Communications on Stochastic Analysis
Vol. 2, No. 1 (2008) 11-26

Serials Publications
www.serialspublications.com

QUADRATIC WIENER FUNCTIONALS OF SQUARE NORMS ON MEASURE SPACES

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Abstract. Stochastic oscillatory integrals associated with quadratic Wiener functionals obtained as square norms on measure spaces of first order Wiener chaos is investigated. As applications, the square norm of the Brownian sheet and quadratic Wiener functional related to the KdV equation will be studied.

1. Introduction

Abstract Wiener spaces were introduced by L. Gross in 1965 [3], and since then, have been playing a key role in infinite dimensional stochastic analysis. In this paper, we investigate the quadratic Wiener functional on an abstract Wiener space (X, H, ν) of the form

$$F = \frac{1}{2}\int_E (\nabla^* f_e)^2 \sigma(de), \tag{1.1}$$

where $(E, \mathcal{E}, \sigma)$ is a σ-finite measure space, $f_e \in H$ for every $e \in E$, and ∇^* stands for the adjoint operator of the Malliavin gradient ∇ on X. For the precise definitions, see Sections 2 and 3. We shall give an exact infinite product expression of stochastic oscillatory integral

$$\int_X e^{\zeta F + \nabla^* h} d\nu$$

for sufficiently small $\zeta \in \boldsymbol{C}$ and any $h \in H$. Moreover, it will be applied to the study of two quadratic Wiener functionals; the first one is

$$\mathfrak{h} = \int_{[0,T]^2} W(s,t)^2 ds dt,$$

the square norm of the Brownian sheet $\{W(s,t)\}_{(s,t)\in[0,T]^2}$ on $[0,T]^2$, and the second is the quadratic Wiener functionals representing reflecting potentials, which plays a key role in the study of soliton solutions to the KdV equation (cf. [9, 10]). We also apply our result to show the corresponding Lévy-Khinchin formulas.

The investigation of such functionals as F goes back to the work of Cameron and Martin ([1]), who studied the functional

$$\int_0^T W(s)^2 ds$$

2000 *Mathematics Subject Classification.* Primary 60H30; Secondary 60H20.
Key words and phrases. quadratic Wiener functional, infinite product, first order Wiener chaos, Brownian sheet, KdV equation.
* This research is supported in part by Grant-in-Aid for Scientific Research (B) 18340038.

of the 1-dimensional Brownian motion $\{W(s)\}_{s\geq 0}$ starting from the origin at time 0. The functional has a deep connection with the Schrödinger operator corresponding to harmonic oscillator; $(d/dx)^2 + x^2$. Recently, Deheuvels, Peccati, and Yor ([2]) studied the functional $\mathfrak{h}$ with the help of Karhunen-Loève expansions and the result due to Cameron-Martin. In their paper, a reason why one is interested in $\mathfrak{h}$ and hence why $\mathfrak{h}$ is not a useless generalization of the functional studied by Cameron-Martin, can be found. Our result, which is based on the complex change of variable formula on the abstract Wiener space obtained via the Malliavin calculus ([5]), covers all their exact expressions. See Section 4

A concrete bijective correspondence between reflectionless potentials and stochastic oscillatory integrals with Ornstein-Uhlenbeck processes in phase function was established in [4, 10]. In particular, in [10], quadratic Wiener functionals obtained as square norms of Wiener integrals with respect to the Brownian sheet played a fundamental role. In Section 5, such quadratic Wiener functionals will be studied as an example of quadratic Wiener functional of the form (1.1).

2. Preliminaries

In this section, we review several results on abstract Wiener spaces.

Let (X, H, ν) be a real abstract Wiener space, i.e., X is a real separable Banach space, H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$, which is imbedded in X densely and continuously, and ν is a Gaussian measure on $(X, \mathcal{B}(X))$, $\mathcal{B}(X)$ being the Borel σ-field of X, such that

$$\int_X e^{\sqrt{-1}\,\ell} d\nu = e^{-\|\ell\|_H^2/2} \quad \text{for any } \ell \in X^*,$$

where X^* is the dual space of X and we have used the standard identification of H^* and H to have the inclusion that $X^* \subset H^* = H \subset X$. For a separable Hilbert space K, we say that a K-valued Wiener functional F belongs to $\boldsymbol{D}^\infty(K)$ if F is infinitely differentiable in the sense of the Malliavin calculus and it and its Malliavin derivatives of all orders are p-integrable with respect to ν for any $p > 1$ (cf. [8]). Denoting by $H \otimes K$ the Hilbert space of Hilbert-Schmidt operators of H to K, we define the adjoint operator $\nabla^* : \boldsymbol{D}^\infty(H \otimes K) \to \boldsymbol{D}^\infty(K)$ of the Malliavin gradient ∇ by

$$\int_X \langle \nabla^* F, G \rangle_K d\nu = \int_X \langle F, \nabla G \rangle_{H\otimes K} d\nu \quad \text{for any } F \in \boldsymbol{D}^\infty(H \otimes K), G \in \boldsymbol{D}^\infty(K).$$

For a symmetric Hilbert-Schmidt operator $U : H \to H$, we set

$$Q_U = (\nabla^*)^2 U,$$

where we have thought of U as a constant function defined on X with values in the Hilbert space $H^{\otimes 2}$ of Hilbert-Schmidt operators on H. If U is of trace class, we can define

$$q_U = Q_U + \operatorname{tr} U.$$

It is easily checked that the third Malliavin derivative of functional $F \in \boldsymbol{D}^\infty(\boldsymbol{R})$ vanishes, i.e., $\nabla^3 F = 0$, if and only if F admits an expression as

$$F = \frac{1}{2} Q_U + \nabla^* h + c, \tag{2.1}$$

where

$$U = \nabla^2 F, \quad h = \int_X \nabla F d\nu, \quad \text{and} \quad c = \int_X F d\nu.$$

Applying the complex change of variables formula shown in [5], we obtain that

Proposition 2.1. *Let $U : H \to H$ be a symmetric Hilbert-Schmidt operator and $h \in H$. For $\zeta \in \boldsymbol{C}$ with $|\zeta| < 1/\|U\|_{\rm op}$, where $\|\cdot\|_{\rm op}$ stands for the operator norm, it holds that*

$$\int_X e^{(\zeta/2)Q_U + \eta\nabla^* h} d\nu = \big\{\det_2(I - \zeta U)\big\}^{-1/2} e^{\eta^2 \langle (I-\zeta U)^{-1}h, h\rangle_H/2}$$

for every $\eta \in \boldsymbol{C}$, where $\det_2$ *is the Carleman-Fredholm determinant, and $\langle\cdot,\cdot\rangle_H$ was extended complex bi-linearly to the complexified Hilbert space $H \otimes \boldsymbol{C}$ of H. If, in addition, U is of trace class, then*

$$\int_{\mathcal{W}} e^{(\zeta/2)q_U + \eta\nabla^* h} d\mu = \big\{\det(I - \zeta U)\big\}^{-1/2} e^{\eta^2 \langle (I-\zeta U)^{-1}h, h\rangle_H/2},$$

where det *is the Fredholm determinant.*

It is routine to extend the above identity to ζ's in much wider domain in $\boldsymbol{C}$ by holomorphic continuation.

