OPTIMAL APPROXIMATION OF SKOROHOD INTEGRALS – EXAMPLES WITH SUBSTANDARD RATES

PETER PARCZEWSKI

Abstract. We consider optimal approximation with respect to the mean square error of Itô integrals and Skorohod integrals given an equidistant discretization of the Brownian motion. We obtain for suitable integrands optimal rates smaller than the standard \( n^{-1} \), where \( n \) denotes the number of evaluations of the Brownian motion. For the Itô integral this is due to the Weyl equidistribution theorem and discontinuities of the integrand. For the Skorohod integral the situation is more complicated and relies on a reformulation of the Wiener chaos expansion. Here, we specify conditions on the integrands to obtain optimal rates \( n^{-1/2} \), respectively, examples of lower rates.

1. Introduction

Suppose a Brownian motion \( (W_t)_{t \in [0,1]} \) on the probability space \( (\Omega, \mathcal{F}, P) \), where the \( \sigma \)-field \( \mathcal{F} \) is generated by the Brownian motion and completed by null sets. Our aim in this note is to investigate the mean square error

\[
e_n := \mathbb{E} \left[ \left( \int_0^1 u_s dW_s - \mathbb{E} \left[ \int_0^1 u_s dW_s \mid W_{1/2}, W_{1}, \ldots, W_1 \right] \right)^2 \right]^{1/2}
\]

for some Itô integrals and Skorohod integrals with sufficiently irregular integrands \( (u_t)_{t \in [0,1]} \). Our main result is that

\[
e_{n_k} \sim c n_k^{-1/2}
\]

for appropriate strictly increasing sequences \( (n_k)_{k \in \mathbb{N}} \) and appropriate integrands. The Skorohod integral is a natural extension of the Itô integral to nonadapted integrands, see e.g. [4, 12]. The asymptotic behaviour in (1.1) is in contrast to the ordinary optimal approximation error

\[
e_n \sim c n^{-1}
\]

for Itô integrable and sufficiently smooth integrands \( u_s = f(s, W_s) \), see e.g. [8, 10, 14]. In [11] we considered simple Skorohod integrands of the type

\[
(u_t = f(t, W_t, W_{\tau_2}, \ldots, W_{\tau_K}))_{t \in [0,1]}
\]
for some $f \in C([0,1] \times \mathbb{R}^K; \mathbb{R})$ and fixed $\tau_2, \ldots, \tau_K \in (0,1]$. Under some smoothness and growth conditions on $f$ and $\tau_2, \ldots, \tau_K \in \frac{1}{n}\mathbb{N}$, we obtained (1.2) where the constant $c$ is a natural extension of the constant in the Itô case. However, treating integrands with the condition $\tau_2, \ldots, \tau_K \in \frac{1}{n}\mathbb{N}$ violated, the situation changes extremely. Starting from the (stationary) Skorohod integrable process

$$u_t = \sum_{k \geq 0} \frac{1}{k!} h_k(\{kT\})(W_{\{kT\}}),$$

for the Hermite polynomials $h_k^k(x)$ (see below), the fractional part $\{kT\} := kT - \lfloor kT \rfloor$ and some fixed $T \in (0,1) \setminus \mathbb{Q}$, we are interested into the optimal approximation problem with integrands beyond the ordinary regularity assumptions. For these integrands no finite extension of the equidistant information to $W_{1/n}, \ldots, W_1, W_{x_1}, \ldots, W_{x_N}$ suffices to evaluate the corresponding Skorohod integral exactly.

Thus, in contrast to (1.2), under some smoothness assumptions on the integrand (1.3), we obtain as the main result in Theorem 3.3 a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ and the asymptotic behaviour (1.1) with the new constant

$$c_2 = \frac{1}{2} \left( \sum_{i=2}^{K} \mathbb{E} \left[ \left( \int_0^1 u_s^i dW_s \right)^2 \right] \right)^{1/2}$$

depending on further Skorohod integrals, where the integrands $(u_t^i)_{t \in [0,1]}$ are specified by the initial process $(u_t)_{t \in [0,1]}$ in (1.3).

We observe the same optimal rate (1.2) for Itô integrals with discontinuous integrands and specify the constant $c$ (Theorem 3.5).

Dealing with stationary integrands of more irregular type as $u_t = |W_T|$ for some fixed $T \in (0,1) \setminus \mathbb{Q}$, we obtain smaller optimal rates, e.g. optimal rates below $n^{-1/2}$.

The recent work [7] presents Itô stochastic differential equations with infinitely often differentiable and globally bounded coefficients such that the strong convergence rate of all numerical schemes based on a finite information of the driving Brownian motion is not polynomial.

However, in contrast to that, our results focus on the Skorohod integral. The integrand requires appropriate continuity assumptions to ensure the existence of the Skorohod integral itself and the elements in the constants in (1.1).

The paper is organized as follows: In Section 2 we give some preliminaries on the Skorohod integral and present some helpful reformulations of the Wiener chaos decomposition. The main results on the optimal approximation of Skorohod integrals of integrands of the type $u_t = f(t, W_t, W_{\tau_2}, \ldots, W_{\tau_K})_{t \in [0,1]}$ are the content of Section 3. The proofs are postponed to Section 4. Finally, in Section 5 we study more irregular cases with optimal rates below $n^{-1/2}$.

### 2. Preliminaries

Here, we give a short introduction to the Skorohod integral. An essential tool is the reformulation of processes via the Wiener chaos decomposition. Aiming
the optimal approximation, we collect some basic properties of conditional expectations and Skorohod integrals and present a simple Itô formula for Skorohod integrals.

We restrict ourselves to the stochastic calculus on the Gaussian Hilbert space \( \{ I(f) : f \in L^2([0,1]) \} \subset L^2(\Omega) \), where \( I(f) \) denotes the Wiener integral. We denote the norm and inner product on \( L^2([0,1]) \) by \( \| \cdot \| \) and \( \langle \cdot , \cdot \rangle \). Due to the totality of the stochastic exponentials \( \exp \left( I(f) - \frac{\| f \|^2}{2} \right) , \ f \in L^2([0,1]) \), in \( L^2(\Omega) \), (see e.g. [6, Corollary 3.40]), for every \( X \in L^2(\Omega, \mathcal{F}, P) \) and \( h \in L^2([0,1]) \), the \( S \)-transform of \( X \) at \( h \) is defined as

\[
(SX)(h) := \mathbb{E}[X \exp \left( I(f) - \frac{\| f \|^2}{2} \right)].
\]

The \( S \)-transform \( (S\cdot)(\cdot) \) is a continuous and injective function on \( L^2(\Omega, \mathcal{F}, P) \) (see e.g. [6, Chapter 16] for more details). As an example, for \( f, g \in L^2([0,1]) \), we have \( (S \exp (I(f) - \frac{\| f \|^2}{2})) (g) = \exp (\langle f, g \rangle) \). In particular the characterization of random variables via the \( S \)-transform can be used to introduce the Skorohod integral, an extension of the Itô integral to nonadapted integrands (cf. e.g. [6, Section 16.4]):

**Definition 2.1.** Suppose \( u = (u_t)_{t \in [0,1]} \in L^2(\Omega \times [0,1]) \) is a (possibly nonadapted) square integrable process on \( (\Omega, \mathcal{F}, P) \) and \( Y \in L^2(\Omega, \mathcal{F}, P) \) such that

\[ \forall g \in L^2([0,1]) : (SY)(g) = \int_0^1 (Su_s)(g(s))ds. \]

Then \( \int_0^1 u_sdW_s = Y \) defines the Skorohod integral of \( u \) with respect to the Brownian motion \( W \).

