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ON CONSTRUCTING SOME MEMBRANES FOR A SYMMETRIC α-STABLE PROCESS

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Abstract. Two kinds of membranes located on a fixed hyperplane S in a Euclidean space are constructed for a symmetric α-stable process with $\alpha \in (1, 2)$. The first one has the property of killing the process at the points of the hyperplane with some given intensity $(r(x))_{x\in S}$. This kind of membranes can be called *an elastic screen* for the process, by analogy to that in the theory of diffusion processes. The second one has the property of delaying the process at the points of S with some given coefficient $(p(x))_{x\in S}$. In other words, the points of S, where $p(x) > 0$, are *sticky* for the process constructed. We show that each one of the membranes is associated with some initial-boundary value problem for pseudo-differential equations related to a symmetric α-stable process.

1. Introduction

Let $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ be a standard Markov process in a d-dimensional Euclidean space $\mathbb{R}^d$ whose transition probability density g_0 (with respect to the Lebesgue measure on $\mathbb{R}^d$) is given by the equality

$$g_0(t, x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\{i(x - y, \xi) - ct|\xi|^{\alpha}\}\, d\xi, \quad t > 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d, \tag{1.1}$$

where $c > 0$ and $\alpha \in (1, 2)$ are fixed parameters (see [4, Theorem 3.14]). This process is called a symmetric (more precisely, rotationally invariant) α-stable process. The generator of it is denoted by $\mathbf{A}$ and this is a pseudo-differential operator with its symbol given by $(-c|\xi|^{\alpha})_{\xi\in\mathbb{R}^d}$.

Let ν be a fixed unit vector in $\mathbb{R}^d$ and S denote the hyperplane in $\mathbb{R}^d$ orthogonal to ν, that is $S = \{x \in \mathbb{R}^d : (x, \nu) = 0\}$. By $\mathbf{B}_\nu$ we denote a pseudo-differential operator with the function $(2ic|\xi|^{\alpha-2}(\xi, \nu))_{\xi\in\mathbb{R}^d}$ as its symbol.

We will consider two kinds of transformations of the process $(x(t))_{t\geq 0}$ (this is a short notation for our process). The first one is connected with the Feynman-Kac formula. Let $(r(x))_{x\in S}$ be a given bounded continuous function with non-negative values. We show that there exists a W-functional $(\eta_t(r))_{t\geq 0}$ of the process $(x(t))_{t\geq 0}$

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such that its characteristic is given by

$$\mathbb{E}_x\eta_t(r) = \int_0^t d\tau \int_S g_0(\tau,x,y)r(y)\,d\sigma_y, \quad t\ge 0,\ x\in\mathbb{R}^d, \tag{1.2}$$

where the inner integral is a surface one.

As is well-known (see [4, Chapter 10]), there exists a standard Markov process $(x^*(t),\mathcal{M}_t^*,\mathbb{P}_x^*,\zeta)$ in $\mathbb{R}^d$ (ζ is the life time of the process) such that the equality

$$\mathbb{E}_x^*(\varphi(x^*(t))1\!\!1_{\zeta>t}) = \mathbb{E}_x(\varphi(x(t))\exp\{-\eta_t(r)\}) \tag{1.3}$$

is valid for $t>0$, $x\in\mathbb{R}^d$ and $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$ (this is the notation for the Banach space of all continuous bounded functions on $\mathbb{R}^d$ with real values and the norm $\|\varphi\| = \sup_{x\in\mathbb{R}^d}|\varphi(x)|$). We show that the function (1.3) (denote it by $u(t,x,\varphi)$, $t\ge 0$, $x\in\mathbb{R}^d$, $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$) is a solution to the following initial-boundary value problem.

Problem A. For a given $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$, a continuous function $(u(t,x))_{t>0,x\in\mathbb{R}^d}$ is being looked for such that it satisfies

(i) the equation $\dfrac{\partial u}{\partial t} = \mathbf{A}u$ in the region $t>0$, $x\notin S$;

(ii) the initial condition $u(0+,x)=\varphi(x)$ for all $x\in\mathbb{R}^d$;

(iii) the boundary condition $\frac{1}{2}\mathbf{B}_\nu u(t,\cdot)(x+) - \frac{1}{2}\mathbf{B}_\nu u(t,\cdot)(x-) = r(x)u(t,x)$ for all $t>0$ and $x\in S$.

