Communications on Stochastic Analysis
Vol. 2, No. 1 (2008) 33-51

Serials Publications
www.serialspublications.com

LOG-SOBOLEV INEQUALITIES WITH POTENTIAL FUNCTIONS ON PINNED PATH GROUPS

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Abstract. We establish a refined version of Gross's log-Sobolev inequalities on pinned path groups. We explain the reason why it is useful in a lower bound estimate of Schrödinger operators on path spaces.

1. Introduction

It is well-known that the hypercontractivity of the diffusion semi-group and the equivalent notion of the validity of the log-Sobolev inequality are used to give a lower bound on the bottom of spectrum of a Schrödinger operator which is given by the sum of the generator of the semi-group and a potential function. We applied this lower bound estimate to study the semi-classical behavior of the bottom of spectrum of Schrödinger operators on path spaces over compact Riemannian manifolds ([3, 4]) partly motivated by an application to $P(\phi)$-type Hamiltonian and an extension of [19] to infinite dimensional curved spaces.

In this paper, we establish a refined version of Gross's log-Sobolev inequalities on a pinned path group. Pinned path group $P_{e,a}(G)$ is a space of continuous paths with values in a compact Lie group G over the time interval $[0,1]$ with a fixed starting point e (unit element) and the fixed end point a. We will apply the log-Sobolev inequalities with potential functions to study the semi-classical behavior of the low lying spectrum of Schrödinger operators on pinned path groups in a separate paper.

The structure of the paper is as follows. In Section 2, we consider smooth pinned path spaces over a general compact Riemannian manifold M. We introduce a Riemannian structure on the pinned path space using a metric connection on M. Next we calculate the gradient of the energy function $E(\gamma)=\frac{1}{2}\int_0^1|\dot{\gamma}(t)|^2dt$. This calculation and a formal argument show that an LSI with a special potential function may be useful for the study of semi-classical behavior of low lying spectrum of Schrödinger operators over a pinned path space. This kind of log-Sobolev inequality with special potential function already appeared in [13, 10]. In Section 3, we consider a pinned path group and introduce an H-derivative on $P_{e,a}(G)$ and the Dirichlet form in L^2-space with respect to the (scaled) pinned Brownian motion measure with scaling (semi-classical) parameter λ. The H-derivative is considered as the gradient operator on the path space which is defined by the right invariant

2000 *Mathematics Subject Classification.* Primary 58J65, Secondary 60H07.
Key words and phrases. Log-Sobolev inequalities, semi-classical analysis, Schrödinger operators, path space.

connection on G. We prove a special kind of LSI which is introduced in Section 2 in the case of the pinned path group with respect to the Dirichlet form. The generator of the Dirichlet form is the Ornstein-Uhlenbeck type operator. However we cannot apply the argument in [3, 4] to the Ornstein-Uhlenbeck type operator on $P_{e,a}(G)$ itself, since the function $\frac{\lambda}{2}|b(1)|^2$ does not satisfy the exponential integrability condition. We refer the reader to Section 2 and Section 3 for the definition of $b(1)$. For that reason, we consider a sequence of bounded domains of $\{\Omega_{L,\varepsilon}\}_{L>0}$ of $P_{e,a}(G)$ which exhaust the path space in Section 4. $\Omega_{L,\varepsilon}$ is the ε-neighborhood in the uniform convergence topology of the level set of the energy function, $\Omega_L = \{\gamma \mid \sqrt{E(\gamma)} \le L\}$. The similar subset appeared in the study of [9]. We prove that $\frac{\lambda}{2}|b(1)|^2$ satisfies good integrability properties on these subsets. This integrability properties will be applied to determine the semi-classical behavior of low lying spectrum of the Ornstein-Uhlenbeck operator with Dirichlet boundary condition on $\Omega_{L,\varepsilon}$ in the forthcoming paper.