For more information on the Skorohod integral we refer to [6], [9] or [12]. We recall that for the Hermite polynomials

\[ h_k(x) = (-\alpha)^k \exp \left( \frac{x^2}{2\alpha} \right) \frac{d^k}{dx^k} \exp \left( \frac{-x^2}{2\alpha} \right) \]

and every \( k \in \mathbb{N} \), the \( k \)-th Wiener chaos \( H^k \) is the \( L^2 \)-completion of \( \{ h_k \| f \| (I(f)) : f \in L^2([0,1]) \} \) in \( L^2(\Omega) \) and these subspaces are orthogonal and fulfill \( L^2(\Omega, \mathcal{F}, P) = \bigoplus_{k \geq 0} H^k \). Thus, for the projections

\[ \pi_k : L^2(\Omega) \rightarrow H^k, \]

for every random variable \( X \in L^2(\Omega) \), we denote the \textit{Wiener chaos decomposition} as

\[ X = \sum_{k=0}^{\infty} \pi_k(X). \]

We refer to [6, 5] for further details and a reformulation in terms of multiple Wiener integrals. We recall that a process \( u \in L^2(\Omega \times [0,1]) \) is Skorohod integrable if and only if

\[ \sum_{k=0}^{\infty} (k+1)\| \pi_k(u)\|_{L^2(\Omega \times [0,1])}^2 < \infty, \]
L products as the Wick-analytic representation. Following [5, 3], we denote the Wiener chaos decomposition in terms of Wick
products as the Wiener chaos representation. For example, for a Gaussian random variable $X$ and Hermite polynomials play the role of monomials in standard calculus as

$$X^{\circ k} = h^0_{E[X^2]}(X).$$

Following [5, 3], we denote the Wiener chaos decomposition in terms of Wick products as the Wick-analytic representation. In particular, for fixed $t_1, \ldots, t_K \in [0,1]$, $f : \mathbb{R}^K \to \mathbb{R}$ and a square integrable left hand side, there exist $a_{i_1,\ldots,i_K} \in \mathbb{R}, l_1, \ldots, l_K \geq 0$, such that

$$f(W_{t_1}, \ldots, W_{t_K}) = \sum_{l_1,\ldots,l_K \geq 0} a_{i_1,\ldots,i_K} W_{t_1}^{\circ l_1} \cdot \cdots \cdot W_{t_K}^{\circ l_K}. \quad (2.1)$$

For a proof of the following reformulations we refer to [6, Chapter 16] or [11, Proposition 7]:

**Proposition 2.2.** Suppose $X \in L^2(\Omega)$ and a Skorohod integrable process $u \in L^2(\Omega \times [0,1])$. Then, if both sides exist in $L^2(\Omega)$:

$$\int_0^1 X \circ u_s dW_s = X \circ \int_0^1 u_s dW_s, \quad \int_0^1 X \circ u_s ds = X \circ \int_0^1 u_s ds.$$

Suppose $f_1, \ldots, f_K \in L^2([0,1])$ and $l_1, \ldots, l_K \in \mathbb{N}$. Then the polynomial

$$h(x_1, \ldots, x_K) = \frac{\partial^{\sum_{i=1}^K l_i}}{\partial a_1^{l_1} \cdots \partial a_K^{l_K}} \exp \left( \sum_{i=1}^K a_i x_i - \| a_i \|^2 / 2 \right) \bigg|_{a_1 = \cdots = a_K = 0},$$

(we omit the dependence on $f_i$ and $l_i$ for short) gives the analytic representation

$$h(I(f_1), \ldots, I(f_K)) = I(f_1)^{\circ l_1} \cdot \cdots \cdot I(f_K)^{\circ l_K}, \quad (2.2)$$

(cf. [6, Chapter 3]). For more details on these multivariate generalizations of Hermite polynomials see e.g. [2] or [13]. Dealing with $L^2$-norms of Gaussian random variables, we will frequently make use of

$$\mathbb{E}[(I(f_1) \cdot \cdots \cdot I(f_n))(I(g_1) \cdot \cdots \cdot I(g_m))] = \delta_{n,m} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n \langle f_i, g_{\sigma(i)} \rangle, \quad (2.3)$$

for all $n, m \in \mathbb{N}, f_1, \ldots, f_n, g_1, \ldots, g_m \in L^2([0,1])$, where $\mathcal{S}_n$ denotes the group of permutations on $\{1, \ldots, n\}$ (see e.g. [6, Theorem 3.9]). In particular

$$\mathbb{E}[(I(f_1)^{\circ l_1} \cdot \cdots \cdot I(f_n)^{\circ l_n})^2] \leq n! \prod_{i=1}^n l_i ! \| f_i \|^{2l_i}. \quad (2.4)$$

We denote the equidistant discretization of the underlying Brownian motion by

$$\mathcal{P}_n := \{ W_{\frac{i}{n}}, W_{\frac{2i}{n}}, \ldots, W_1 \}$$
and the corresponding linear interpolation of the Brownian motion as
\[ W^\text{lin}_t := W_{i/n} + n(t - i/n)(W_{(i+1)/n} - W_{i/n}), \]
for \( t \in [i/n, (i+1)/n) \), \( i = 0, \ldots, n - 1 \). Obviously, we have \( W^\text{lin}_t = \mathbb{E}[W_t | \mathcal{P}_n] \).

The advantage of the Wick product is preserving the conditional expectation (see [6, Corollary 9.4]):

**Proposition 2.3.** For \( X, Y, X \diamond Y \in L^2(\Omega) \) and the sub-\( \sigma \)-field \( \mathcal{G} \subseteq \mathcal{F} \):
\[ \mathbb{E}[X \circ Y | \mathcal{G}] = \mathbb{E}[X | \mathcal{G}] \circ \mathbb{E}[Y | \mathcal{G}]. \]

**Definition 2.4.** We define the class of Wick-analytic functionals as
\[ F^\diamond(W_t_1, \ldots, W_{t_K}) = \sum_{k=0}^{\infty} a_{1,k} W^{\diamond k}_{t_1} \cdots \sum_{k=0}^{\infty} a_{K,k} W^{\diamond k}_{t_K}, \max_k \sup_k \sqrt[k]{k! |a_{i,k}|} < \infty. \]

In particular, the analytic representation
\[ f^\diamond(t_1, \ldots, t_K, W_t_1, \ldots, W_{t_K}) \]
of a Wick-analytic functional via (2.2) fulfills \( f^\circ \in C^\infty([0,1]^K \times \mathbb{R}^K, \mathbb{R}) \) and all derivatives of Wick-analytic functionals are Wick-analytic as well (cf. [11, Proposition 10]). Combining the previous propositions, we obtained the following reformulation (Remark 11 in [11]):

**Proposition 2.5.** Suppose \( f \in C^{1;2,\ldots,2}([0,1] \times \mathbb{R}^K) \) and fixed \( \tau_2, \ldots, \tau_K \in [0,1] \). Then, via (2.1) and (2.2), the analytic representation
\[ f^\circ(t, \tau_2, \ldots, \tau_K, W_t, W_{\tau_2}, \ldots, W_{\tau_K}) = f(t, W_t, W_{\tau_2}, \ldots, W_{\tau_K}) \]
satisfies \( f^\circ \in C^{1;1;2,\ldots,2}([0,1]^K \times \mathbb{R}^K) \).