The symbol $\mathbf{B}_\nu u(t,\cdot)(x+)$ (respectively, $\mathbf{B}_\nu u(t,\cdot)(x-)$) for $t>0$ and $x\in S$ means the limit value of the function $\mathbf{B}_\nu u(t,\cdot)(z)$, as z approaches x along any curve lying in a finite closed cone $\mathcal{K}$ in $\mathbb{R}^d$ with vertex at x such that $\mathcal{K}\subset\{z\in\mathbb{R}^d:(z,\nu)>0\}\cup\{x\}$ (respectively, $\mathcal{K}\subset\{z\in\mathbb{R}^d:(z,\nu)<0\}\cup\{x\}$).

The second transformation is connected with some random change of time. Let a continuous bounded function $(p(x))_{x\in S}$ with non-negative values be given. For $t\ge 0$, we put

$$\zeta_t = \inf\{s\ge 0: s+\eta_s(p)\ge t\}, \quad \hat{x}(t) = x(\zeta_t), \quad \hat{\mathcal{M}}_t = \mathcal{M}_{\zeta_t}.$$

It is well-known (see, for example, [4, Chapter 10]) that the process $(\hat{x}(t),\hat{\mathcal{M}}_t,\mathbb{P}_x)$ is also a standard Markov process in $\mathbb{R}^d$. We show that the function

$$\hat{u}(t,x,\varphi) = \mathbb{E}_x\varphi(\hat{x}(t)), \quad t\ge 0,\ x\in\mathbb{R}^d, \tag{1.4}$$

is a solution to the following problem.

Problem B. For a given $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$, a continuous function $(u(t,x))_{t>0,x\in\mathbb{R}^d}$ is being looked for such that it satisfies the condition (i), the initial condition (ii) and the following boundary condition (for $t>0$ and $x\in S$)

(iii') $p(x)\dfrac{\partial u}{\partial t}(t,x) = \frac{1}{2}\mathbf{B}_\nu u(t,\cdot)(x+) - \frac{1}{2}\mathbf{B}_\nu u(t,\cdot)(x-)$.

If $\alpha=2$ (and $c=\frac{1}{2}$), then our process is a standard Brownian motion, and the operator $\mathbf{A}$ coincides with $\frac{1}{2}\Delta$ (Δ is the Laplace operator) and $\mathbf{B}_\nu$ coincides with $\frac{\partial}{\partial\nu}$ (the derivative in the direction ν). The facts that in this case the functions (1.3) and (1.4) solve *Problems A* and *B*, respectively, are well-known (some results of the kind can be found in the books [4, 6] and also in [1, 2, 7] and many others).

The article is organized as follows. In Section 2 some auxiliary results are presented. Sections 3 and 4 are devoted to solving the *Problems A* and *B*, respectively.

2. Single-layer Potentials for a Symmetric α-stable Process and the Feynman-Kac Formula.

2.1. The function g_0 defined by (1.1) is continuous in the region $t > 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$. Moreover, it is uniformly continuous in any region of the form $(t, x, y) \in [\gamma, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ for $\gamma > 0$. As follows from [3], it satisfies the inequality

$$g_0(t, x, y) \le N \frac{t}{(t^{1/\alpha} + |y - x|)^{d+\alpha}}, \quad t > 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d, \tag{2.1}$$

where N is a positive constant. The inequalities of the kind in more general situations including similar inequalities for (fractional) derivatives of g_0 can be found in [5].

2.2. Let $\nu \in \mathbb{R}^d$ be a fixed unit vector and S be the hyperplane in $\mathbb{R}^d$ orthogonal to ν. The following formula

$$\int_S e^{i(\xi, y)} g_0(t, x, y)\, d\sigma_y = \frac{1}{\pi} \int_0^\infty e^{-ct(|\xi|^2 + \rho^2)^{\alpha/2}} \cos(\rho(x, \nu))\, d\rho \tag{2.2}$$

holds true for all $t > 0$, $x \in \mathbb{R}^d$ and $\xi \in S$ (see [8]). Combining (2.1) and (2.2) (for $\xi = 0$),we arrive at the inequality

$$\int_S g_0(t, x, y)\, d\sigma_y \le N \frac{t}{(t^{1/\alpha} + |(x, \nu)|)^{1+\alpha}} \tag{2.3}$$

valid for all $t > 0$ and $x \in \mathbb{R}^d$ with some positive constant N.

