in \mathcal{L}^1(\mathbb{R}^n)$. Then it holds,*

$$\pi_{t,s}(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \lambda^{(n)}\big(B^{d_{Q_{t,s}}}(0,\sqrt{r})\big)e^{-r}\,\mathrm{d}r. \tag{4.11}$$

If we have in addition,

$$\beta_0 q(t_0,\xi) \leq q(t,\xi) \leq \beta_1 q(t_0,\xi), \tag{4.12}$$

for some $t_0 > 0$, all $t > 0$ and $\xi \in \mathbb{R}^n$ with constants $0 < \beta_0 \leq \beta_1$, and if the metric measure space $\big(\mathbb{R}^n, d_{q(t_0,\cdot)}, \lambda^{(n)}\big)$ has the volume doubling property then we get,

$$\pi_{t,s}(0) \asymp \lambda^{(n)}\Big(B^{d_{Q_{t,s}}}\big(0,\sqrt{\beta_1/\beta_0}\big)\Big). \tag{4.13}$$

Remark 4.2. Note that (4.12) implies $e^{-\int_s^t q(\tau,\cdot)\,\mathrm{d}\tau} \in \mathcal{L}^1(\mathbb{R}^n)$.

Further note that we can always use the examples from [14] to construct examples for Theorem 4.1 provided we introduce a t-dependence respecting (4.12).

Proof. Since,

$$\begin{aligned}(2\pi)^n \pi_{t,s}(0) &= \int_{\mathbb{R}^n} e^{-Q_{t,s}(\xi)}\,\mathrm{d}\xi \\ &= \int_0^\infty \lambda^{(n)}\big(\{\xi \in \mathbb{R}^n : e^{-Q_{t,s}(\xi)} \geq \rho\}\big)\,\mathrm{d}\rho \\ &= \int_0^1 \lambda^{(n)}\big(\{\xi \in \mathbb{R}^n : Q_{t,s}(\xi) \leq -\ln\rho\}\big)\,\mathrm{d}\rho,\end{aligned}$$

we get,

$$(2\pi)^n \pi_{t,s}(0) = -\int_\infty^0 \lambda^{(n)}\big(\{\xi \in \mathbb{R}^n : Q_{t,s}(\xi) \le r\}\big)e^{-r}\,\mathrm{d}r$$
$$= \int_0^\infty \lambda^{(n)}\big(\{\xi \in \mathbb{R}^n : Q_{t,s}(\xi) \le r\}\big)e^{-r}\,\mathrm{d}r,$$

and (4.11) is proved. Next, since $\big(\mathbb{R}^n, d_{q(t_0,\cdot)}, \lambda^{(n)}\big)$ has the volume doubling property, by [14, Corollary 3.10] we get $e^{-uq(t_0,\cdot)} \in \mathcal{L}^1(\mathbb{R}^n)$ for all $u > 0$, hence $e^{-\beta_0(t-s)q(t_0,\cdot)} \in \mathcal{L}^1(\mathbb{R}^n)$ for all $t > s \ge 0$. Now, for all $\xi \in \mathbb{R}^n$ we have,

$$\beta_0(t-s)q(t_0,\xi) \le \int_s^t q(\tau,\xi)\,\mathrm{d}\tau,$$

or,

$$e^{-\beta_0(t-s)q(t_0,\xi)} \ge e^{-\int_s^t q(\tau,\xi)\,\mathrm{d}\tau},$$

i.e., $e^{-\int_s^t q(\tau,\cdot)\,\mathrm{d}\tau} \in \mathcal{L}^1(\mathbb{R}^n)$ for all $t > s \ge 0$. Using the monotonicity of $r \mapsto \lambda^{(n)}\big(B^{d_{Q_{t,s}}}(0,\sqrt{r})\big)$ we find,

$$(2\pi)^n \pi_{t,s}(0) \ge \int_{\beta_1/\beta_0}^\infty \lambda^{(n)}\big(B^{d_{Q_{t,s}}}(0,\sqrt{r})\big)e^{-r}\,\mathrm{d}r$$
$$\ge \lambda^{(n)}\Big(B^{d_{Q_{t,s}}}\big(0,\sqrt{\beta_1/\beta_0}\big)\Big)\int_{\beta_1/\beta_0}^\infty e^{-r}\,\mathrm{d}r$$
$$= \frac{1}{e^{\beta_1/\beta_0}}\lambda^{(n)}\Big(B^{d_{Q_{t,s}}}\big(0,\sqrt{\beta_1/\beta_0}\big)\Big).$$