This yields a simple Skorohod Itô formula ([11, Theorem 15]):

**Theorem 2.6.** Suppose \( K \in \mathbb{N}, f \in C^{1;2,\ldots,2}([0,1] \times \mathbb{R}^K) \) and fixed \( \tau_2, \ldots, \tau_K \in [0,1] \). Then \( \left( \frac{\partial}{\partial x_1} f(t, W_t, W_{\tau_2}, \ldots, W_{\tau_K}) \right)_{t \in [0,1]} \) is Skorohod integrable and
\[ f(1, W_1, W_{\tau_2}, \ldots, W_{\tau_K}) - f(0, W_0, W_{\tau_2}, \ldots, W_{\tau_K}) = \int_0^1 \frac{\partial}{\partial x_1} f^\circ(t, \tau_2, \ldots, W_t, \ldots, W_{\tau_K}) dW_t + \int_0^1 \mathcal{L} f^\circ(t, \tau_2, \ldots, W_t, \ldots, W_{\tau_K}) dt, \]
with \( f^\circ \in C^{1;1;2,\ldots,2}([0,1]^K \times \mathbb{R}^K) \) from Proposition 2.5 and the differential operator \( \mathcal{L} \) is defined as
\[ \mathcal{L} := \left( \sum_{1 \leq k \leq K} \frac{\partial}{\partial t_k} + \frac{1}{2} \sum_{1 \leq k, l \leq K} \frac{\partial^2}{\partial x_k \partial x_l} \right). \]
It is mainly due to the following integration by parts formula which can be checked by S-transform and will be useful below:

\[
\begin{align*}
& a_{l_1, \ldots, l_K}(1) W_{\tau_K}^{\gamma_{l_1}} \cdots \gamma_{l_K} - a_{l_1, \ldots, l_K}(0) W_{\tau_K}^{\gamma_{l_1}} \cdots \gamma_{l_K} \\
& = \int_0^1 l_1 a_{l_1, \ldots, l_K}(t) W_t^{\gamma_{l_1}-1} \cdots \gamma_{l_K} dW_t \\
& \quad + \int_0^1 \left(a'_{l_1, \ldots, l_K}(t)\right) W_t^{\gamma_{l_1}} \cdots \gamma_{l_K} dt,
\end{align*}
\]

where \( a_{l_1, \ldots, l_K}(\cdot) \in C^1([0, 1]) \) for all \( l_1, \ldots, l_K \geq 0 \).

3. The Main Result

In [11] we specified some sufficient and equivalent assumptions for Skorohod integrands to ensure exact simulation, i.e. \( e_n = 0 \). Among them we had (Theorem 17):

(1) There exists a Wick-analytic representation of \((u_t)_{t \in [0,1]}\) as in Definition 2.4.

(2) The analytic representation fulfills \( f^\circ \in C^1, \ldots, C^2 \times [0,1] \times \mathbb{R}^K \) and \( L f^\circ = 0 \) on \([0,1] \times \mathbb{R}^K\).

This motivated the following optimal approximation result which covers the ordinary Itô case ([11, Theorem 21]):

**Theorem 3.1.** Suppose \( K \in \mathbb{N}, f^\circ \in C^1, \ldots, C^2([0, 1] \times \mathbb{R}^K) \) with some linear and Hölder growth conditions on \( L f^\circ, \tau_2, \ldots, \tau_K \in [0,1] \) are fixed and let

\[
c_1 := \frac{1}{\sqrt{12}} \left( \int_0^1 \mathbb{E}[|L f^\circ(s, \tau_2, \ldots, \tau_K, W_s, W_{\tau_2}, \ldots, W_{\tau_K})|^2] ds \right)^{1/2}.
\]

Then it is

\[
\lim_{n \to \infty} n \cdot \mathbb{E}\left[\left( \int_0^1 u_s dW_s - \mathbb{E}\left[\int_0^1 u_s dW_s|\mathcal{P}_n \cup \{W_{\tau_2}, \ldots, W_{\tau_K}\}\right]\right]^2 \right]^{1/2} = c_1.
\]

The convergence rate in Theorem 3.1 is essentially based on the exact approximation of the nonadapted parts \( W_{\tau_2}, \ldots, W_{\tau_K} \) in \( \mathbb{E}[|I - \mathbb{E}[I]| \mathcal{P}_n \cup \{W_{\tau_2}, \ldots, W_{\tau_K}\}]\).

Let us consider the following example of a stochastic exponential random variable without a finite nonadapted input:

**Example 3.2.** For a fixed \( T \in (0,1) \setminus \mathbb{Q} \) we denote the fractional part as

\[
\{kT\} := kT - \lfloor kT \rfloor
\]

and notice \( \{kT\} \in (0,1) \setminus \mathbb{Q} \) for all \( k \in \mathbb{N} \). Then we define the infinite chaos random variable depending on an infinite number of different nonadapted parts \( W_{\{kT\}}, k \in \mathbb{N} \), as

\[
X_T = \sum_{k \geq 0} \frac{1}{k!} h_{\{kT\}}^T(W_{\{kT\}}) = \sum_{k \geq 0} \frac{1}{k!} W_{\{kT\}}^{\gamma_{kT}}.
\]
Due to (2.3), we observe
\[ \sum_{k=0}^{\infty} (k + 1) \| \pi_k(X_T) \|^2_{L^2(\Omega)} = \sum_{k \geq 0} \frac{k + 1}{k!} \{ kT \}^k < \infty, \]
which gives the existence of the Skorohod integral \( \int_0^1 u_s \, dW_s \) for the stationary integrand \( (u_t = X_T)_{t \in [0,1]} \). One can easily check by (2.4) that \( X_T \in L^p(\Omega) \) for all \( p > 0 \). Notice that \( X_T \) is in some sense an infinite order Wick-analytic functional and by [11, Proposition 10] the analytic representation
\[ X_T = f(W_0, W_{\tau_2}, W_{\tau_K}, \ldots) \]
is infinitely differentiable in all variables. Thus, in contrast to Theorem 3.1, no finite set \( \{ W_{\tau_2}, \ldots, W_{\tau_K} \} \) is sufficient for the proof technique in Theorem 3.1.

This motivates to consider optimal approximation of Skorohod integrals with unattainable nonadapted parts. Surprisingly we have the following low convergence rate result in contrast to Theorem 3.1. We consider the optimal approximation error with respect to the equidistant discretization
\[ e_n := \mathbb{E} \left[ \left( \int_0^1 u_s \, dW_s - \mathbb{E} \left[ \int_0^1 u_s \, dW_s \bigg| \mathcal{P}_n \right] \right)^2 \right]^{1/2}. \]
The proofs of the following theorems are postponed to the next section.

**Theorem 3.3.** Suppose \( K \in \mathbb{N}, f^o \in C^{1,\ldots,1;3,\ldots,3}([0,1]^K \times \mathbb{R}^K), \tau_2, \ldots, \tau_K \in [0,1] \) are fixed and let
\[ c_2 = \frac{1}{2} \left( \sum_{\tau_i \not \in \mathbb{Q}} \mathbb{E} \left[ \left( \int_0^1 \frac{\partial}{\partial x_i} f^o(s, \tau_2, \ldots, \tau_K, W_s, W_{\tau_2}, \ldots, W_{\tau_K}) \, dW_s \right)^2 \right] \right)^{1/2}. \]
Then there exists a strictly increasing sequence of integers \( (n_k)_{k \in \mathbb{N}} \), such that
\[ \lim_{k \to \infty} n_k^{1/2} e_{n_k} = c_2. \]

**Remark 3.4.** The smoothness assumptions and Theorem 2.6 ensure the existence of all Skorohod integrals involved.

A similar optimal rate appears for Itô integrals of discontinuous integrands:

**Theorem 3.5.** Suppose \( N \subset (0,1) \setminus \mathbb{Q} \) a countable set of discontinuities, \( f \in C^{1,2}([0,1] \setminus N \times \mathbb{R}) \) such that \( f(t, \cdot) \) is nonlinear for all fixed \( t \in [0,1] \setminus N \). Then for the Itô integral \( I = \int_0^1 f(s, W_s) \, dW_s \),
and the constant
\[ c_2 = \frac{1}{2} \left( \int_0^1 \mathbb{E} \left[ \left( \frac{\partial}{\partial x} f(s, W_s) \right)^2 \right] \, ds \right)^{1/2}, \]
there exists a strictly increasing sequence of integers \( (n_k)_{k \in \mathbb{N}} \), such that
\[ \lim_{k \to \infty} n_k^{1/2} e_{n_k} = c_2. \]
Example 3.6. In particular, we obtain strictly increasing sequences \((n_k)_{k \in \mathbb{N}}\) such that, as \(k\) tends to infinity:

1. For the Itô integral \(I = \int_0^{1/\pi} e^{W_s} dW_s\),
   \[ n_k e_{n_k}^2 \sim \frac{1}{8} (1 - e^{-2/\pi}), \]

2. For the Skorohod integral \(I = \int_0^{1} e^{W_{1/s}} dW_s\),
   \[ n_k e_{n_k}^2 \sim \frac{1}{4} (1 + \pi^{-2}) e^{2/\pi}. \]

4. Proofs of Theorem 3.3 and Theorem 3.5

In the proofs we will frequently make use of:

Remark 4.1. The Brownian bridge is denoted by \(B^n_t := W_t - W^\text{lin}_t\) and yields the expansion

\[ W^{\circ k} - (W^\text{lin})^{\circ k} = B^n_t \circ \sum_{j=1}^{k} W^{\circ k-j} \circ (W^\text{lin})^{\circ j-1}. \tag{4.1} \]

Some elementary computations via (2.3) give for all \(s, t \in [0, 1], i, j \in \{1, \ldots, n\}:

\[ E[W^n_s W^\text{lin}_t] = \begin{cases} \frac{nt}{n} + n(t - \frac{nt}{n})(s - \frac{nt}{n}) & [ns] = [nt] \\ \frac{nt}{n} & [ns] \neq [nt] \end{cases} \leq s \wedge t, \tag{4.2} \]

\[ E[B^n_t W^\text{lin}_t] = \delta_{[nt], [nt]} (s \wedge t - \frac{nt}{n})(1 - n(s \vee t - \frac{nt}{n})) \leq \frac{1}{4n}, \tag{4.3} \]

\[ \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} E[B^n_s B^n_t] ds dt = \delta_{i,j} \frac{1}{12n^3}. \tag{4.5} \]

The asymptotic behaviour relies on the following worst case convergence rate of \(E[(B^n_t)^2]\) as \(n\) tends to infinity:

Proposition 4.2. Suppose \(t \in (0, 1) \setminus \mathbb{Q}\). Then there exists a strictly increasing sequence of integers \((n_k)_{k \in \mathbb{N}}\) such that

\[ E[(B^n_t)^2] = E[B^n_t W_t] \sim \frac{1}{4n_k}. \]

Proof. Due to the Weyl equidistribution theorem [15], for every irrational \(t \in (0, 1)\) there exists a strictly increasing sequence of integers \((n_k)_{k \in \mathbb{N}}\) with

\[ \lim_{k \to \infty} (n_k t - [n_k t]) = 1/2. \]

Hence, by (4.3),

\[ E[(B^n_t)^2] = E[B^n_t W_t] = (t - [nt]/n_k)(1 - n_k t - [n_k t]) \sim \frac{1}{4n_k}. \]

\[ \square \]
We firstly consider the simpler Theorem 3.5.

Proof of Theorem 3.5. We consider only the integrand \( f(s, x) \) for some \( f \in C^1([0, 1] \times \mathbb{R}) \) and a fixed \( t \in (0, 1) \setminus \mathbb{Q} \). Then the statement of the theorem follows by linearity and the Itô isometry. The Wick-analytic representation (2.1) gives

\[
 f(s, W_s) = \sum_{k \geq 0} a_k(s)W_s^\circ_k
\]

with \( a_k(\cdot) \in C^1([0, 1]) \). Due to the Wiener chaos decomposition it suffices to consider the individual chaoses \( a_k(s)W_s^\circ_k \). Thanks to (2.3) and Remark 4.1, for all \( s, t \in [0, 1] \), we have

\[
 \mathbb{E} \left[ (W_t^\circ_k - (W_t^\text{lin})^\circ_k) (W_s^\circ_k - (W_s^\text{lin})^\circ_k) \right] - \mathbb{E}[B_t^n B_s^n] \mathbb{E} \left[ (kW_t^\circ_{k-1}) (kW_s^\circ_{k-1}) \right] \to 0
\]
as \( n \) tends to infinity. Thus, by the integration by parts formula in (2.5) and Remark 4.1 along the subsequence in Proposition 4.2, we conclude

\[
 \begin{align*}
 \mathbb{E} \left[ \left( \int_0^t a_k(s)W_s^\circ_k dW_s - \mathbb{E}[\int_0^t a_k(s)W_s^\circ_k dW_s | \mathcal{F}_n] \right)^2 \right] \\
 &\quad = \mathbb{E} \left[ \left( \frac{a_k(t)}{k+1} (W_t^\circ_{k+1} - (W_t^\text{lin})^\circ_{k+1}) \\
 &\quad - \sum_{i=1}^n \int_{\frac{i-1}{n} \wedge t}^{\frac{i}{n} \wedge t} a_k'(s) (W_s^\circ_{k+1} - (W_s^\text{lin})^\circ_{k+1}) ds \right)^2 \right] \\
 &\quad \sim \mathbb{E}[(B_t^n)^2] \mathbb{E} \left[ (a_k(t)W_t^\circ_k - \int_0^t a_k'(s)W_s^\circ_k ds)^2 \right] \\
 &\quad \sim \frac{1}{4n} \mathbb{E} \left[ \left( \int_0^t a_k(s)kW_s^\circ_{k-1} dW_s \right)^2 \right].
\end{align*}
\]

Hence, \( a_k(s)kW_s^\circ_{k-1} = \frac{\partial}{\partial x}a_k(s)W_s^\circ_k \) and the Itô isometry yield the asserted constant. \( \square \)

Proof of Theorem 3.3. The proof is divided into three steps. In the first step we consider the simplest case of nonadapted integrands

\[
 F^\circ(W_s, W_{\tau_2})
\]

for some Wick-analytic functional \( F^\circ \). Then, in the second step, we conclude the statement for Wick-analytic functionals

\[
 F^\circ(W_s, W_{\tau_2}, \ldots, W_{\tau_K}).
\]

Finally, in the last step, we complete the proof to arbitrary integrands in the theorem.

Step 1: Let the simple integrand

\[
 F^\circ(W_s, W_{\tau_2}) = \sum_{k \geq 0, l \geq 0} a_{k,l} W_s^\circ_{l-1} \circ W_{\tau_2}^\circ_k.
\]
(This strange form is chosen to simplify the integration by parts formula in (2.5)) Firstly, due to the expansion in (4.1), (2.3) and (4.2)-(4.4), for \( k, k', l, l' \in \mathbb{N}, t \in [0, 1] \), we observe

\[
\begin{align*}
\mathbb{E} \left[ (W_t^\circ k - (W_t^\text{lin})^\circ k) \circ W_t^\circ l \right] &= \sum_{i=1}^{k} \mathbb{E} \left[ (B_t^n \circ W_t^\circ k - i \circ (W_t^\text{lin})^\circ i - 1 \circ W_t^\circ l) \left( B_t^n \circ W_t^\circ k' - j \circ (W_t^\text{lin})^\circ j - 1 \circ W_t^\circ l' \right) \right] \\
&= \mathbb{E} (B_t^n)^2 \sum_{i=1}^{k} \sum_{j=1}^{k'} (k - i)(k' - j) \times \mathbb{E} \left[ (W_t^\circ k - i \circ (W_t^\text{lin})^\circ i - 1 \circ W_t^\circ l) \left( W_t^\circ k' - j \circ (W_t^\text{lin})^\circ j - 1 \circ W_t^\circ l' \right) \right]. \quad (4.6)
\end{align*}
\]