2.3. In accordance with the definition of $\mathbf{B}_\nu$ (see Section 1), the following equality (for fixed $t > 0$ and $y \in \mathbb{R}^d$)

$$\mathbf{B}_\nu g_0(t, \cdot, y)(x) = \frac{2ic}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\{i(x - y, \xi) - ct|\xi|^\alpha\} |\xi|^{\alpha - 2} (\xi, \nu)\, d\xi$$

is fulfilled for all $x \in \mathbb{R}^d$. Integrating by parts leads us to the formula

$$\mathbf{B}_\nu g_0(t, \cdot, y)(x) = \frac{2(y - x, \nu)}{\alpha t} g_0(t, x, y) \tag{2.4}$$

2.4. Let $(\psi(t, x))_{t \ge 0, x \in S}$ be a continuous function with real values satisfying the inequality $|\psi(t, x)| \le C t^{-\beta}$ for all $t > 0$ and $x \in S$ with some constants $C > 0$ and $\beta < 1$. We put

$$V_0(t, x) = \int_0^t d\tau \int_S g_0(t - \tau, x, y) \psi(\tau, y)\, d\sigma_y, \quad t > 0,\ x \in \mathbb{R}^d. \tag{2.5}$$

This function is well-defined, as the following estimations show

$$|V_0(t,x)| \le C \int_0^t \tau^{-\beta}\, d\tau \int_S g_0(t-\tau,x,y)\, d\sigma_y$$
$$\le CN \int_0^t \tau^{-\beta}(t-\tau)^{-1/\alpha}\, d\tau$$
$$= CN \frac{\Gamma(1-\beta)\Gamma(1-1/\alpha)}{\Gamma(2-\beta-1/\alpha)} t^{1-\beta-1/\alpha}.$$

Moreover, this function is continuous in the region $t > 0$ and $x \in \mathbb{R}^d$. It is called a single-layer potential.

The following properties of the function V_0 are proved in [8].

2.4.A. The function V_0 is a solution of the equation $\dfrac{\partial V_0}{\partial t} = \mathbf{A}V_0$ in the region $t > 0$ and $x \notin S$.

2.4.B. The following relations $\mathbf{B}_\nu V_0(t,\cdot)(x\pm) = \mp\psi(t,x)$ are held for all $t > 0$ and $x \in S$ (the sense of the left hand side is explained in Section 1).

Remark 2.1. Relation 2.4.B are some analogy to the well-known theorem on the jump of the (co-)normal derivative of a single-layer potential in the classical theory of potentials. The term analogous to the so-called direct value of the derivative vanishes in 2.4.B, since $\mathbf{B}_\nu g_0(t,\cdot,y)(x) = 0$ for $y \in S$ and $x \in S$ (see (2.4)).

2.5. Let $(v(x))_{x\in\mathbb{R}^d}$ be a continuous bounded function with real values. We put for $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, $t > 0$ and $x \in \mathbb{R}^d$

$$Q(t,x,\varphi) = \mathbb{E}_x \left(\varphi(x(t)) \exp\left\{ \int_0^t v(x(\tau))\, d\tau \right\} \right).$$

The well-known Feynman-Kac formula asserts that Q satisfies the equation

$$\frac{\partial Q}{\partial t} = \mathbf{A}Q + v(x)Q$$

in the region $(t,x) \in (0,\infty) \times \mathbb{R}^d$ and the initial condition $Q(0+,x,\varphi) = \varphi(x)$ for all $x \in \mathbb{R}^d$.

An intermediate stage of this result is the following integral equation for Q

$$Q(t,x,\varphi) = \int_{\mathbb{R}^d} g_0(t,x,y)\varphi(y)\, dy + \int_0^t d\tau \int_{\mathbb{R}^d} g_0(t-\tau,x,y) Q(\tau,y,\varphi) v(y)\, dy,$$

where $t > 0$, $x \in \mathbb{R}^d$.