We now recall the Lévy-Khinchin formulas of the distributions of Q_U and q_U given in [6]. Let $\{a_n\}_{n=1}^\infty$ be a sequence of eigenvalues of U repeated according to multiplicity. Define

$$f_U(x) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{n:\, xa_n > 0} \frac{1}{|x|} \exp(-x/a_n), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Proposition 2.2. *For any $\lambda \in \boldsymbol{R}$, it holds that*

$$\int_X e^{\sqrt{-1}\,\lambda Q_U/2} d\nu = \exp\bigg(\int_{\boldsymbol{R}} \{e^{\sqrt{-1}\,\lambda x} - 1 - \sqrt{-1}\,\lambda x\} f_U(x)dx\bigg). \tag{2.2}$$

If, in addition, U is of trace class, then

$$\int_X e^{\sqrt{-1}\,\lambda q_U/2} d\nu = \exp\bigg(\int_{\boldsymbol{R}} \{e^{\sqrt{-1}\,\lambda x} - 1\} f_U(x)dx\bigg). \tag{2.3}$$

Proof. The identity (2.2) was shown in [6, Theorem 2]. To see (2.3), it suffices to note that $\int_{\boldsymbol{R}} x f_U(x)dx = \operatorname{tr} U/2$. □

3. Square Norm on Measure Space

Let (X, H, ν) be a real abstract Wiener space, E a topological space, $\mathcal{E}$ its Borel σ-field, and σ a σ-finite measure on $(E, \mathcal{E})$. Consider a continuous mapping $E \ni e \mapsto f_e \in H$, where the topology of H is the strong one, i.e., comes from the norm. Assume that

(A) $$\int_E \|f_e\|_H^2 \sigma(de) < \infty.$$

Every $h \in H$ is an element of $\boldsymbol{D}^\infty(H)$, being thought of as a constant Wiener functional. The functional $\nabla^* h$ satisfies that

$$\|\nabla^* h\|_{L^p(\nu)} = C_p \|h\|_H, \quad p > 0, \quad \text{where } C_p = \left(\int_{\boldsymbol{R}} \frac{|x|^p}{\sqrt{2\pi}} e^{-x^2/2} dx \right)^{1/p}, \tag{3.1}$$

and $\|\cdot\|_{L^p(\nu)}$ denotes the norm of the L^p-space $L^p(\nu)$ associated with ν. It follows from this identity that

$$\|(\nabla^* f_e)^2 - (\nabla^* f_{e'})^2\|_{L^2(\nu)} \le C_4^2 \|f_e - f_{e'}\|_H \|f_e + f_{e'}\|_H, \quad e, e' \in E.$$

Thus the mapping $E \ni e \mapsto (\nabla^* f_e)^2 \in L^2(\nu)$ is strongly continuous, and hence strongly measurable. By virtue of Assumption (A) and (3.1),

$$\int_E \|(\nabla^* f_e)^2\|_{L^2(\nu)} \sigma(de) \le C_4^2 \int_E \|f_e\|_H^2 \sigma(de) < \infty.$$

Hence the function $E \ni e \mapsto (\nabla^* f_e)^2 \in L^2(\nu)$ is Bochner integrable. Define $F \in L^2(\nu)$ by the Bochner integral

$$F = \frac{1}{2} \int_E (\nabla^* f_e)^2 \sigma(de).$$

Due to Assumption (A), we can define $A : H \to H$ by

$$\langle Ah, g \rangle_H = \int_E \langle f_e, h \rangle_H \langle f_e, g \rangle_H \sigma(de), \quad h, g \in H.$$

It also follows from Assumption (A) that A is a non-negative definite, symmetric Hilbert-Schmidt operator of trace class and

$$\operatorname{tr} A = \int_E \|f_e\|_H^2 \sigma(de).$$

Proposition 3.1. *F belongs to $\boldsymbol{D}^\infty(\boldsymbol{R})$ and coincides with $q_A/2$.*

Proof. Let $p > 2$. Every bounded and measurable $G : X \to \boldsymbol{R}$ is in the dual space of $L^2(\nu)$, and due to (3.1), satisfies that

$$\left| \int_X F G d\nu \right| \le \frac{1}{2} \int_E \left| \int_X (\nabla^* f_e)^2 G d\nu \right| \sigma(de) \le \frac{C_{2p}^2}{2} \left(\int_E \|f_e\|_H^2 \sigma(de) \right) \|G\|_{L^q(\nu)},$$

where $q = p/(p-1)$. By Assumption (A), this implies that $F \in L^p(\nu)$.

For the H-valued functional $(\nabla^* f_e) f_e$, by virtue of Assumption (A) and (3.1), we have that

$$\|(\nabla^* f_e) f_e - (\nabla^* f_{e'}) f_{e'}\|_{L^2(\nu;H)} \le C_2 \{\|f_e\|_H + \|f_{e'}\|_H\} \|f_e - f_{e'}\|_H, \quad e, e' \in E,$$

$$\int_E \|(\nabla^* f_e) f_e\|_{L^2(\nu;H)} \sigma(de) \le C_2 \int_E \|f_e\|_H^2 \sigma(de) < \infty,$$

where $\|\cdot\|_{L^p(\nu;H)}$ stands for the norm of $L^p(\nu; H)$, the space of pth integrable H-valued functionals with respect to ν. Hence we can define $L^2(\nu; H)$-valued F' by the Bochner integral

$$F' = \int_E (\nabla^* f_e) f_e \, \sigma(de).$$

The same argument as given in the previous paragraph implies that $F' \in L^p(\nu; H)$ for any $p > 1$.

Let $G \in \boldsymbol{D}^\infty(H)$ and $K \in \boldsymbol{D}^\infty(H^{\otimes 2})$, where $H^{\otimes 2}$ stands for the Hilbert space of Hilbert-Schmidt operators on H. Since $\nabla(\nabla^* f_e) = f_e$, it holds that

$$\int_X F\nabla^* G d\nu = \int_X \langle F', G\rangle_H d\nu, \quad \int_X F\left((\nabla^*)^2 K\right) d\nu = \int_X \langle A, K\rangle_{H^{\otimes 2}} d\nu.$$

Thus $F \in \boldsymbol{D}^\infty(\boldsymbol{R})$ and

$$\nabla F = \int_E (\nabla^* f_e) f_e \sigma(de), \quad \nabla^2 F = A, \quad \text{and} \quad \nabla^k F = 0,\ k \geq 3. \tag{3.2}$$

By (3.1), we see that

$$\int_X F d\nu = \frac{1}{2}\int_E \left(\int_X (\nabla^* f_e)^2 d\nu\right)\sigma(de) = \frac{1}{2}\int_E \|f_e\|_H^2 \sigma(de) = \frac{1}{2}\mathrm{tr}\, A.$$

Moreover, the expression of ∇F in (3.2) yields that $\int_X \nabla F d\nu = 0$. Then by (3.2) and (2.1), we have that $F = q_A/2$. □

Proposition 3.2. *Let $\{\phi_n\}_{n=1}^\infty$ be an orthonormal basis of H consisting of eigenvectors of A and put*

$$a_n = \int_E \langle f_e, \phi_n\rangle_H^2 \sigma(de), \quad n = 1, 2, \ldots$$

For sufficiently small $\zeta \in \boldsymbol{C}$, it holds that

$$\int_X e^{\zeta F + \nabla^* h} d\nu = \left\{\prod_{n=1}^\infty (1 - \zeta a_n)\right\}^{-1/2} \exp\left(\frac{1}{2}\sum_{n=1}^\infty \frac{\langle h, \phi_n\rangle_H^2}{1 - \zeta a_n}\right)$$

for every $h \in H$.