2. Path Integral and Logarithmic Sobolev Inequalities

Let (M, g) be a d-dimensional complete connected Riemannian manifold. Let Γ be a metric connection whose torsion T satisfies that

$$g(T(X,Y),Z) = -g(Y, T(X,Z))$$

for any vector fields X, Y, Z. We refer the reader to [8] for the notion, "torsion skew symmetric connection". Let $x \in M$ and consider a smooth path $\gamma(t)$ $(0 \le t \le 1)$ on M starting at x. Along γ, the parallel translation operator $\tau(\gamma)_t : T_x(M) \to T_{\gamma(t)}M$ is defined by the connection Γ. Let $P_{x,y,H^1}(M)$ be the space of H^1 maps from $[0,1]$ to M with $\gamma(0) = x, \gamma(1) = y \in M$. Let $T_\gamma P_{x,y,H^1}(M)$ be the tangent space at γ which consists of mapping h from $[0,1]$ to TM such that $h(t) \in T_{\gamma(t)}M$, $h(0)$ and $h(1)$ is 0, and its H^1-norm

$$\|h\|_{T_\gamma P_{x,y,H^1}(M)} = \left\{ \int_0^1 \left| \frac{d}{dt} \left(\tau(\gamma)_t^{-1} h(t) \right) \right|^2 dt \right\}^{1/2}$$

is finite. This Hilbert norm defines a Riemannian metric on $P_{x,y,H^1}(M)$. The gradient operator ∇ which is naturally defined by the metric is given explicitly for a smooth cylindrical function $f(\gamma) = F(\gamma(t_1), \dots, \gamma(t_n)) \in \mathfrak{F}C_b^\infty(P_{x,y,H^1}(M))$ by

$$(\nabla f)(\gamma)_t = \sum_{i=1}^n \tau(\gamma)_{t_i}^{-1} \nabla^{(i)} F(\gamma)(t \wedge t_i - tt_i). \tag{2.1}$$

Here $t \wedge t_i = \min(t, t_i)$ and $\nabla^{(i)} F(\gamma) \in T_{\gamma(t_i)}M$ denotes the covariant derivative with respect to the i-th variable. Now we consider measures on path spaces. Formally, we consider the Riemannian measure (Feynman measure) $d\gamma$ on $P_{x,y,H^1}(M)$ which is defined by the Riemannian metric. On the other hand, the Brownian bridge measure which is denoted by $\nu_{\lambda,x,y}$ is rigorously defined on $P_{x,y}(M)$ which is a space of continuous map from $[0,1]$ to M such that $\gamma(0) = x, \gamma(1) = y$. The formal expression of $\nu_{\lambda,x,y}$ is given by

$$d\nu_{\lambda,x,y}(\gamma) = Z_\lambda^{-1} \exp\left(-\lambda E(\gamma)\right) d\gamma,$$

where $E(\gamma) = \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2 dt$ ($\gamma \in P_{x,y}(M)$) and λ is a positive number. See Andersson and Driver [6] for the study of Brownian motion measure from this view point. Let

$$\mathcal{E}_\lambda(f,f) = \int_{P_{x,y}(M)} |(\nabla f)(\gamma)|^2_{T_\gamma P_{x,y}(M)} d\nu_{\lambda,x,y}(\gamma),$$

where $f \in \mathfrak{F}C_b^\infty(P_{x,y}(M))$. The gradient operator ∇ is also defined by the formula (2.1) using the stochastic parallel translation. We denote the generator by $-L_\lambda$. Let V be a real-valued measurable function on $P_{x,y}(M)$ and consider a Schrödinger operator $-L_{\lambda,V} = -L_\lambda + \lambda^2 V$ on $L^2(P_{x,y}(M), d\nu_{\lambda,x,y})$. Set $E(\lambda, V) = \inf \sigma(-L_{\lambda,V})$. We are interested in determining the asymptotics of $E(\lambda, V)$ when $\lambda \to \infty$. Let Δ be the Laplace-Bertlami operator on $P_{x,y}(M)$ which is formally defined by the Riemannian metric. Then $-L_{\lambda,V}$ on $L^2(P_{x,y}(M), d\nu_{\lambda,x,y})$ is formally unitary equivalent to

$$-H_{\lambda,V} = -\Delta + \lambda^2\left(\frac{1}{4}|(\nabla E)(\gamma)|^2_{T_\gamma(P_{x,y}(M))} + V(\gamma)\right) - \frac{\lambda}{2}(\Delta E)(\gamma)$$

on $L^2(P_{x,y}(M), d\gamma)$ by the mapping

$$f \in L^2(P_{x,y}(M), d\nu_{\lambda,x,y}) \to f \cdot Z_\lambda^{-1/2} e^{-\lambda E/2} \in L^2(P_{x,y}(M), d\gamma).$$