3. Solving Problem A

3.1. Let the hyperplane S and the bounded continuous function $(r(x))_{x\in S}$ be such as above. One can easily verify that the function

$$f_t(x) = \int_0^t d\tau \int_S g_0(t-\tau,x,y) r(y)\, d\sigma_y$$

is a W-function for the process $(x(t))_{t\geq 0}$ (see [4, Chapter 6, §3]) satisfying the inequality

$$f_t(x) \leq N\|r\|\frac{\alpha}{\alpha-1}t^{1-1/\alpha}$$

for all $t \geq 0$ and $x \in \mathbb{R}^d$ (see (2.4)), where $\|r\| = \sup_{x\in S} r(x)$. Therefore, according to Theorem 6.6 from [4], there exists a W-functional $(\eta_t(r))_{t\geq 0}$ of the process $(x(t))_{t\geq 0}$ such that $\mathbb{E}_x\eta_t(r) = f_t(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$.

For $r_0(x) \equiv 1$ we put $\eta_t = \eta_t(r_0)$, $t \geq 0$. The functional $(\eta_t)_{t\geq 0}$ is called the local time on S for the process $(x(t))_{t\geq 0}$. It is evident that $\eta_t(r) = \int_0^t r(x(s))\,d\eta_s$, $t \geq 0$.

3.2. We now approximate the functional $(\eta_t(r))_{t\geq 0}$ by somewhat simpler ones. For $h > 0$, we define a function $(v_h(x))_{x\in\mathbb{R}^d}$ by setting $v_h(x) = \int_S g_0(h, x, y)r(y)\,d\sigma_y$, $x \in \mathbb{R}^d$, and a functional $(\eta_t^{(h)}(r))_{t\geq 0}$ by the equality $\eta_t^{(h)}(r) = \int_0^t v_h(x(s))\,ds$, $t \geq 0$.

The function v_h for fixed $h > 0$ is continuous and bounded, so the W-functional $(\eta_t^{(h)}(r))_{t\geq 0}$ is well-defined. Its characteristic is given by

$$\begin{aligned} f_t^{(h)}(x) = \mathbb{E}_x\eta_t^{(h)}(r) &= \int_0^t d\tau \int_{\mathbb{R}^d} g_0(\tau, x, y)v_h(y)\,dy \\ &= \int_h^{t+h} d\tau \int_S g_0(\tau, x, y)r(y)\,d\sigma_y. \end{aligned}$$

Hence,

$$\mathbb{E}_x\eta_t^{(h)}(r) - \mathbb{E}_x\eta_t(r) = \int_t^{t+h} d\tau \int_S g_0(\tau, x, y)r(y)\,d\sigma_y - \int_0^h d\tau \int_S g_0(\tau, x, y)r(y)\,d\sigma_y.$$

Taking into account (2.4), we arrive at the inequality

$$\begin{aligned} \sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}^d} |\mathbb{E}_x\eta_t^{(h)}(r) - \mathbb{E}_x\eta_t(r)| \leq N\|r\|\frac{\alpha}{\alpha-1}\Big[h^{1-1/\alpha} \\ + \sup_{0\leq t\leq T}\Big((t+h)^{1-1/\alpha} - t^{1-1/\alpha}\Big)\Big] \end{aligned}$$

valid for all $T > 0$ and $h > 0$. Denote by $q_T(h)$ the expression on the right-hand side of this inequality. Obviously, $q_T(h) \to 0$, as $h \to 0+$, for any fixed $T > 0$. According to Lemma 6.5 from [4], the following inequality

$$\mathbb{E}_x(\eta_t^{(h)}(r) - \eta_t(r))^2 \leq 2(f_t^{(h)}(x) + f_t(x))q_T(h)$$

holds true for all $t \in [0, T]$ and $x \in \mathbb{R}^d$. Since for those (t, x) we have

$$f_t^{(h)}(x) \leq N\|r\|\frac{\alpha}{\alpha-1}(T+h)^{1-1/\alpha}; \quad f_t(x) \leq N\|r\|\frac{\alpha}{\alpha-1}T^{1-1/\alpha},$$

we can assert that the inequality

$$\mathbb{E}_x(\eta_t^{(h)}(r) - \eta_t(r))^2 \leq 4N\|r\|\frac{\alpha}{\alpha-1}(T+h_0)^{1-1/\alpha}q_T(h) \tag{3.1}$$

is fulfilled for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $h \in (0, h_0]$ ($T > 0$ and $h_0 > 0$ are arbitrary fixed numbers).