Proof. By Proposition 2.1, for sufficiently small $\zeta \in \boldsymbol{C}$, it holds that

$$\int_X e^{\zeta F + \nabla^* h} d\nu = \{\det(I - \zeta A)\}^{-1/2} \exp(\langle (I - \zeta A)^{-1} h, h\rangle_H / 2).$$

Due to the eigenfunction expansion $A = \sum_{n=1}^\infty a_n \phi_n \otimes \phi_n$, where $\phi_n \otimes \phi_n$ denotes the Hilbert-Schmidt operator such that $(\phi_n \otimes \phi_n)(h) = \langle \phi_n, h\rangle_H \phi_n$, $h \in H$, we obtain that

$$\det(I - \zeta A) = \prod_{n=1}^\infty (1 - \zeta a_n), \quad \langle (I - \zeta A)^{-1} h, h\rangle_H = \sum_{n=1}^\infty \frac{\langle h, \phi_n\rangle_H^2}{1 - \zeta a_n},$$

which implies the the desired expression. □

4. Brownian Sheet

In this section we investigate the stochastic oscillatory integrals associated with the square norm of Brownian sheet.

4.1. Abstract Wiener spaces. We introduce the abstract Wiener spaces associated with the Brownian sheet.

Let $T>0$ and set

$$\mathcal{W}=\{w:[0,T]^2\to \boldsymbol{R}\,|\,w \text{ is continuous and } w(t,0)=w(0,t)=0,\ t\in[0,T]\}.$$

Denote by $\mathcal{H}$ the set of all $h\in\mathcal{W}$ of the form

$$h(s,t)=\int_{[0,s]\times[0,t]} h'(u,v)dudv, \quad (s,t)\in[0,T]^2$$

for some $h'\in L^2([0,T]^2)$ ($\equiv$ the space of all real square integrable functions on $[0,T]^2$ with respect to the Lebesgue measure). $\mathcal{W}$ is a real separable Banach space with the norm

$$\|w\|=\sup_{(s,t)\in[0,T]^2}|w(s,t)|, \quad w\in\mathcal{W},$$

and $\mathcal{H}$ is a real separable Hilbert space with the inner product

$$\langle h,g\rangle_{\mathcal{H}}=\int_{[0,T]^2} h'(s,t)g'(s,t)dsdt, \quad h,g\in\mathcal{H}.$$

We denote by $\|\cdot\|_{\mathcal{H}}$ the associated norm of $\mathcal{H}$. For $(s,t)\in[0,T]^2$, define the coordinate function $W(s,t):\mathcal{W}\to\boldsymbol{R}$ by

$$W(s,t)(w)=W(s,t;w)=w(s,t), \quad w\in\mathcal{W}.$$

There exists a unique probability measure μ on $\mathcal{W}$ such that $\{W(s,t)\}_{(s,t)\in[0,T]^2}$ is a continuous Gaussian field with mean 0 and covariance function

$$\int_{\mathcal{W}} W(s,t)W(u,v)d\mu=(s\wedge u)(t\wedge v) \quad \text{for any } (s,t),(u,v)\in[0,T]^2. \tag{4.1}$$

It is easily seen that $(\mathcal{W},\mathcal{H},\mu)$ is a real abstract Wiener space.

Put

$$\mathcal{W}_0=\{w\in\mathcal{W}\,|\,w(T,t)=w(t,T)=0, t\in[0,T]\}, \quad \mathcal{H}_0=\mathcal{H}\cap\mathcal{W}_0.$$

$\mathcal{W}_0$ is a real separable Banach space with the same norm as $\mathcal{W}$, and $\mathcal{H}_0$ is a real separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_0}$ inherited from $\mathcal{H}$. Define $\pi:\mathcal{W}\to\mathcal{W}_0$ by

$$(\pi w)(s,t)=w(s,t)-\frac{s}{T}w(T,t)-\frac{t}{T}w(s,T)+\frac{st}{T^2}w(T,T), \quad w\in\mathcal{W}.$$

Note that $\pi(\mathcal{H})=\mathcal{H}_0$. Let μ_0 be the induced measure of μ on $\mathcal{W}_0$ through π;

$$\mu_0=\mu\circ\pi^{-1}.$$

Write $B(s,t)$ for the restriction of $W(s,t)$ onto $\mathcal{W}_0$ to emphasize working on $\mathcal{W}_0$. Under μ_0, $\{B(s,t)\}_{(s,t)\in[0,T]^2}$ is a continuous Gaussian field with mean 0 and covariance function

$$\int_{\mathcal{W}_0} B(s,t)B(u,v)d\mu_0=\left\{s\wedge u-\frac{su}{T}\right\}\left\{t\wedge v-\frac{tv}{T}\right\}, \quad (s,t),(u,v)\in[0,T]^2, \tag{4.2}$$

and $(\mathcal{W}_0,\mathcal{H}_0,\mu_0)$ is an abstract Wiener space.

We introduce the third abstract Wiener space associated with Brownian sheet. Let

$$\mathcal{W}'=\{w\in\mathcal{W}\,|\,w(s,T)=0, s\in[0,T]\}, \quad \mathcal{H}'=\mathcal{H}\cap\mathcal{W}'.$$

$\mathcal{W}'$ is a separable Banach space equipped with the norm of uniform convergence, and $\mathcal{H}'$ is a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}'}$ inherited from $\mathcal{H}$. Define $\pi' : \mathcal{W} \to \mathcal{W}'$ by

$$(\pi' w)(s,t) = w(s,t) - \frac{t}{T} w(s,T), \quad w \in \mathcal{W}.$$

Then $\pi'(\mathcal{H}) = \mathcal{H}'$. Set

$$\mu' = \mu \circ (\pi')^{-1}.$$

Denote the restriction of $W(s,t)$ to $\mathcal{W}'$ by $B'(s,t)$ to emphasize dealing with $\mathcal{W}'$. Under μ', $\{B'(s,t)\}_{(s,t)\in[0,T]^2}$ is a continuous Gaussian field with mean 0 and covariance function

$$\int_{\mathcal{W}'} B'(s,t)B'(u,v)d\mu' = (s \wedge u)\left\{t \wedge v - \frac{tv}{T}\right\}, \quad (s,t),(u,v) \in [0,T]^2, \tag{4.3}$$

and $(\mathcal{W}', \mathcal{H}', \mu')$ is an abstract Wiener space.