For the upper estimate we split the integral according to,

$$(2\pi)^n \pi_{t,s}(0) = \int_0^{\beta_1/\beta_0} \lambda^{(n)}\big(B^{d_{Q_{t,s}}}(0,\sqrt{r})\big)e^{-r}\,\mathrm{d}r$$
$$+ \int_{\beta_1/\beta_0}^\infty \lambda^{(n)}\big(B^{d_{Q_{t,s}}}(0,\sqrt{r})\big)e^{-r}\,\mathrm{d}r,$$

and note that,

$$\int_0^{\beta_1/\beta_0} \lambda^{(n)}\big(B^{d_{Q_{t,s}}}(0,\sqrt{r})\big)e^{-r}\,\mathrm{d}r \le \lambda^{(n)}\Big(B^{d_{Q_{t,s}}}\big(0,\sqrt{\beta_1/\beta_0}\big)\Big)\int_0^{\beta_1/\beta_0} e^{-r}\,\mathrm{d}r$$
$$= \left(1 - \frac{1}{e^{\beta_1/\beta_0}}\right)\lambda^{(n)}\Big(B^{d_{Q_{t,s}}}\big(0,\sqrt{\beta_1/\beta_0}\big)\Big).$$

On the other hand, by (4.12) we get,

$$\int_{\beta_1/\beta_0}^\infty \lambda^{(n)}\big(B^{d_{Q_{t,s}}}(0,\sqrt{r})\big)e^{-r}\,\mathrm{d}r$$
$$\le \int_{\beta_1/\beta_0}^\infty \lambda^{(n)}\left(\left\{\xi \in \mathbb{R}^n : \left(\int_s^t \beta_0 q(t_0,\xi)\,\mathrm{d}\tau\right)^{1/2} \le \sqrt{r}\right\}\right)e^{-r}\,\mathrm{d}r$$
$$\le \int_{\beta_1/\beta_0}^\infty \lambda^{(n)}\big(\{\xi \in \mathbb{R}^n : \sqrt{\beta_0(t-s)q(t_0,\xi)} \le \sqrt{r}\}\big)e^{-r}\,\mathrm{d}r.$$

By the volume doubling property it follows for $k \ge 1$ that,

$$\lambda^{(n)}\big(B^{d_{q(t_0,\cdot)}}(0,r)\big) \le c(t_0,r)\lambda^{(n)}\big(B^{d_{q(t_0,\cdot)}}(0,1)\big),$$

where $c(t_0, r) \leq r^{\alpha(t_0)} c_0(t_0, 1)$ for all $k \geq 1$ and some $\alpha(t_0) \geq 0$. This implies,

$$\begin{aligned}
&\int_{\beta_1/\beta_0}^{\infty} \lambda^{(n)}\big(B^{d_{Q_{t,s}}}(0, \sqrt{r})\big) e^{-r}\, \mathrm{d}r \\
&\leq \int_{\beta_1/\beta_0}^{\infty} c(t_0, r) \lambda^{(n)}\big(\big\{\xi \in \mathbb{R}^n : \sqrt{\beta_0 (t-s) q(t_0, \xi)} \leq 1\big\}\big) e^{-r}\, \mathrm{d}r \\
&\leq\ c(t_0, 1) \int_{\beta_1/\beta_0}^{\infty} \lambda^{(n)}\big(\big\{\xi \in \mathbb{R}^n : \sqrt{(t-s) q(t_0, \xi)} \leq \sqrt{1/\beta_0}\big\}\big) r^{\alpha(t_0)/2} e^{-r}\, \mathrm{d}r \\
&\leq\ c(t_0, 1) \int_{\beta_1/\beta_0}^{\infty} \lambda^{(n)}\left(\left\{\xi \in \mathbb{R}^n : \left(\int_s^t q(t, \xi)\, \mathrm{d}\tau\right)^{1/2} \leq \sqrt{\beta_1/\beta_0}\right\}\right) r^{\alpha(t_0)/2} e^{-r}\, \mathrm{d}r \\
&=\ d(t_0) \lambda^{(n)}\Big(B^{d_{Q_{t,s}}}\big(0, \sqrt{\beta_1/\beta_0}\big)\Big),
\end{aligned}$$