3.3. For $t>0$, $x\in\mathbb{R}^d$ and $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$, we put

$$u^{(h)}(t,x,\varphi)=\mathbb{E}_x\left(\varphi(x(t))e^{-\eta_t^{(h)}(r)}\right),\quad u(t,x,\varphi)=\mathbb{E}_x\left(\varphi(x(t))e^{-\eta_t(r)}\right).$$

Proposition 3.1. *There exists a sequence $(h_n)_{n\ge1}$ such that $h_n\to0$, as $n\to\infty$, and*

$$\lim_{n\to\infty}u^{(h_n)}(t,x,\varphi)=u(t,x,\varphi)$$

uniformly with respect to $x\in\mathbb{R}^d$ and locally uniformly with respect to $t\in[0,\infty)$.

Proof. Since $|e^{-a}-e^{-b}|\le|a-b|$ for all $a\ge0$ and $b\ge0$, we can write down the chain of inequalities (for an arbitrary $T>0$)

$$\begin{aligned}&|u^{(h)}(t,x,\varphi)-u(t,x,\varphi)|\le\|\varphi\|\mathbb{E}_x|\eta_t^{(h)}(r)-\eta_t(r)|\\&\le\|\varphi\|\left[\mathbb{E}_x(\eta_t^{(h)}(r)-\eta_t(r))^2\right]^{1/2}\le K_T(h_0)(q_T(h))^{1/2}\|\varphi\|\end{aligned}$$

valid for all $(t,x)\in[0,T]\times\mathbb{R}^d$ and $h\in(0,h_0]$, where $K_T(h_0)$ is a constant finite for $T<\infty$. To complete the proof one should make use of the diagonal method. □

3.4. The function $u^{(h)}$ (for a fixed $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$) is a unique bounded solution to the integral equation (see Section 2.5)

$$u^{(h)}(t,x,\varphi)=\int_{\mathbb{R}^d}g_0(t,x,y)\varphi(y)\,dy-\int_0^t d\tau\int_{\mathbb{R}^d}g_0(t-\tau,x,y)u^{(h)}(\tau,y,\varphi)v_h(y)\,dy.\tag{3.2}$$

It is an easy exercise to verify that the relation

$$\lim_{h\to0+}\int_{\mathbb{R}^d}\psi(y)v_h(y)\,dy=\int_S\psi(y)r(y)\,d\sigma_y\tag{3.3}$$

is fulfilled for any continuous function $(\psi(y))_{y\in\mathbb{R}^d}$ such that $\int_{\mathbb{R}^d}|\psi(y)|\,dy<\infty$.

Proposition 3.2. *For a given $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$, the function $(u(t,x,\varphi))_{t\ge0,x\in\mathbb{R}^d}$ is a unique bounded solution of the equation*

$$u(t,x,\varphi)=\int_{\mathbb{R}^d}g_0(t,x,y)\varphi(y)\,dy-\int_0^t d\tau\int_S g_0(t-\tau,x,y)u(\tau,y,\varphi)r(y)\,d\sigma_y.\tag{3.4}$$

Proof. In order to pass to the limit, as $h_n\to0$, in equation (3.2) (written for $h=h_n$), one should observe that

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t d\tau\int_{\mathbb{R}^d}g_0(t-\tau,x,y)u(\tau,y,\varphi)v_{h_n}(y)\,dy\\&=\int_0^t d\tau\int_S g_0(t-\tau,x,y)u(\tau,y,\varphi)r(y)\,d\sigma_y\end{aligned}$$

according to (3.3). Besides,

$$\int_0^t d\tau\int_{\mathbb{R}^d}g_0(t-\tau,x,y)v_h(y)\,dy=f_t^{(h)}(x)\le N\|r\|\frac{\alpha}{\alpha-1}(T+h)^{1-1/\alpha},$$

as was established in Section 3.2. Taking into account Proposition 3.1, we arrive at equation (3.4) for the function u.