For the sake of simplicity of notation, in what follows, all Malliavin gradients and their adjoints on $\mathcal{W}$, $\mathcal{W}_0$, and $\mathcal{W}'$ will be denoted by the same symbol ∇ and ∇^*, respectively.

4.2. Square norm of Brownian sheet. Define $\mathfrak{h} : \mathcal{W} \to \boldsymbol{R}$ by

$$\mathfrak{h} = \frac{1}{2}\int_{[0,T]^2} W(s,t)^2 dsdt,$$

and denote by $\mathfrak{h}_0$ and $\mathfrak{h}'$ its restriction to $\mathcal{W}_0$ and $\mathcal{W}'$, respectively;

$$\mathfrak{h}_0 = \frac{1}{2}\int_{[0,T]^2} B(s,t)^2 dsdt \quad \text{and} \quad \mathfrak{h}' = \frac{1}{2}\int_{[0,T]^2} B'(s,t)^2 dsdt.$$

Define $f_{s,t} \in \mathcal{W}^*$ by $f_{s,t}(w) = w(s,t)$, $w \in \mathcal{W}$. Recall that on an abstract Wiener space (X, H, ν), it holds that $\nabla^*\ell = \ell$ for $\ell \in X^*$. Hence we have that

$$\mathfrak{h} = \frac{1}{2}\int_{[0,T]^2} (\nabla^* f_{s,t})^2 dsdt.$$

Following the convention that we use the same symbol ∇ and ∇^* for the Malliavin gradients and its adjoint operators on $\mathcal{W}$, $\mathcal{W}_0$, and $\mathcal{W}'$, we denote the restrictions $f_{s,t}$ to $\mathcal{W}_0$ and $\mathcal{W}'$ by the same letter $f_{s,t}$. Then $f_{s,t}$ belong to $\mathcal{W}_0^*$ and $(\mathcal{W}')^*$, and we have the similar expression as above;

$$\mathfrak{h}_0 = \frac{1}{2}\int_{[0,T]^2} (\nabla^* f_{s,t})^2 dsdt \quad \text{and} \quad \mathfrak{h}' = \frac{1}{2}\int_{[0,T]^2} (\nabla^* f_{s,t})^2 dsdt.$$

Define $A : \mathcal{H} \to \mathcal{H}$, $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$, and $A' : \mathcal{H}' \to \mathcal{H}'$ by

$$(Ah)(s,t) = \int_{[0,s]\times[0,t]} \left(\int_{[u,T]\times[v,T]} h(a,b)dadb\right)dudv, \quad h \in \mathcal{H},\ (s,t) \in [0,T]^2$$

and

$$A_0 = \pi_0 \circ A \quad \text{and} \quad A' = \pi' \circ A.$$

For $m,n \in \boldsymbol{Z}_+ = \{0,1,2,\dots\}$ and $j,k \in \boldsymbol{N}$, set

$$\phi_{m,n}(s,t) = \frac{2T}{(m+\frac{1}{2})(n+\frac{1}{2})\pi^2} \sin\Big(\frac{(m+\frac{1}{2})\pi s}{T}\Big) \sin\Big(\frac{(n+\frac{1}{2})\pi t}{T}\Big),$$

$$\psi_{j,k}(s,t) = \frac{2T}{jk\pi^2} \sin\Big(\frac{j\pi s}{T}\Big) \sin\Big(\frac{k\pi t}{T}\Big),$$

$$\psi'_{m,k}(s,t) = \frac{2T}{(m+\frac{1}{2})k\pi^2} \sin\Big(\frac{(m+\frac{1}{2})\pi s}{T}\Big) \sin\Big(\frac{k\pi t}{T}\Big), \quad (s,t) \in [0,T]^2.$$

Lemma 4.1. *(i) $\{\phi_{m,n}\}_{m,n\in\boldsymbol{Z}_+}$, $\{\psi_{m,n}\}_{m,n\in\boldsymbol{N}}$, and $\{\psi'_{m,n}\}_{m\in\boldsymbol{Z}_+,n\in\boldsymbol{N}}$ are orthonormal bases of $\mathcal{H}$, $\mathcal{H}_0$, and $\mathcal{H}'$, respectively.*
(ii) A, A_0, and A' admit the following eigenfunction expansions;

$$A = \sum_{m,n\in\boldsymbol{Z}_+} \Big(\frac{T}{(m+\frac{1}{2})\pi}\Big)^2 \Big(\frac{T}{(n+\frac{1}{2})\pi}\Big)^2 \phi_{m,n}\otimes\phi_{m,n},$$

$$A_0 = \sum_{m,n\in\boldsymbol{N}} \Big(\frac{T}{m\pi}\Big)^2 \Big(\frac{T}{n\pi}\Big)^2 \psi_{m,n}\otimes\psi_{m,n},$$

$$A' = \sum_{m\in\boldsymbol{Z}_+,n\in\boldsymbol{N}} \Big(\frac{T}{(m+\frac{1}{2})\pi}\Big)^2 \Big(\frac{T}{n\pi}\Big)^2 \psi'_{m,n}\otimes\psi'_{m,n}.$$

(iii) It holds that $\mathfrak{h} = q_A/2$, $\mathfrak{h}_0 = q_{A_0}/2$, and $\mathfrak{h}' = q_{A'}/2$.

Proof. (i) Obviously $\phi_{m,n} \in \mathcal{H}$, $\psi_{m,n} \in \mathcal{H}_0$, and $\psi'_{m,n} \in \mathcal{H}_0$. It is an elementary exercise of Fourier series to show that $\{\phi_{m,n}\}_{m,n\in\boldsymbol{Z}_+}$, $\{\psi_{m,n}\}_{m,n\in\boldsymbol{N}}$, and $\{\psi'_{m,n}\}_{m\in\boldsymbol{Z}_+,n\in\boldsymbol{N}}$ are orthonormal bases of $\mathcal{H}$, $\mathcal{H}_0$, and $\mathcal{H}'$, respectively, once one notices that $h \in \mathcal{H}_0$ and $g \in \mathcal{H}'$ satisfy that

$$\int_0^T h'(s,t)ds = 0 \quad \text{a.e. } t, \qquad \int_0^T h'(s,t)dt = \int_0^T g'(s,t)dt = 0 \quad \text{a.e. } s.$$

(ii) It is easily checked that

$$A\phi_{m,n} = \Big(\frac{T}{(m+\frac{1}{2})\pi}\Big)^2 \Big(\frac{T}{(n+\frac{1}{2})\pi}\Big)^2 \phi_{m,n}, \quad A_0\psi_{m,n} = \Big(\frac{T}{m\pi}\Big)^2 \Big(\frac{T}{n\pi}\Big)^2 \psi_{m,n},$$

$$A'\psi'_{m,n} = \Big(\frac{T}{m\pi}\Big)^2 \Big(\frac{T}{n\pi}\Big)^2 \psi'_{m,n}.$$

In conjunction with (i), these identities imply the desired expansions.
(iii) This is an immediate consequence of Proposition 3.1 and the integration by parts formula on $[0,T]$. □