where $d(t_0) = c_0(t_0, 1) \int_{\beta_1/\beta_0}^{\infty} r^{\alpha(t_0)/2} e^{-r}\, \mathrm{d}r < \infty$, and finally we obtain,

$$(2\pi)^n \pi_{t,s}(0) \leq \left(1 - \frac{1}{e^{\beta_1/\beta_0}} + d(t_0)\right) \lambda^{(n)}\Big(B^{d_{Q_{t,s}}}\big(0, \sqrt{\beta_1/\beta_0}\big)\Big).$$

□

5. Transition Functions and Geometry II. The Off-Diagonal Term

Suppose that we are given a symmetric Lévy process $(Y_t)_{t \geq 0}$ associated with a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfying Ai)- Aiii). We denote by d_ψ or $d_{\psi,t}$ the corresponding metrics $d_\psi(\xi, \eta) = \psi^{1/2}(\xi - \eta)$ or $d_{\psi,t}(\xi, \eta) = \sqrt{t\psi(\xi - \eta)}$. We can now write the corresponding transition density as,

$$\tilde{p}_t(x, y) := p_t(x - y) = p_t(0) \frac{p_t(x - y)}{p_t(0)}, \tag{5.1}$$

and further as,

$$\tilde{p}_t(x, y) = p_t(0) e^{\ln \frac{p_t(x-y)}{p_t(0)}} = p_t(0) e^{-\left(-\ln \sigma_{\frac{1}{t}}(x-y)\right)}, \tag{5.2}$$

where we used (2.12). We need the following observation: Let $q(t, \xi)$ be defined as in (2.20) and σ_t as in (2.12). If the function $\xi \mapsto q(\tau, \xi)$ is for all $\tau > 0$ a continuous negative definite function then the function $\xi \mapsto -\ln \sigma_\tau(\xi)$ is also for all $\tau > 0$ a continuous negative definite function. We have seen that $\xi \mapsto \int_s^t q(\tau, \xi)\, \mathrm{d}\tau$ is a continuous negative definite function if $\xi \mapsto q(\tau, \xi)$ is. Since,

$$-\int_s^t \frac{\partial}{\partial \tau} \ln \frac{p_{\frac{1}{\tau}}(\xi)}{p_{\frac{1}{\tau}}(0)}\, \mathrm{d}\tau = -\ln \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)} + \ln \frac{p_{\frac{1}{s}}(\xi)}{p_{\frac{1}{s}}(0)},$$

a further application of the ratio limit theorem yields,

$$\int_0^t q(\tau, \xi)\, \mathrm{d}\tau = -\ln \frac{p_{\frac{1}{t}}(\xi)}{p_{\frac{1}{t}}(0)},$$

i.e., the negative definiteness of $-\ln \sigma_t(\cdot)$ for all $t > 0$. Thus we can write,

$$\tilde{p}_t(x, y) = p_t(0) e^{-\left(\ln \sigma_{\frac{1}{t}}(x-y)\right)}, \tag{5.3}$$

where $\eta \mapsto -\ln \sigma_{\frac{1}{t}}(\eta)$ is a continuous negative definite function and from previous considerations it follows that if $p_{\frac{1}{t}}(\eta_0) = p_{\frac{1}{t}}(0)$ then $\eta_0 = 0$, hence $\delta_{\psi,t}(x,y) = \big(-\ln \sigma_{\frac{1}{t}}(x-y)\big)^{1/2}$ is a further metric on $\mathbb{R}^n$. So, if $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function satisfying Ai)-Aiii) for which $\xi \mapsto -\frac{\partial}{\partial t} \ln \big(p_{\frac{1}{t}}(\xi)/p_{\frac{1}{t}}(0)\big)$ is also continuous negative definite, then the transition density p_t is controlled by two families of (time-dependent) metrics on $\mathbb{R}^n$ namely $d_{\psi,t}$ and $\delta_{\psi,t}$. In particular, if $\big(\mathbb{R}^n, d_{\psi,t}, \lambda^{(n)}\big)$ has the volume doubling property then it holds that,