Making use of (2.3) and (4.6), we have the two sums

\[
\begin{align*}
e_n^2 &= \mathbb{E} \left[ \left( \sum_{k,l \geq 0} a_{k,l} (W_{\tau_2}^\circ k - (W_{\tau_2}^\text{lin})^\circ k) \circ W_{\tau_2}^\circ l \right)^2 \right] \\
&= \mathbb{E} (B_{\tau_2}^n)^2 \sum_{k+l=k'+l' \geq 1} a_{k,l} a_{k',l'} \times \mathbb{E} \left[ \left( \sum_{i=1}^{k} W_{\tau_2}^\circ k - i \circ (W_{\tau_2}^\text{lin})^\circ i - 1 \circ W_{\tau_2}^\circ l \right) \left( \sum_{j=1}^{k'} W_{\tau_2}^\circ k' - j \circ (W_{\tau_2}^\text{lin})^\circ j - 1 \circ W_{\tau_2}^\circ l' \right) \right] \\
&\quad + \mathbb{E} (B_{\tau_2}^n W_{\tau_2})^2 \sum_{k+l=k'+l' \geq 1} a_{k,l} a_{k',l'} \mathbb{E} \left[ \sum_{i=1}^{k} (k - i)W_{\tau_2}^\circ k - i - 1 \circ (W_{\tau_2}^\text{lin})^\circ i - 1 \circ W_{\tau_2}^\circ l \right.
\end{align*}
\]

\[
\times \left. \sum_{j=1}^{k'} (k' - j') W_{\tau_2}^\circ k' - j' - 1 \circ (W_{\tau_2}^\text{lin})^\circ j' - 1 \circ W_{\tau_2}^\circ l' \right]
\]

\[
= X_2^n + X_2^n. \quad (4.7)
\]

Obviously, if \( \tau_2 \in \mathbb{Q} \), then we can choose a strictly increasing sequence of indices \( n \in \mathbb{N} \) such that \( B_{\tau_2}^n = 0 \) and \( e_n^2 = 0 \).

A computation via (2.3) and (4.1) shows that the \( L^2 \)-norm of the difference

\[
\sum_{i=1}^{k} W_{\tau_2}^\circ k - i \circ (W_{\tau_2}^\text{lin})^\circ i - 1 \circ W_{\tau_2}^\circ l - (kW_{\tau_2}^\circ (k-1) \circ W_{\tau_2}^\circ l)
\]

behaves in upper bounds as the \( L^2 \)-norm of

\[
B_{\tau_2}^n \circ k(k - 1)W_{\tau_2}^\circ (k-1) \circ W_{\tau_2}^\circ l.
\]
Hence, via (4.3)-(4.5) in Remark 4.1 and the assumptions on the integrand, we conclude for
\[ f_n^2 := \mathbb{E}[(B^n_{\tau_2})^2] \sum_{k+l=k'+l' \geq 1} a_{k,l}^n a_{k',l'}^n \mathbb{E} \left[ \left( k W_{\tau_2}^{(k-1)} \circ W_1^{(k')} \right) \left( k' W_{\tau_2}^{(k'-1)} \circ W_1^{(k')} \right) \right] \]
\[ = \mathbb{E}[(B^n_{\tau_2})^2] \mathbb{E} \left[ \left( \int_0^1 \frac{\partial}{\partial x_2} F^\circ(W_s,W_{\tau_2})dW_s \right)^2 \right], \]
that
\[ |X_n^1 - f_n^2| \in O(n^{-2}). \tag{4.8} \]

By a similar reasoning, via (2.3) and Remark 4.1, it is
\[ \mathbb{E} \left[ \left( \int_0^1 \frac{\partial^2}{\partial x_2^2} F^\circ(W_s,W_{\tau_2})dW_s \right)^2 \right] \]
\[ = \sum_{k+l=k'+l' \geq 1} a_{k,l}^n a_{k',l'}^n \mathbb{E} \left[ \left( \begin{pmatrix} k \\ 2 \end{pmatrix} W_{\tau_2}^{(k-2)} \circ W_1^{(k')} \right) \left( \begin{pmatrix} k' \\ 2 \end{pmatrix} W_{\tau_2}^{(k'-2)} \circ W_1^{(k')} \right) \right], \]
and for
\[ g_n^2 := \mathbb{E}[(B^n_{\tau_2} W_{\tau_2})^2] \mathbb{E} \left[ \left( \int_0^1 \frac{\partial}{\partial x_2} F^\circ(W_s,W_{\tau_2})dW_s \right)^2 \right] \]
we have
\[ |X_n^2 - g_n^2| \in O(n^{-3}). \tag{4.9} \]

The Skorohod integrals \( \int_0^1 \frac{\partial}{\partial x_2} F^\circ(W_s,W_{\tau_2})dW_s \) and \( \int_0^1 \frac{\partial^2}{\partial x_2^2} F^\circ(W_s,W_{\tau_2})dW_s \) involved exist by the continuity assumptions and Theorem 2.6. Hence, by (4.7)-(4.9), we conclude
\[ e_n^2 = f_n^2 + O(n^{-2}). \]

With the sequence in Proposition 4.2, this yields
\[ \lim_{k \to \infty} n_k \cdot e_n^2 = \frac{1}{4} \mathbb{E} \left[ \left( \int_0^1 \frac{\partial}{\partial x_2} F^\circ(W_s,W_{\tau_2})dW_s \right)^2 \right]. \tag{4.10} \]

**Step 2:** Now we make use of the shorthand notations
\[ \bar{l} := (l_1, \ldots, l_K) \in \mathbb{N}^K, \quad |\bar{l}| := \sum l_i, \quad \partial_{\bar{l}} := \frac{\partial}{\partial s} a_{l_1, \ldots, l_K}, \]
dealing with a Wick-analytic functional integrand
\[ F^\circ(W_s,W_{\tau_2},\ldots,W_{\tau_K}) = \sum_{l_1,\ldots,l_K \geq 0} a_{l_1,\ldots,l_K} W_{\tau_2}^{(l_1-1)} \circ W_{\tau_2}^{(l')} \circ \cdots \circ W_{\tau_K}^{l_K}, \]
we make use of the multivariate expansion of (4.1) as
\[ W_{\tau_2}^{(l')} \circ \cdots \circ W_{\tau_K}^{(l_K)} = (W_{\tau_2}^{(l_1)})^{(l')} \circ \cdots \circ (W_{\tau_K}^{(l_k)})^{(l_K)}, \]
Then, via (2.3), Remark 4.1 and (4.11), for appropriate and sufficiently large \( n \) we obtain similarly to (4.7),

\[
\epsilon_n^2 = \sum_{\tau \not\in \mathbb{Q}} \mathbb{E}[(B_{\tau}^n)^2] \sum_{|\ell| = |\ell'| \geq 1} a_\ell a_{\ell'} \\
\times \mathbb{E} \left[ \sum_{i=1}^{l_\ell} W_{\tau_i}^{\text{olt}_{i-1}} \circ (W_{\tau_i}^{\text{lin}})^{\text{olt}_{i-1}} \circ W_{\tau_{i-1}}^{\text{olt}_{i-1}} \circ \cdots \circ (W_{\tau_1}^{\text{lin}})^{\text{olt}_{1}} \circ W_1^{\text{olt}_{1}} \\
\times \sum_{i'=1}^{l_{\ell'}} W_{\tau_i}^{\text{olt}'_{i'-1}} \circ (W_{\tau_i}^{\text{lin}})^{\text{olt}'_{i'-1}} \circ W_{\tau_{i-1}}^{\text{olt}'_{i-1}} \circ \cdots \circ (W_{\tau_1}^{\text{lin}})^{\text{olt}'_{1}} \circ W_1^{\text{olt}'_{1}} \\
+ \sum_{\tau \not\in \mathbb{Q}} \mathbb{E}[(B_{\tau}^n W_{\tau})^2] \sum_{|\ell| = |\ell'| \geq 1} a_\ell a_{\ell'} \\
\times \mathbb{E} \left[ \sum_{i=1}^{l_\ell} (l_i - i) W_{\tau_i}^{\text{olt}_{i-1}} \circ (W_{\tau_i}^{\text{lin}})^{\text{olt}_{i-1}} \circ W_{\tau_{i-1}}^{\text{olt}_{i-1}} \circ \cdots \circ W_1^{\text{olt}_{1}} \\
\times \sum_{i'=1}^{l_{\ell'}} (l'_i - i') W_{\tau_i}^{\text{olt}'_{i'-1}} \circ (W_{\tau_i}^{\text{lin}})^{\text{olt}'_{i'-1}} \circ W_{\tau_{i-1}}^{\text{olt}'_{i-1}} \circ \cdots \circ W_1^{\text{olt}'_{1}} \\
=: X_1^n + X_2^n.
\]