The first goal of this section is

Theorem 4.2. *For sufficiently small $\zeta \in \boldsymbol{C}$, the following identities hold.*

$$\begin{aligned} &\int_{\mathcal{W}} e^{\zeta^2\mathfrak{h}} d\mu = \mathcal{C}(T^2\zeta)^{-1/2}, \quad \int_{\mathcal{W}_0} e^{\zeta^2\mathfrak{h}_0} d\mu_0 = \mathcal{S}(T^2\zeta)^{-1/2}, \\ &\int_{\mathcal{W}'} e^{\zeta^2\mathfrak{h}'} d\mu' = \widetilde{\mathcal{C}}(T^2\zeta)^{-1/2}, \end{aligned} \tag{4.4}$$

where

$$\mathcal{C}(x)=\prod_{m=0}^{\infty}\cos\Big(\frac{2x}{(2m+1)\pi}\Big),\quad \mathcal{S}(x)=\prod_{n=1}^{\infty}\frac{n\pi\sin(x/(n\pi))}{x},$$

$$\widetilde{\mathcal{C}}(x)=\prod_{n=1}^{\infty}\cos\Big(\frac{x}{n\pi}\Big).$$

Proof. By Lemma 4.1 (ii) and the well known identities that

$$\prod_{n=0}^{\infty}\Big\{1-\frac{x^2}{(2n+1)^2}\Big\}=\cos\Big(\frac{\pi x}{2}\Big),\quad \prod_{m=1}^{\infty}\Big(1-\frac{x^2}{m^2}\Big)=\frac{\sin(\pi x)}{\pi x}$$

we see that

$$\det(I-\zeta^2A)=\prod_{m=0}^{\infty}\prod_{n=0}^{\infty}\Big\{1-\zeta^2\Big(\frac{T}{(m+\frac12)\pi}\Big)^2\Big(\frac{T}{(n+\frac12)\pi}\Big)^2\Big\}=\mathcal{C}(T^2\zeta),$$

$$\det(I-\zeta^2A_0)=\prod_{m,n=1}^{\infty}\Big\{1-\zeta^2\Big(\frac{T}{m\pi}\Big)^2\Big(\frac{T}{n\pi}\Big)^2\Big\}=\mathcal{S}(T^2\zeta),$$

$$\det(I-\zeta^2A')=\prod_{m=0}^{\infty}\prod_{n=1}^{\infty}\Big\{1-\zeta^2\Big(\frac{T}{(m+\frac12)\pi}\Big)^2\Big(\frac{T}{n\pi}\Big)^2\Big\}=\widetilde{\mathcal{C}}(T^2\zeta).$$

In conjunction with Lemma 4.1(iii) and Proposition 3.2, these implies the identities described in (4.4). □

As another application of Lemma 4.1, we obtain the following Lévy-Khinchin formulas.

Proposition 4.3. *For $\lambda\in\boldsymbol{R}$, it holds that*

$$\int_{\mathcal{W}}e^{\sqrt{-1}\lambda\mathfrak{h}}d\mu=\exp\Big(\int_{(0,\infty)}\{e^{\sqrt{-1}\,\lambda x}-1\}\times \times\frac{1}{4x}\sum_{m=0}^{\infty}\Big\{\Theta\Big(\frac{(2m+1)^2\pi^4x}{16T^4}\Big)-\Theta\Big(\frac{(2m+1)^2\pi^4x}{4T^4}\Big)\Big\}dx\Big),$$

$$\int_{\mathcal{W}_0}e^{\sqrt{-1}\lambda\mathfrak{h}_0}d\mu_0=\exp\Big(\int_{(0,\infty)}\{e^{\sqrt{-1}\,\lambda x}-1\}\frac{1}{4x}\sum_{n=1}^{\infty}\Big\{\Theta\Big(\frac{n^2\pi^4x}{T^4}\Big)-1\Big\}dx\Big),$$

$$\int_{\mathcal{W}'}e^{\sqrt{-1}\lambda\mathfrak{h}'}d\mu'=\exp\Big(\int_{(0,\infty)}\{e^{\sqrt{-1}\,\lambda x}-1\}\frac{1}{4x}\sum_{m=0}^{\infty}\Big\{\Theta\Big(\frac{(2m+1)^2\pi^4x}{4T^4}\Big)-1\Big\}dx\Big),$$

where $\Theta(u)$ is Jacobi's Theta function;

$$\Theta(u)=\sum_{n=-\infty}^{\infty}e^{-n^2u},\quad u\in\boldsymbol{R}.$$

Proof. By Lemma 4.1 (ii) and the very definition, $f_A(x) = f_{A_0}(x) = f_{A'}(x) = 0$ if $x \le 0$. For $x > 0$, by Lemma 4.1 again, we have that

$$\begin{aligned} f_A(x) &= \frac{1}{2}\sum_{m,n=0}^{\infty} \frac{1}{x}\exp\biggl(-x\biggl(\frac{T}{(m+\frac{1}{2})\pi}\biggr)^{-2}\biggl(\frac{T}{(n+\frac{1}{2})\pi}\biggr)^{-2}\biggr) \\ &= \frac{1}{4x}\sum_{m=0}^{\infty}\biggl\{\Theta\biggl(\frac{(2m+1)^2\pi^4 x}{16T^4}\biggr) - \Theta\biggl(\frac{(2m+1)^2\pi^4 x}{4T^4}\biggr)\biggr\}, \end{aligned}$$

$$f_{A_0}(x) = \frac{1}{2}\sum_{m,n=1}^{\infty}\frac{1}{x}\exp\biggl(-x\biggl(\frac{T}{m\pi}\biggr)^{-2}\biggl(\frac{T}{n\pi}\biggr)^{-2}\biggr) = \frac{1}{4x}\sum_{m=1}^{\infty}\biggl\{\Theta\biggl(\frac{m^2\pi^4 x}{T^4}\biggr) - 1\biggr\},$$

$$\begin{aligned} f_{A'}(x) &= \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{x}\exp\biggl[-x\biggl(\frac{T}{(m+\frac{1}{2})\pi}\biggr)^{-2}\biggl(\frac{T}{n\pi}\biggr)^{-2}\biggr] \\ &= \frac{1}{4x}\sum_{m=1}^{\infty}\biggl\{\Theta\biggl(\frac{(2m+1)^2\pi^4 x}{4T^4}\biggr) - 1\biggr\}. \end{aligned}$$

Combined with Proposition 2.2, these imply the desired identities. □

We next consider an application to conditional expectations of $e^{\zeta\mathfrak{h}}$.