A solution to the equation (3.4) can be constructed by the method of successive approximations. If we put

$$u_0(t,x,\varphi) = \int_{\mathbb{R}^d} g_0(t,x,y)\varphi(y)\,dy, \quad t>0,\ x\in\mathbb{R}^d,\ \varphi\in\mathbb{C}_b(\mathbb{R}^d),$$

and for $k \geq 1$

$$u_k(t,x,\varphi) = \int_0^t d\tau \int_S g_0(t-\tau,x,y)u_{k-1}(\tau,y,\varphi)r(y)\,d\sigma_y,$$

then by induction on k, we can easily obtain the following estimate

$$|u_k(t,x,\varphi)| \leq \frac{\|\varphi\|\|r\|^k}{\left(c^{1/\alpha}\alpha\sin\frac{\pi}{\alpha}\right)^k}\frac{t^{k(1-1/\alpha)}}{\Gamma(k(1-1/\alpha)+1)} \tag{3.5}$$

held true for all $t>0$, $x\in\mathbb{R}^d$, $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$ and $k=0,1,2,\ldots$. As a consequence of (3.5), we have that the series

$$\sum_{k=0}^{\infty}(-1)^k u_k(t,x,\varphi) \tag{3.6}$$

is a continuous solution of (3.4) satisfying the condition

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}|u(t,x,\varphi)|<\infty$$

for any $T>0$. Another consequence of (3.5) is that such a solution is unique. Therefore, the function u can be represented by the series (3.6). The proposition is proved. □

3.5. We now can formulate the main result of Section 3

Theorem 3.3. *For a fixed $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$ the function*

$$u(t,x,\varphi) = \mathbb{E}_x\left(\varphi(x(t))e^{-\eta_t(r)}\right), \quad t\geq 0,\ x\in\mathbb{R}^d,$$

solves the Problem A.

Proof. The first item on the right hand side of (3.4) satisfies the equation (i) in the whole region $t>0$ and $x\in\mathbb{R}^d$. It also satisfies the initial condition (ii). The second item on the right-hand side of (3.4) is a single-layer potential. According to 2.4.A, it satisfies (i) and its initial value vanishes. The relations 2.4.B imply now the equalities

$$\mathbf{B}_\nu u(t,\cdot,\varphi)(x\pm) = \frac{2}{\alpha t}\int_{\mathbb{R}^d}(y,\nu)\varphi(y)g_0(t,x,y)\,dy \pm r(x)u(t,x,\varphi)$$

valid for $t>0$ and $x\in S$, and the condition (iii) follows from these relations immediately. The theorem has been proved. □

3.6. If $d=1$, then $S=\{0\}$ and $r=r(0)$ is a non-negative number. The equation for the function u in this case can be written as follows

$$u(t,x,\varphi)=\int_{\mathbb{R}^1} g_0(t,x,y)\varphi(y)\,dy - r\int_0^t g_0(t-\tau,x,0)u(\tau,0,\varphi)\,d\tau. \tag{3.7}$$

Denote by $\tilde{u}$ and $\tilde{g}_0$ the Laplace transformations of the functions u and g_0, respectively ($\lambda>0$)

$$\tilde{u}(\lambda,x,\varphi)=\int_0^\infty u(t,x,\varphi)e^{-\lambda t}\,dt,\quad \tilde{g}_0(\lambda,x,y)=\int_0^\infty g_0(t,x,y)e^{-\lambda t}\,dt.$$

Then (3.7) implies the equality

$$\tilde{u}(\lambda,x,\varphi)=\int_{\mathbb{R}^1}\left[\tilde{g}_0(\lambda,x,y)-\frac{r\tilde{g}_0(\lambda,x,0)\tilde{g}_0(\lambda,0,y)}{1+r\tilde{g}_0(\lambda,0,0)}\right]\varphi(y)\,dy,$$

where $\tilde{g}_0(\lambda,0,0)=\left[c^{1/\alpha}\alpha\sin\frac{\pi}{\alpha}\right]^{-1}\lambda^{1/\alpha-1}$. It means that the resolvent kernel $\tilde{g}^*(\lambda,x,y)$ of the process $(x^*(t))_{t\geq 0}$ (see Section 1) is given by

$$\tilde{g}^*(\lambda,x,y)=\tilde{g}_0(\lambda,x,y)-\frac{r\tilde{g}_0(\lambda,x,0)\tilde{g}_0(\lambda,0,y)}{1+r\tilde{g}_0(\lambda,0,0)}$$

for $\lambda>0$, $x\in\mathbb{R}^1$ and $y\in\mathbb{R}^1$. One can obtain from this equality, in particular, the Laplace transform for the distribution function of ζ (the life time of the process $(x^*(t))_{t\geq 0}$)

$$\mathbb{E}_x^* e^{-\lambda\zeta}=\frac{r\tilde{g}_0(\lambda,x,0)}{1+r\tilde{g}_0(\lambda,0,0)},\quad x\in\mathbb{R}^1,\lambda>0.$$