Here the indices \( I \) with \( \tau \in \mathbb{Q} \) are ignored due to \( B_{\tau_1}^n = 0 \) for an appropriate sequence of indices. Moreover, for sufficiently large \( n \) we have \([n \tau_i] \neq [n \tau_{i'}] \) for all \( \tau_i \neq \tau_{i'} \), and thus, via (4.3),

\[
\mathbb{E}[B_{\tau_i}^n W_{\tau_i}] = \mathbb{E}[B_{\tau_i}^n, B_{\tau_{i'}}^n] = 0.
\]

Thus, by an analogous reasoning as for (4.8) and the assumptions on the integrand, we conclude

\[
\mathbb{E} \left[ \left( \int_0^1 \frac{\partial}{\partial x_1} F^\circ(W_s, W_{\tau_2}, \ldots, W_{\tau_K})dW_s \right)^2 \right]
= \sum_{|\ell| = |\ell'| \geq 1} a_\ell a_{\ell'} \mathbb{E} \left[ (W_{\tau_1}^{\text{olt}_{1}} \circ \cdots \circ l_\ell W_{\tau_{l-1}}^{\text{olt}_{l-1}} \circ \cdots \circ (W_{\tau_1}^{\text{lin}})^{\text{olt}_{1}} \circ W_1^{\text{olt}_{1}}) \\
\times (W_{\tau_1}^{\text{olt}_{1}} \circ \cdots \circ l'_{\ell'} W_{\tau_{l'-1}}^{\text{olt}'_{l'-1}} \circ \cdots \circ (W_{\tau_1}^{\text{lin}})^{\text{olt}'_{1}} \circ W_1^{\text{olt}'_{1}}) \right]
\]

and

\[
|X_1^n - f_{11}^2| \in \mathcal{O}(n^{-2})
\]

for

\[
f_{11}^2 := \sum_{\tau_i \not\in \mathbb{Q}} \mathbb{E}[(B_{\tau_i}^n)^2] \mathbb{E} \left[ \left( \int_0^1 \frac{\partial}{\partial x_1} F^\circ(W_s, W_{\tau_2}, \ldots, W_{\tau_K})dW_s \right)^2 \right].
\]
Similarly to (4.9), we have
\[ |X^n_2 - g^n_1| \in O(n^{-3}) \]
for
\[ g^n_2 := \sum_{\tau_i \notin \mathbb{Q}} \mathbb{E}[B^n_{\tau_2}W_{\tau_2}]^2 \mathbb{E}\left[ \left( \int_0^1 \frac{\partial^2}{\partial x_t^2} F^0(W_s, W_{\tau_2}, \ldots, W_{\tau_K}) dW_s \right)^2 \right]. \]

Proposition 4.2 yields again an appropriate sequence with
\[ \lim_{k \to \infty} n_k \cdot e^n_{n_k} = \frac{1}{4} \sum_{\tau_i \notin \mathbb{Q}} \mathbb{E}\left[ \left( \int_0^1 \frac{\partial}{\partial x_t} F^0(W_s, W_{\tau_2}, \ldots, W_{\tau_K}) dW_s \right)^2 \right]. \]

We notice that all Skorohod integrals above exist by the Skorohod Itô formula in Theorem 2.6.

**Step 3:** The generalization to arbitrary integrands follows by a straightforward extension of the computations above and the Wiener chaos decomposition. For the arbitrary integrand, we have
\[ f(t, \tau_2, \ldots, \tau_k, W_t, \ldots, W_{\tau_K}) = \sum_{i_1, \ldots, i_k \geq 0} a_{i_1, \ldots, i_k}(t) W^{\odot i_1}_t \circ W^{\odot i_2}_{\tau_2} \circ \cdots \circ W^{\odot i_k}_{\tau_k}, \]
where the continuity assumption implies for all coefficients \( a_{i_1, \ldots, i_k} \in C^1([0, 1]). \)

We consider exemplary the integrand
\[ f(s, t, W_s, W_t) = a(s) \cdot W^{\odot (l-1)}_s \circ W^{\odot m}_t \]
for some fixed \( t \in (0, 1) \setminus \mathbb{Q}, m, l \geq 1. \) The generalization is straightforward.

Thanks to the integration by parts formula (2.5), we have
\[ e^n_s = \mathbb{E}\left[ (a(1)W^{\odot l}_t \circ (W^{\odot m}_s - W^{\odot m}_t)^{\odot m}) - \sum_{i=1}^n \int_{(i-1)/n}^{i/n} a'(s)(W^{\odot l}_s \circ W^{\odot m}_s - (W^{\odot l}_s \circ (W^{\odot m}_s)^{\odot m})) ds \right]^2 \]
\[ = \mathbb{E}\left[ (a(1)W^{\odot l}_1 \circ (W^{\odot m}_s - (W^{\odot m}_1)^{\odot m})^2 \right] - 2 \sum_{i=1}^n \int_{(i-1)/n}^{i/n} a'(s)a(1) \times \mathbb{E}\left[ (W^{\odot l}_1 \circ ((W^{\odot m}_s - (W^{\odot m}_1)^{\odot m})) (W^{\odot l}_s \circ W^{\odot m}_s - (W^{\odot l}_s \circ (W^{\odot m}_s)^{\odot m}) ds \right] \]
\[ + \mathbb{E}\left[ \left( \sum_{i=1}^n \int_{(i-1)/n}^{i/n} a'(s)(W^{\odot l}_s \circ W^{\odot m}_s - (W^{\odot l}_s \circ (W^{\odot m}_s)^{\odot m})) ds \right)^2 \right] \]
\[ =: I^n_1 - 2I^n_2 + I^n_3 \].