Theorem 4.4. *(i) Let $x \in \boldsymbol{R}$. For sufficiently small $\zeta \in \boldsymbol{C}$, it holds that*

$$\boldsymbol{E}[e^{\zeta^2\mathfrak{h}}\,|\,W(T,T) = x] = \{\mathcal{C}(T^2\zeta)\mathcal{T}(T^2\zeta)\}^{-1/2}\exp\biggl(-\frac{x^2}{2T^2}\biggl\{\frac{1}{\mathcal{T}(T^2\zeta)} - 1\biggr\}\biggr),$$

where

$$\mathcal{T}(a) = \frac{4}{a}\sum_{m=0}^{\infty}\frac{1}{(2m+1)\pi}\tan\biggl(\frac{2a}{(2m+1)\pi}\biggr).$$

(ii) Let $\alpha, \beta \in C([0,T])$ satisfy $\alpha(0) = \beta(0) = 0$ and $\alpha(T) = \beta(T)$. For sufficiently small $\zeta \in \boldsymbol{C}$, it holds that

$$\begin{aligned} &\boldsymbol{E}_\mu\bigl[\exp(\zeta^2\mathfrak{h})\big|W(\cdot,T) = \alpha, W(T,\cdot) = \beta\bigr] \\ &\quad = \mathcal{S}(T^2\zeta)^{-1/2}\exp\biggl(\frac{\zeta^2}{2}I(\alpha,\beta) + 2\zeta^4T^4\sum_{m,n=1}^{\infty}\frac{K_{m,n}^2}{m^2n^2\pi^4 - \zeta^2T^4}\biggr), \end{aligned}$$

where

$$\begin{aligned} I(\alpha,\beta) = \frac{T}{3}\int_0^T\{\alpha(u)^2 + \beta(u)^2\}du + \frac{T^2}{9}\alpha(T)^2 + \frac{2}{T^2}\int_0^T s\alpha(s)ds\int_0^T t\beta(t)dt \\ - \frac{2\alpha(T)}{3}\int_0^T\{u\alpha(u) + u\beta(u)\}du, \end{aligned}$$

$$\alpha_m = \int_0^T \alpha(s)\sin\biggl(\frac{m\pi s}{T}\biggr)ds, \quad \beta_n = \int_0^T \beta(t)\sin\biggl(\frac{n\pi t}{T}\biggr)dt,$$

$$K_{m,n} = \frac{(-1)^{m+1}\beta_n}{m\pi} + \frac{(-1)^{n+1}\alpha_m}{n\pi} - \frac{(-1)^{m+n}T\alpha(T)}{mn\pi^2}.$$

(iii) Suppose that $\gamma \in C([0,T])$ satisfies that $\gamma(0)=0$. For sufficiently small $\zeta\in\boldsymbol{C}$, it holds that

$$\begin{aligned}&\boldsymbol{E}_\mu\big[\exp(\zeta^2\mathfrak{h})\big|W(\cdot,T)=\gamma\big]\\&\quad=\widetilde{\mathcal{C}}(T^2\zeta)^{-1/2}\exp\bigg(\frac{\zeta^2TI(\gamma)}{6}+2\zeta^4T^4\sum_{m=0}^\infty\sum_{n=1}^\infty\frac{\gamma_m^2}{(m+\frac12)^2n^2\pi^4-\zeta^2T^4}\frac{1}{n^2\pi^2}\bigg),\end{aligned}$$

where

$$I(\gamma)=\int_0^T\gamma(s)^2ds,\quad \gamma_m=\int_0^T\gamma(s)\sin\bigg(\frac{(m+\frac12)\pi s}{T}\bigg)ds.$$

Remark 4.5. The expression in the second assertion is different from that given in [2]. Their formula seems incomplete because the right hand side of the identity given at the bottom of [2, p.526], on which their formula is based, has no boundary values $y(\cdot,1)$ and $y(1,\cdot)$.

Proof. (i) Since $W(T,T)$ obeys the normal distribution with mean 0 and variance T^2, it holds that

$$\boldsymbol{E}[e^{\zeta^2\mathfrak{h}}\,|\,W(T,T)=x]=\sqrt{2\pi T^2}e^{x^2/(2T^2)}\int_{\mathcal{W}}e^{\zeta\mathfrak{h}}\delta_x(W(T,T))d\mu,$$

where $\delta_x(W(T,T))$ is Watanabe's pull-back of the Dirac measure δ_x concentrated at $x\in\boldsymbol{R}$ through the smooth and non-degenerate Wiener functional $W(T,T):\mathcal{W}\to\boldsymbol{R}$. Thus it suffices to show that

$$\int_{\mathcal{W}}e^{\zeta^2\mathfrak{h}}\delta_x(W(T,T))d\mu=\{2\pi T^2\mathcal{C}(T^2\zeta)\mathcal{T}(T^2\zeta)\}^{-1/2}\exp\bigg(-\frac{x^2}{2T^2\mathcal{T}(T^2\zeta)}\bigg).\quad(4.5)$$

To see (4.5), define $\ell\in\mathcal{W}^*$ by $\ell=f_{T,T}$; $\ell(w)=w(T,T)$, $w\in\mathcal{W}$. Then $\nabla^*\ell=W(T,T)$. By Proposition 3.2, Lemma 4.1, and Theorem 4.2, we have that

$$\begin{aligned}\int_{\mathcal{W}}e^{\zeta^2\mathfrak{h}}\delta_x(W(T,T))d\mu&=\frac{1}{2\pi}\int_{\boldsymbol{R}}e^{-\sqrt{-1}\,\eta x}\bigg(\int_{\mathcal{W}}e^{\zeta^2\mathfrak{h}+\sqrt{-1}\,\eta\nabla^*\ell}d\mu\bigg)d\eta\\&=\mathcal{C}(T^2\zeta)^{-1/2}\big\{2\pi\langle(I-\zeta^2A)^{-1}\ell,\ell\rangle_{\mathcal{H}}\big\}^{-1/2}\exp\bigg(-\frac{x^2}{2}\langle(I-\zeta^2A)^{-1}\ell,\ell\rangle_{\mathcal{H}}^{-1}\bigg).\end{aligned}$$

By Lemma 4.1 (ii), we see that

$$\begin{aligned}\langle(I-\zeta^2A)^{-1}\ell,\ell\rangle_{\mathcal{H}}&=\sum_{m,n\in\boldsymbol{Z}_+}\bigg\{1-\zeta^2\bigg(\frac{T}{(m+\frac12)\pi}\bigg)^2\bigg(\frac{T}{(n+\frac12)\pi}\bigg)^2\bigg\}^{-1}\phi_{m,n}(T,T)^2\\&=\sum_{m,n\in\boldsymbol{Z}_+}\frac{4T^2}{(m+\frac12)^2(n+\frac12)^2\pi^4-\zeta^2T^4}\\&=\frac{4}{\zeta}\sum_{m\in\boldsymbol{Z}_+}\frac{1}{(2m+1)\pi}\tan\bigg(\frac{2\zeta T^2}{(2m+1)\pi}\bigg)=T^2\mathcal{T}(T^2\zeta).\end{aligned}$$

Thus (4.5) holds.