$$p_t(x-y) \asymp \lambda^{(n)}\big(B^{d_{\psi,t}}(0,1)\big)e^{-\delta^2_{\psi,t}(x,y)}. \tag{5.4}$$

Now we assume that we can associate with ψ also the additive process $(X_t)_{t\geq 0}$ constructed in Section 3. The density of the distribution $P_{X_t - X_0}$ is given by,

$$\pi_{t,0}(x-y) = \frac{e^{-\frac{1}{t}\psi(x-y)}}{(2\pi)^n p_{\frac{1}{t}}(0)}, \tag{5.5}$$

compare with (4.10). Assuming in addition (4.12) we arrive at,

$$\pi_{t,0}(x-y) \asymp \lambda^{(n)}\Big(B^{d_{Q_{t,0}}}\big(0, \sqrt{\beta_1/\beta_0}\big)\Big)e^{-\delta^2_{Q_{t,0}}(x,y)},$$

where $\delta_{Q_{t,0}}$ is the metric,

$$\delta_{Q_{t,0}}(x,y) = \sqrt{\frac{1}{t}\psi(x-y)} = d_{\psi,\frac{1}{t}}(x,y),$$

and for $d_{Q_{t,0}}$ we find,

$$Q_{t,0}(x,y) = -\ln \frac{p_{\frac{1}{t}}(x-y)}{p_{\frac{1}{t}}(0)} = -\ln \sigma_t(x-y),$$

thus,

$$d_{Q_{t,0}}(x,y) = \big(-\ln \sigma_t(x-y)\big)^{1/2} = \delta_{\psi,\frac{1}{t}}(x,y),$$

which yields,

$$\pi_{t,0}(x-y) \asymp \lambda^{(n)}\Big(B^{\delta_{\psi,\frac{1}{t}}}\big(0, \sqrt{\beta_1}\beta_0\big)\Big)e^{-d^2_{\psi,\frac{1}{t}}(x,y)}.$$

Summing our considerations up we have proved,

Theorem 5.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function satisfying Ai)-Aiii) and assume that $q(t,\xi) = -\frac{\partial}{\partial t}\ln\big(p_{\frac{1}{t}}(\xi)/p_{\frac{1}{t}}(0)\big)$ is with respect to ξ a continuous negative definite function. Moreover assume that the metric measure space $\big(\mathbb{R}^n, d_\psi, \lambda^{(n)}\big)$ has the volume doubling property and for $q(t,\xi)$ we have that (4.12) holds. Denote by $(Y_t)_{t\geq 0}$ the Lévy process associated with ψ and by $(X_t)_{t>0}$ the additive process constructed in Section 3. With*

$$d_{\psi,t}(\xi,\eta) = \sqrt{t\psi(\xi-\eta)}$$

and

$$\delta_{\psi,t}(x,y) = \Big(-\ln\frac{p_{\frac{1}{t}}(x-y)}{p_{\frac{1}{t}}(0)}\Big)^{1/2} = \big(-\ln \sigma_{\frac{1}{t}}(x-y)\big)^{1/2}$$

we find,

$$p_t(x-y) \asymp \lambda^{(n)}\big(B^{d_{\psi,t}}(0,1)\big)e^{-\delta^2_{\psi,t}(x,y)}, \tag{5.6}$$

and,

$$\pi_{t,0}(x-y) \asymp \lambda^{(n)}\Big(B^{\delta_{\psi,\frac{1}{t}}}\big(0,\sqrt{\beta_1/\beta_0}\big)\Big)e^{-d^2_{\psi,\frac{1}{t}}(x,y)}, \tag{5.7}$$

hold.

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