4. Solving Problem B

4.1. We are now given by a continuous bounded function $(p(x))_{x\in S}$ with non-negative values. Consider the Markov process $(\hat{x}(t),\hat{\mathcal{M}}_t,\mathbb{P}_x)$ defined in Section 1. The resolvent operator for this process can be calculated in the following way (see [6, Chapter II, §6])

$$\begin{aligned}\mathbb{E}_x\int_0^\infty e^{-\lambda t}\varphi(\hat{x}(t))\,dt&=\mathbb{E}_x\int_0^\infty e^{-\lambda t}\varphi(x(\zeta_t))\,dt\\ &=\mathbb{E}_x\int_0^\infty e^{-\lambda(t+\eta_t(p))}\varphi(x(t))\,dt+\mathbb{E}_x\int_0^\infty e^{-\lambda(t+\eta_t(p))}\varphi(x(t))\,d\eta_t(p),\end{aligned} \tag{4.1}$$

where $x\in\mathbb{R}^d$, $\lambda>0$, $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$ (we have taken into account that the equality $\zeta_t=t'$ implies $t=t'+\eta_{t'}(p)$).

4.2. If we put

$$Q_\lambda(t,x,\varphi)=\mathbb{E}_x(\varphi(x(t))e^{-\lambda\eta_t(p)}),\quad t>0,\ \lambda>0,\ x\in\mathbb{R}^d,\ \varphi\in\mathbb{C}_b(\mathbb{R}^d),$$

then in accordance with Section 3, we have the following equation for Q_λ

$$Q_\lambda(t,x,\varphi)=\int_{\mathbb{R}^d} g_0(t,x,y)\varphi(y)\,dy-\lambda\int_0^t d\tau\int_S g_0(t-\tau,x,y)Q_\lambda(\tau,y,\varphi)p(y)\,d\sigma_y.$$

Multiplying both sides of this equation by $e^{-\lambda t}$ and integrating with respect to t over $(0,\infty)$, we get the equation

$$U_1(\lambda,x,\varphi)=\int_{\mathbb{R}^d}\tilde g_0(\lambda,x,y)\varphi(y)\,dy-\lambda\int_S\tilde g_0(\lambda,x,y)U_1(\lambda,y,\varphi)p(y)\,d\sigma_y, \tag{4.2}$$

where $\tilde g_0(\lambda,x,y)=\int_0^\infty g_0(t,x,y)e^{-\lambda t}\,dt$ and

$$U_1(\lambda,x,\varphi)=\int_0^\infty Q_\lambda(t,x,y)e^{-\lambda t}\,dt=\mathbb{E}_x\int_0^\infty e^{-\lambda(t+\eta_t(p))}\varphi(x(t))\,dt.$$

4.3. To calculate the second item on the right hand side of (4.1), we observe that

$$\mathbb{E}_x\int_0^\infty e^{-\lambda(t+\eta_t(p))}\varphi(x(t))\,d\eta_t(p)=\lim_{h\to0+}\mathbb{E}_x\int_0^\infty e^{-\lambda(t+\eta_t(p))}\varphi(x(t))v_h(x(t))\,dt,$$

where this time $v_h(x)=\int_S g_0(h,x,y)p(y)\,d\sigma_y$, $h>0$, $x\in\mathbb{R}^d$. According to Section 4.2, we have

$$\mathbb{E}_x\int_0^\infty e^{-\lambda(t+\eta_t(p))}\varphi(x(t))v_h(x(t))\,dt=U_1(\lambda,x,\varphi\cdot v_h).$$