(ii) Since πW and $(I-\pi)W$ are independent, for $y\in(I-\pi)(\mathcal{W})$, the conditional distribution of W under μ given the condition that $(I-\pi)W=y$ coincides with the distribution of $B+y$ under μ_0;

$$\mu(W\in\bullet|(I-\pi)W=y)=\mu_0(B+y\in\bullet).$$

This implies that

$$\mu(W\in\bullet|W(\cdot,T)=y(\cdot,T),W(T,\cdot)=y(T,\cdot))=\mu_0(B+y\in\bullet).$$

Define $y\in(I-\pi)\mathcal{W}$ and $\ell\in\mathcal{W}_0^*$ by

$$y(s,t)=\frac{s}{T}\beta(t)+\frac{t}{T}\alpha(s)-\frac{st}{T^2}\alpha(T),\quad (s,t)\in[0,T]^2,$$

$$\ell(w)=\int_{[0,T]^2}w(s,t)y(s,t)dsdt,\quad w\in\mathcal{W}_0.$$

By Proposition 2.1 and Theorem 4.2, we have that

$$\begin{aligned}&\boldsymbol{E}_\mu\big[\exp(\zeta^2\mathfrak{h})\big|W(\cdot,T)=\alpha,W(T,\cdot)=\beta\big]\\&=\exp\biggl(\frac{\zeta^2}{2}\int_{[0,T]^2}y(s,t)^2dsdt\biggr)\int_{\mathcal{W}_0}\exp\bigl(\zeta^2\mathfrak{h}_0+\zeta^2\ell\bigr)d\mu_0\\&=\exp\biggl(\frac{\zeta^2}{2}\int_{[0,T]^2}y(s,t)^2dsdt\biggr)\mathcal{S}(T^2\zeta)^{-1/2}\exp\bigl(\zeta^4\langle(I-\zeta^2A_0)^{-1}\ell,\ell\rangle_{\mathcal{H}_0}/2\bigr).\end{aligned}$$

It holds that $\int_{[0,T]^2}y(s,t)^2dsdt=I(\alpha,\beta)$. By virtue of Lemma 4.1 (ii), we see that

$$\langle(I-\zeta^2A_0)^{-1}\ell,\ell\rangle_{\mathcal{H}_0}=\sum_{m,n=1}^\infty\frac{m^2n^2\pi^4}{m^2n^2\pi^4-\zeta^2T^4}\biggl\{\int_{[0,T]^2}\psi_{m,n}(s,t)y(s,t)dsdt\biggr\}^2.$$

A direct computation yields that

$$\int_{[0,T]^2}\psi_{m,n}(s,t)y(s,t)dsdt=\frac{2T^2}{mn\pi^2}K_{m,n}.$$

Thus the desired identity follows.
(iii) Taking the advantage of the independence of $\pi'W$ and $(I-\pi')W$, we see that

$$\mu(W\in\bullet|W(\cdot,T)=\gamma)=\mu'(B'+z\in\bullet),$$

where $z\in\mathcal{W}$ is given by

$$z(s,t)=\frac{t}{T}\gamma(s),\quad (s,t)\in[0,T]^2.$$

Define $\ell\in(\mathcal{W}')^*$ by

$$\ell(w)=\int_{[0,T]^2}w(s,t)z(s,t)dsdt,\quad w\in\mathcal{W}'.$$

Then, due to Proposition 2.1 and Theorem 4.2, we have that

$$\begin{aligned}&\boldsymbol{E}_\mu\big[\exp(\zeta^2\mathfrak{h})\big|W(\cdot,T)=\gamma\big]\\&=\exp\biggl(\frac{\zeta^2}{2}\int_{[0,T]^2}z(s,t)^2dsdt\biggr)\widetilde{C}(T^2\zeta)^{-1/2}\exp\bigl(\zeta^4\langle(I-\zeta^2A')^{-1}\ell,\ell\rangle_{\mathcal{H}'}/2\bigr).\end{aligned}$$

It follows from Lemma 4.1 that

$$\int_{[0,T]^2} z(s,t)^2 dsdt = \frac{T}{3}\int_0^T \gamma(s)^2 ds,$$

$$\langle (I-\zeta^2 A')^{-1}\ell,\ell\rangle_{\mathcal{H}'} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{(m+\frac{1}{2})^2 n^2\pi^4}{(m+\frac{1}{2})^2 n^2\pi^4 - \zeta^2 T^4}\langle \psi'_{m,n},\ell\rangle_{\mathcal{H}'}^2,$$

and

$$\langle \psi'_{m,n},\ell\rangle_{\mathcal{H}'} = \frac{2T^2}{(m+\frac{1}{2})n\pi^2}\,\frac{\gamma_m(-1)^{n+1}}{n\pi}.$$

Hence the desired identity follows. □

5. Reflectionless Potential

In this section, we continue to work on the abstract Wiener space $(\mathcal{W},\mathcal{H},\mu)$ associated with the Brownian sheet.

A reflectionless potential $u:\boldsymbol{R}\to\boldsymbol{R}$ with scattering data $\{\eta_i,m_i\}_{1\le i\le n}$, where $\eta_i, m_i>0$, $1\le i\le n$, and $\eta_i\ne\eta_j$ if $i\ne j$, is the function defined by

$$u(x) = -2\frac{d^2}{dx^2}\log\det(I+G(x)),$$

where

$$G(x) = \left(\frac{\sqrt{m_i m_j}\, e^{-(\eta_i+\eta_j)x}}{\eta_i+\eta_j}\right)_{1\le i,j\le n}.$$

It is widely known that $u(x,t)=-q(x,t)$, q being the reflectionless potential with scattering data $\{\eta_i, m_i\exp[-2\eta_i^3 t]\}_{1\le i\le n}$ gives a rise of an n-soliton solution of the KdV equation

$$\frac{\partial v}{\partial t} = \frac{3}{2}\,v\,\frac{\partial v}{\partial x} + \frac{1}{4}\,\frac{\partial^3 v}{\partial x^3}. \tag{5.1}$$

For examples, see [7].

In [10], using the Brownian sheet, quadratic Wiener functionals related to reflectionless potentials are studied. More precisely, let $\{(p_j,c_j)\}_{j=1}^n\subset(\boldsymbol{R}^2)^n$ satisfy that $p_i\ne p_j$ if $i\ne j$ and $c_j>0$, $1\le j\le n$. Define q_j, $0\le j\le n$ by

$$q_0=0,\quad q_j=\sum_{k=1}^{j}|p_k-p_{k-1}|,$$

where $p_0=-|p_1|-1$. Assume $q_n\le T$. Define $e_1,\dots,e_n\in L^2[0,T]$, the space of square integrable functions on $[0,T]$ with respect to the Lebesgue measure, by

$$e_j(s) = \frac{1}{\sqrt{q_j-q_{j-1}}}\chi_{[q_{j-1},q_j)}(s),\quad s\in[0,T],$$

where $\chi_{[a,b)}$ stands for the indicator function of $[a,b)$. Let $x\le T$. For $y\in[0,x]$, define $f_y\in\mathcal{H}$ by

$$f'_y(s,t) = \sum_{j=1}^{n} c_j e_j(s) e^{(y-t)p_j}\chi_{[0,y)}(t),\quad (s,t)\in[0,T]^2.$$

The function

$$u(x) = 4\frac{d^2}{dx^2}\,\log\biggl(\int_{\mathcal{W}} \exp\biggl(-\frac{1}{2}\int_0^x (\nabla^* f_y)^2 dy\biggr) d\mu\biggr)$$

determines a reflectionless potential, and conversely, every reflectionless potential admits such an expression ([10, 4]).