By (4.10) in Step 1, for the sequence of integers in Proposition 4.2, we have
\[ n_k \cdot I^n_1 - \frac{1}{4} \mathbb{E}\left[ (a(1)W^{\odot l}_1 \circ mW^{\odot m-1}_t)^2 \right] \to 0, \]
as $k$ tends to infinity. By the expansion (4.11) in Step 2 and (4.10), we conclude similarly

$$n_k \cdot I_2^n - \frac{1}{4} E \left[ (a(1)W_1^{cl} \circ mW_t^{\circ m-1}) \left( \int_0^1 a'(s)W_s^{cl} \circ mW_t^{\circ m-1} ds \right) \right] \to 0,$$

$$n_k \cdot I_3^n - \frac{1}{4} E \left[ \left( \int_0^1 a'(s)W_s^{cl} \circ mW_t^{\circ m-1} ds \right)^2 \right] \to 0.$$

Due to Proposition 2.2 and the integration by parts formula in (2.5),

$$a(1)W_1^{cl} \circ mW_t^{\circ m-1} - \int_0^1 a'(s)W_s^{cl} \circ mW_t^{\circ m-1} ds = \int_0^1 a(s)W_s^{cl} \circ mW_t^{\circ m-1} dW_s.$$

Hence we obtain

$$\lim_{k \to \infty} n_k e_{n_k}^2 = \frac{1}{4} E \left[ \left( \int_0^1 a(s)W_s^{cl} \circ mW_t^{\circ m-1} dW_s \right)^2 \right],$$

$$= \frac{1}{4} E \left[ \left( \int_0^1 \frac{\partial}{\partial x_2} f(s,t,W_s,W_t) dW_s \right)^2 \right].$$

Thanks to the same arguments as above and in Step 1 and Step 2, the Wiener chaos decomposition completes the proof of the statement for arbitrary integrands $f \in C^{1,3,3}(0,1)^K \times \mathbb{R}^K)$. The existence of all Skorohod integrals involved is justified by Theorem 2.6. □

5. Examples of Further Irregularity

In this section we discuss optimal approximation for Skorohod integrals of integrands beyond the continuity conditions in Theorem 3.1 and Theorem 3.3. In fact, we present Skorohod integrals $I$ such that the mean square error

$$e_n = E[(I - E[I|\mathcal{F}_n])^2]^{1/2}$$

exhibits a lower convergence rate than $n^{-1/2}$. Moreover we construct Skorohod integrals with infinite nonadapted part (i.e. an integrand which depends on an infinite set $\{W_t, t > 0\}$) and optimal rates $n^{-\alpha}$ with $\alpha \in (0,1)$.

**Proposition 5.1.** Let $t \in [0,1]$ and the Skorohod integral

$$I = \int_0^1 W_t dW_s = |W_t| \circ W_1.$$

Then there exists in each case a strictly increasing sequence of integers $(n_k)_{k \in \mathbb{N}}$ such that:

(i) If $t \in \mathbb{Q}$, then $e_{n_k} = 0$.

(ii) If $t \notin \mathbb{Q}$, then for all $\varepsilon \in (0,1/4)$,

$$\frac{t^{3/4}}{\sqrt{2\pi}} n_k^{-1/4} \leq e_{n_k} \in \mathcal{O}(n_k^{-1/4+\varepsilon}). \quad (5.1)$$
Proof. (i) The Wiener chaos decomposition (see e.g. [9, p. 65]) gives
\[ |W_t| = \sqrt{\frac{2t}{\pi}} \sum_{m \geq 0} \frac{(-1)^{m+1}t^{-m}}{(2m-1)(2m)!} W_t^{(m)} . \] (5.2)
Applying Proposition 2.3 on (5.2), we have
\[ \mathbb{E}[I|\mathcal{P}_n] = \sqrt{\frac{2t}{\pi}} \sum_{m \geq 0} \frac{(-1)^{m+1}t^{-m}}{(2m-1)(2m)!} (W_t^{(m)})^2 \circ W_1 . \] (5.3)
Thus, for \( t \in \frac{1}{m} \mathbb{N} \) and \( n \in m \mathbb{N} \) we obtain \( W_t^{(m)} = W_t \) and exact simulation.

(ii) We firstly observe by the orthogonality of \((W_t^{(m)} - (W_t^{(m)})^2) \circ W_1 \) for different \( m \),
\[ e_n^2 = \frac{2t}{\pi} \sum_{m \geq 1} \frac{t^{-2m}}{(2m-1)(2m)!^2} \mathbb{E} \left[ (W_t^{(m)} - (W_t^{(m)})^2) \circ W_1 \right]^2 . \] (5.4)
Due to (4.6), the proof is based on upper and lower bounds of
\[ \mathbb{E} \left[ (W_t^{(m)} - (W_t^{(m)})^2) \circ W_1 \right] \left( (W_t^{(m)} - (W_t^{(m)})^2) \circ W_1 \right] . \]

Lower bounds: We define for shorthand \( z := \frac{t - \mathbb{E}(B_t^n)^2}{t} \leq 1 \).
Due to Remark 4.1, it is
\[ \min_{X,Y \in \{W_t,W_t^{(m)}\}} \mathbb{E}[XY] = \mathbb{E}[(W_t^{(m)})^2] = t - \mathbb{E}(B_t^n)^2 = tz . \] (5.5)
Hence, by (2.3) and (5.5), we have
\[ \mathbb{E} \left[ (W_t^{(m)} - (W_t^{(m)})^2) \circ W_1 \right] \left( (W_t^{(m)} - (W_t^{(m)})^2) \circ W_1 \right] \geq \mathbb{E} \left[ (W_t^{(m+1)} - (W_t^{(m+1)})^2) \circ W_1 \right] \geq k! (tz)^k . \]
Thanks to (4.6), ignoring the second term on the right hand side in (4.6), we obtain
\[ \mathbb{E} \left[ (W_t^{(m+1)} - (W_t^{(m+1)})^2) \circ W_1 \right]^2 \geq \mathbb{E}(B_t^n)^2 (2m)^2 (2m)! (tz)^{2m} . \] (5.6)
We observe for sufficiently large \( n \) that \( z \geq 0.96 \) and therefore
\[ \frac{1}{\sqrt{1-z^2}} - 1 = \frac{1}{\sqrt{1-z}} \left( \frac{z^2}{1+z} \right) \geq \frac{1}{2\sqrt{1-z}} . \]
Then, making use of (5.4), (5.6), the series
\[ \sum_{m \geq 1} \frac{(2m)^2}{(2m)!^2} = \sum_{m \geq 0} \left( \frac{2m}{m} \right) \left( \frac{z^2}{4} \right)^m = \frac{1}{\sqrt{1-z^2}} . \] (5.7)
(see e.g. [1, 2.1]) and \( \mathbb{E}(B_t^n) = t(1-z) \), we conclude for all \( \varepsilon \in (0,1/2) \) and sufficiently large \( n \):
\[ e_n^2 = \frac{2t}{\pi} \sum_{m \geq 1} \frac{t^{-2m}}{(2m-1)^2(2m)!^2} \mathbb{E} \left[ (W_t^{(m)} - (W_t^{(m)})^2) \circ W_1 \right]^2 . \]
Proposition 4.2, for all $\varepsilon$ and we obtain $e$

Thanks to (2.3) and (4.2), we have Since \[
\sum_{j=1}^{n} (j - 1)z^{j-1} \leq k!t^{k-1}z^{ijj-1}, \quad (5.10)\]
Since \[
\sum_{j=1}^{k} z^{j-1} \leq k, \text{ it remains to control the first sum on the right hand side. For every } \varepsilon \in (0, 1/2), \text{ the H"older inequality implies} \]
\[
\sum_{j=1}^{k} (j - 1)z^{j-1} \leq \left( \sum_{j=1}^{k} z^{j-1} \right)^{1/2+\varepsilon} \left( \sum_{j=1}^{k} (j - 1)z^{j-1} \right)^{1/2-\varepsilon} \]
\[
\leq \left( \frac{1 - z^{1/\varepsilon}}{1 - z^{1/2-\varepsilon}} \right)^{1/2+\varepsilon} \left( \int_{0}^{z} x^{j-1} \, dx \right)^{1/2-\varepsilon}. \quad (5.12)\]
For every $t \in (0, 1)$ it is $z \in (0, 1]$ for sufficiently large $n$. Thus, by $1/(1/2 + \varepsilon) > 1$ we observe
\[
(1 - z)^{1/2+\varepsilon} \left( \frac{1 - z^{1/\varepsilon}}{1 - z^{1/2-\varepsilon}} \right)^{1/2+\varepsilon} \leq 1.
\]
Hence, by (5.11)-(5.12), \[
\left( \int_{0}^{k} x^{1/2-\varepsilon} \, dx \right)^{1/2-\varepsilon} < k^{3/2-\varepsilon} \text{ and } \mathbb{E}(B_t^n)^2 = t(1 - z), \text{ we obtain} \]
\[
\mathbb{E}(B_t^n)^2 \sum_{i,j=1}^{k} z^{ijj-1} \leq \mathbb{E}(B_t^n)^2 z^{1/2+\varepsilon} 1^{1/2+\varepsilon} 3 k^{3/2-\varepsilon}. \]
We used the simple bounds
\[ \text{Remark 5.2} \]
metric series \( \varepsilon \in P \) we obtain a constant \( c > 0 \) due to (5.4) and the Stirling formula (2.5), it is
to the Wiener chaos decomposition in (5.2), we have and thus, making use of the arguments in the proof of Theorem 3.1, which carry
experiments we conjecture for a constant \( c > 0 \) and the subtle handling of the sum in (5.14).