It is a very simple conclusion that for $\lambda>0$, $x\in\mathbb{R}^d$ and $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$, the relation $\lim_{h\to0+}U_1(\lambda,x,\varphi\cdot v_h)=U_2(\lambda,x,\varphi)$ fulfilled, where U_2 is the solution to the equation

$$U_2(\lambda,x,\varphi)=\int_S\tilde g_0(\lambda,x,y)\varphi(y)p(y)\,d\sigma_y-\lambda\int_S\tilde g_0(\lambda,x,y)U_2(\lambda,y,\varphi)p(y)\,d\sigma_y, \tag{4.3}$$

4.4. As a consequence of 2.4.B, we have the following relations

$$\mathbf{B}_\nu\left(\int_S\tilde g_0(\lambda,\cdot,y)\tilde\psi(\lambda,y)\,d\sigma_y\right)(x\pm)=\mp\tilde\psi(\lambda,x)$$

valid for $\lambda>0$, $x\in S$ and any continuous function $(\psi(t,x))_{t\geq0,x\in S}$ such as in Section 2.4. These relations imply the following ones ($x\in S$, $\lambda>0$)

$$\mathbf{B}_\nu U_1(\lambda,\cdot,\varphi)(x\pm)=\int_{\mathbb{R}^d}\mathbf{B}_\nu\tilde g_0(\lambda,\cdot,y)(x)\varphi(y)\,dy\pm\lambda p(x)U_1(\lambda,x,\varphi),$$
$$\mathbf{B}_\nu U_2(\lambda,\cdot,\varphi)(x\pm)=\mp p(x)\varphi(x)\pm\lambda p(x)U_2(\lambda,x,\varphi).$$

4.5. We put $U(\lambda,x,\varphi)=U_1(\lambda,x,\varphi)+U_2(\lambda,x,\varphi)$. Then

$$\mathbb{E}_x\int_0^\infty e^{-\lambda t}\varphi(\hat x(t))\,dt=U(\lambda,x,\varphi).$$

It follows from the equations (4.2), (4.3) that the function U satisfies the equation

$$\mathbf{A}U=\lambda U-\varphi(x)$$

in the region $x\notin S$. Besides, it satisfies the boundary condition ($\lambda>0$, $x\in S$)

$$\frac{1}{2}\mathbf{B}_\nu U(\lambda,\cdot,\varphi)(x+)-\frac{1}{2}\mathbf{B}_\nu U(\lambda,\cdot,\varphi)(x-)=p(x)(\lambda U(\lambda,x,\varphi)-\varphi(x)).$$

We have thus proved the following assertion

Theorem 4.1. *The function*

$$\hat{U}(t,x,\varphi) = \mathbb{E}_x \varphi(\hat{x}(t)), \quad t>0,\ x\in\mathbb{R}^d$$

solves the Problem B.

4.6. If $d=1$, then

$$\mathbb{E}_x \int_0^\infty e^{-\lambda t}\varphi(\hat{x}(t))\,dt = \frac{p\,\tilde{g}_0(\lambda,x,0)}{1+\lambda\,p\,\tilde{g}_0(\lambda,0,0)}\varphi(0)$$
$$+\int_{\mathbb{R}^1}\left[\tilde{g}_0(\lambda,x,y) - \frac{\lambda\,p\,\tilde{g}_0(\lambda,x,0)\tilde{g}_0(\lambda,0,y)}{1+\lambda\,p\,\tilde{g}_0(\lambda,0,0)}\right]\varphi(y)\,dy$$

for all $\lambda>0$, $x\in\mathbb{R}^1$ and $\varphi\in\mathbb{C}_b(\mathbb{R}^1)$, where $p=p(0)$ is a non-negative number.

In the case of $p\to\infty$ the point $x=0$ becomes an absorbing one. In this case

$$\mathbb{E}_x \int_0^\infty e^{-\lambda t}\varphi(\hat{x}_\infty(t))\,dt = \frac{\tilde{g}_0(\lambda,x,0)}{\lambda\,\tilde{g}_0(\lambda,0,0)}\varphi(0)$$
$$+\int_{\mathbb{R}^1}\left[\tilde{g}_0(\lambda,x,y) - \frac{\tilde{g}_0(\lambda,x,0)\tilde{g}_0(\lambda,0,y)}{\tilde{g}_0(\lambda,0,0)}\right]\varphi(y)\,dy.$$

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