Define the Hilbert-Schmidt operator $A : \mathcal{H} \to \mathcal{H}$ by

$$A = \nabla^2 F = \int_0^x f_y \otimes f_y dy, \quad \text{where } F = \frac{1}{2}\int_0^x (\nabla^* f_y)^2 dy$$

(cf. Proposition 3.1). The aim of this section is to specify eigenvalues and eigenvectors of A.

5.1. ker $\boldsymbol{A}$. Let $\mathcal{K}$ be the subspace of $\mathcal{H}$ consisting of all $h \in \mathcal{H}$ of the form

$$h'(s,t) = \sum_{i=1}^n e_i(s) h_i(t), \quad (s,t) \in [0,T]^2 \tag{5.2}$$

for some $h_1, \dots, h_n \in L^2[0,T]$. Denote by $\mathcal{P}$ the set of all $h \in \mathcal{K}$ satisfying that

$$h_i = g_i - p_i \int_0^\bullet g_i(s) ds \quad \text{a.e. on } [0,x],\ 1 \le i \le n, \tag{5.3}$$

for some $g_1, \dots, g_n \in L^2[0,T]$ with $\sum_{i=1}^n c_i g_i = 0$ a.e. on $[0,x]$. Let $\mathcal{G}$ be the space of all $h \in \mathcal{H}$ such that $h'(s,t) = u(s)v(t)$ for some $u, v \in L^2[0,T]$ with $\int_0^T u(s) e_i(s) ds = 0$, $1 \le i \le n$. We shall show that

$$\ker A = \overline{\mathcal{G}} \oplus \mathcal{P}, \tag{5.4}$$

where $\overline{\mathcal{G}}$ is the closure of $\mathcal{G}$.

Notice that

$$\mathcal{K} = \mathcal{G}^\perp \quad \text{and} \quad (\ker A)^\perp \subset \mathcal{K}, \tag{5.5}$$

where $\mathcal{A}^\perp$ stands for the orthogonal complement of $\mathcal{A}$ in $\mathcal{H}$. Hence, the proof of (5.4) completes once one has shown that

$$\mathcal{P} = \ker A \cap \mathcal{K}. \tag{5.6}$$

To show (5.6), we first give an expression of Ah for $h \in \mathcal{K}$. Let $h \in \mathcal{K}$ and represent it as in (5.2). Putting

$$\xi^i(y;h) = e^{yp_i} \int_0^y e^{-zp_i} h_i(z) dz, \quad i = 1, \dots, n,$$

we see that $\langle f_y, h \rangle_{\mathcal{H}} = \sum_{i=1}^n c_i \xi^i(y;h)$, and hence that, for $(s,t) \in [0,T]^2$,

$$(Ah)'(s,t) = \sum_{j=1}^n \biggl(e^{-tp_j} \int_t^x e^{yp_j} c_j \biggl(\sum_{i=1}^n c_i \xi^i(y;h) \biggr) dy \biggr) e_j(s) \chi_{[0,x]}(t). \tag{5.7}$$

Let $h \in \mathcal{K}$ and represent it as in (5.2). By the very definition, it holds that

$$(\xi^i)'(y;h) = h_i(y) + p_i \xi^i(y;h). \tag{5.8}$$

Suppose first that $h \in \mathcal{P}$. Then, substituting the expression (5.3) into (5.8), we obtain that $\xi^i(y;h) = \int_0^y g_i(z)dz$, $y \in [0,x]$. Since $\sum_{i=1}^n c_i g_i = 0$ a.e. on $[0,x]$,

$$\sum_{i=1}^n c_i \xi^i(y;h) = 0, \quad y \in [0,x]. \tag{5.9}$$

Plugging this into (5.7), we see that $h \in \ker A$. Next suppose that $h \in \ker A$. Since $e_j(s) \neq 0$ if and only if $s \in [q_{j-1}, q_j)$, it follows from (5.7) that (5.9) holds. Moreover, due to (5.8), we have that

$$h_i(y) = (\xi^i)'(y;h) - p_i \int_0^y (\xi^i)'(z;h)dz, \quad \text{a.e. } y \le x,$$

which means that $h \in \mathcal{P}$. Thus (5.6) has been verified.

5.2. Non-zero eigenvalues. Let $\lambda \neq 0$. By (5.5) and (5.7), λ is an eigenvalue of A if and only if there exists $h \in \mathcal{K} \setminus \{0\}$ such that, under the representation as in (5.2), it holds that

$$\lambda \begin{pmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{pmatrix} = e^{-tD} \int_t^x e^{yD} (\boldsymbol{c} \otimes \boldsymbol{c}) \xi(y;h) dy \, \chi_{[0,x]}(t), \quad t \in [0,T], \tag{5.10}$$

where D is the diagonal matrix with elements $p_1, \dots, p_n$, $\boldsymbol{c} \otimes \boldsymbol{c}$ denotes the $n \times n$ matrix $\big(c_i c_j\big)_{1 \le i,j \le n}$, and

$$\xi(y;h) = \begin{pmatrix} \xi^1(y;h) \\ \vdots \\ \xi^n(y;h) \end{pmatrix}.$$

The identity (5.10) implies immediately that

$$h_i(t) = 0, \quad t > x, \, i = 1, \dots, n.$$

By virtue of (5.8), the identity (5.10) reads as

$$\lambda e^{tD} \{\xi'(t;h) - D\xi(t;h)\} = \int_t^x e^{yD} (\boldsymbol{c} \otimes \boldsymbol{c}) \xi(y;h) dy, \quad t \le x.$$

This is equivalent to the ordinary differential equation that

$$\xi'' - B(1/\lambda)\xi = 0 \quad \text{with } \xi(0) = 0 \text{ and } \xi'(x) - D\xi(x) = 0,$$

where $B(a) = D^2 - a(\boldsymbol{c} \otimes \boldsymbol{c})$.

Let $s(t;\lambda) = \sinh(tB(1/\lambda)^{1/2})$ and $c(t;\lambda) = \cosh(tB(1/\lambda)^{1/2})$. The above ordinary differential equation has a solution $\xi(t;h)$ if and only if there exists a $u \in \boldsymbol{R}^n$ such that

$$\big\{c(x;\lambda) - Ds(x;\lambda)\big\}u = 0,$$

and then $\xi(t;h) = s(t;\lambda)u$, $t \le x$. Moreover, $h \neq 0$ if and only if $u \neq 0$.

5.3. Eigenvalues and eigenvectors. Summing up the above observations, we have that

Proposition 5.1. *(i)* $\ker A = \overline{\mathcal{G}} \oplus \mathcal{P}$.
(ii) $\lambda \in \boldsymbol{R} \setminus \{0\}$ *is an eigenvalue of* A *if and only if* $\det\big[c(x;\lambda) - Ds(x;\lambda)\big] = 0$. *Moreover, in this case, the corresponding eigenvector* $h \in \mathcal{H}$ *is of the form*

$$h'(s,t) = \sum_{i=1}^{n} e_i(s) h_j(t),$$

with

$$\begin{pmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{pmatrix} = \begin{cases} \{c(t;\lambda) - Ds(t;\lambda)\}u, & t \le x, \\ 0, & t > x, \end{cases}$$

for some $u \in \ker[c(x;\lambda) - Ds(x;\lambda)] \setminus \{0\}$.

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