By the sequence \((n_k)_{k \in \mathbb{N}}\) in Proposition 4.2, this gives \( c_{n_k}^2 \in \mathcal{O}(n_k^{-1/2+\varepsilon}) \) for all \( \varepsilon \in (0, 1/2) \).

**Remark 5.2.** Due to (2.3) and (4.2), for all \( 0 \leq i, j \leq k \), we observe the hypergeometric series
\[
\mathbb{E}\left[ (W_t^{\circ k-i} \circ (W_t^{\text{lin}})^{\circ j})(W_t^{\circ k-j} \circ (W_t^{\text{lin}})^{\circ j}) \right] = \sum_{l=0}^{k-(i+j)} t^{k-l} \frac{(k-i)!(k-j)!i!j!}{l!(k-i-l)!(k-j-l)!(i+j+k-l)!}.
\]

We used the simple bounds \( \geq k! t^k z^k \) and \( \leq k! t^k z^{i+j} \). From the proof and computer experiments we conjecture
\[
e_{n_k} \sim c n_k^{-1/4}
\]
for a constant \( c > 0 \). The computation of the constant \( c \) must rely on a more subtle handling of the sum in (5.14).

**Remark 5.3.** The following example shows that the regularity conditions in Theorem 3.1 are less sensitive. Let
\[
I = \int_0^1 g(s)|W_t|dW_s = \int_0^1 g(s)dW_s \circ |W_t|
\]
for some \( g \in C^1([0, 1]) \) and \( t \in \frac{1}{m} \mathbb{N} \) fixed. Then, by Proposition 2.2, \( n \in m\mathbb{N} \) and (2.5), it is
\[
\mathbb{E}[I|\mathcal{F}_n] = \left( g(1)W_1 - \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} g'(s)W_s^{\text{lin}}ds \right) \circ |W_t|
\]
and thus, making use of the arguments in the proof of Theorem 3.1, which carry over to the Wiener chaos decomposition in (5.2), we have
\[
e_n^2 = \sum_{i,j=1}^{n} \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} g'(s)g'(u)\mathbb{E}[(B_u^n \circ |W_t|)(B_u^n \circ |W_t|)]ds du
\]
\[
= \sum_{i=1}^{n} \int_{(i-1)/n}^{i/n} \int_{(i-1)/n}^{i/n} \mathbb{E}[B_u^n B_u^n]ds du g'((i-1)/n)^2\mathbb{E}[|W_t|^2] + \mathcal{O}(n^{-3})
\]
\[
= \frac{1}{12n^2} \left( \int_0^1 g'(s)^2ds \right) \mathbb{E}[|W_t|^2] + \mathcal{O}(n^{-3}).
\]
We notice that the constant \( \int_0^1 g'(s)^2 ds \mathbb{E}[|W_t|^2] \) equals \( \int_0^1 \mathbb{E}[\mathcal{L}f(s,t,W_s,W_t)^2] ds \) for \( f(s,t,W_s,W_t) = g(s)|W_t| \) in Theorem 3.1 if we ignore the discontinuities. This indicates further extensions of Theorem 3.1 to weaker assumptions. Under the same assumptions, by analogous arguments, for

\[
I = \int_0^1 g(s)|W_t| \circ W_s dW_s = \int_0^1 g(s)W_s dW_s \circ |W_t|,
\]

it is

\[
e_n^2 = \frac{1}{12n^2} \left( \int_0^1 g'(s)^2 \mathbb{E}\left[ (|W_t| \circ W_s)^2 \right] ds \right) + \mathcal{O}(n^{-3}).
\]

We notice that \( |W_t| \circ W_s^{\infty} \notin L^2(\Omega) \) due to (5.2) and (5.13). Hence for the next integrand of this type, \( (g(s)|W_t| \circ W_s^{\infty})_{s \in [0,1]} \), the Skorohod integral does not exist.

**Remark 5.4.** Inspired by the Wiener chaos decomposition in (5.3), for a fixed \( T \in (0,1) \setminus \mathbb{Q} \), we define the random variable

\[
X_q = \sum_{m \geq 0} \frac{1}{(2m)^{1+q}((2m)!!)^2} W_{r^{2m}}^{(mT)}.
\]

One can easily check by (5.7) that \( X_q \in L^2(\Omega) \) for all \( q \in (-1/2,1/2) \) and the Skorohod integral

\[
I_s = \int_0^1 X_q dW_s = X_q \circ W_1
\]

exists in \( L^2 \) with

\[
\mathbb{E}[I_s^2] \leq \sum_{m \geq 0} \frac{(2m+1)!}{(2m)^{2+2q}(2m)!!^2} (mT)^{2m} < \infty.
\]

Due to Proposition 4.2 there is no finite extension of \( \mathcal{P}_n \) such that \( I_q \) is exactly simulated. We sketch the optimal approximation results since the proofs can be done following the lines in the proof of Proposition 5.1. Similarly to (5.4), it is

\[
e_n^2 = \sum_{m \geq 1} \frac{1}{(2m)^{2+2q}(2m)!!^2} \mathbb{E}\left[ \left( W_{r^{2m}}^{(mT)} - (W_{r^{2m}}^{(mT)} \circ W_1) \right)^2 \right].
\]

Due to fractional calculus on (5.7), we conjecture for the contained sum

\[
\sum_{m \geq 0} (2m)^{-2q} \left( \frac{2m}{m} \right)^{2m} \{mT\}^{2m} \asymp (1 - \{mT\}^2)^{-1/2-2q}.
\]

Following the proof of Proposition 5.1, we assume with the the sequence from Proposition 4.2 for all \( q \in (-1/2,1/2) \) the asymptotic behaviour

\[
e_{nk}^2 \asymp n_k^{-1/2+2q}.
\]

**Remark 5.5.** Simpler Skorohod integrals such that no finite extension of \( \mathcal{P}_n \) is sufficient for exact approximation can be constructed as follows: Let \( g : [0,1] \to [0,1] \) be a function with \( g([0,1]) \notin \mathbb{Q}, |g([0,1])| = \infty \). Then the following Skorohod integral exists (is an element in the second Wiener chaos) and fulfills \( \int_0^1 W_{g(s)} dW_s = \).
\[ \sum_{j \in J} a_j W_{s_j} \diamond W_t \] for an infinite index set \( J \). Such examples let us assume that Skorohod integrals of the following type

\[ \int_0^1 f(W_{g(s)}) dW_s \]

for sufficiently irregular or discontinuous function \( g : [0, 1] \to [0, 1] \) and appropriate \( f \in C^2(\mathbb{R}) \), do not allow a finite extension of \( \mathcal{P}_n \) with exact approximation and the optimal approximation behaves according to the rather irregular case in Theorem 3.3.

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References


Peter Parczewski: Institute of Mathematics A5, 6, University of Mannheim, D-68131 Mannheim, Germany  
E-mail address: parczewski@math.uni-mannheim.de