For $\gamma \in P_{x,y,H^1}(M)$, let $b(t) = \int_0^t \tau(\gamma)_s^{-1}\dot{\gamma}(s)ds$. Then $E(\gamma) = \frac{1}{2}\int_0^1 |\dot{b}(t)|^2 dt$. Note that

$$\begin{aligned}\nabla_h b(t) &= h(t) + \int_0^t \left(\int_0^s \overline{R(\gamma)}_u(h(u), \dot{b}(u))du\right)\dot{b}(s)ds \\ &\quad - \int_0^t \overline{T(\gamma)}_s(h(s), \dot{b}(s))ds,\end{aligned}$$

where $R(X,Y)Z$ denotes the curvature tensor and for any $\xi_i \in T_xM$,

$$\begin{aligned}\overline{R(\gamma)}_t(\xi_1,\xi_2)\xi_3 &= \tau(\gamma)_t^{-1}R(\gamma(t))(\tau(\gamma)_t\xi_1, \tau(\gamma)_t\xi_2)\tau(\gamma)_t\xi_3, \\ \overline{T(\gamma)}_u(\xi_1,\xi_2) &= \tau(\gamma)_t^{-1}T(\gamma(t))(\tau(\gamma)_t\xi_1, \tau(\gamma)_t\xi_2).\end{aligned}$$

By the skew symmetric property of the curvature tensor and the torsion, we have $(\nabla E)(\gamma)_t = b(t) - tb(1)$ and

$$\frac{1}{4}|(\nabla E)(\gamma)|^2_{T_\gamma(L_x(M))} = \frac{1}{4}\int_0^1 |\dot{b}(t)|^2_{T_x(M)}dt - \frac{1}{4}|b(1)|^2_{T_x(M)}.$$

Thus

$$-H_{\lambda,V} = -\Delta + \lambda^2\left(\frac{1}{2}E(\gamma) - \frac{1}{4}|b(1)|^2_{T_x(M)} + V(\gamma)\right) - \frac{\lambda}{2}\Delta E(\gamma).$$

We assume that the following LSI holds:

$$\int_{P_{x,y}(M)} f(\gamma)^2 \log\left(\frac{f(\gamma)^2}{\|f\|^2_{L^2(\nu_{\lambda,x,y})}}\right) d\nu_{\lambda,x,y} \leq \frac{2}{\lambda}\left(1 + \frac{C}{\lambda}\right)\mathcal{E}_{\lambda,V_{\lambda,x,y}}(f,f), \tag{2.2}$$

where

$$\begin{aligned}\mathcal{E}_{\lambda,V_{\lambda,x,y}}(f,f) &= \int_{P_{x,y}(M)} |(\nabla f)(\gamma)|_H^2 d\nu_{\lambda,x,y} \\ &\quad + \int_{P_{x,y}(M)} \lambda^2 V_{\lambda,x,y}(\gamma) f(\gamma)^2 d\nu_{\lambda,x,y} \\ V_{\lambda,x,y}(\gamma) &= \frac{1}{4}\left\{|b(1)|^2_{T_x(M)} + \frac{2}{\lambda}\log\left(\lambda^{-d/2} p\left(\frac{1}{\lambda},x,y\right)\right) + \frac{1}{\lambda}W(\gamma)\right\}\end{aligned}$$

and C is a positive constant and W is a real-valued measurable function. Getzler [10] and Gross [13] proved LSIs in the above forms in the case where M is a compact Lie group and $x=y$ are unit element e. They point out that $\frac{1}{4}|b(1)|^2$ is main term in the inequalities. LSI on a loop space $P_{x,x}(M)$ over general compact Riemannian manifolds were studied in [2, 11]. However the potential functions in their inequalities are very different from the above. Let $-L_{\lambda,V_{\lambda,x,y}}$ be the generator of the form $\mathcal{E}_{\lambda,V_{\lambda,x,y}}$. Then by the results in [12],

$$\begin{aligned}E(\lambda,V) &= \inf \sigma\left(-L_{\lambda,V_{\lambda,x,y}} - \lambda^2 V_{\lambda,x,y}(\gamma) + \lambda^2 V\right) \\ &\geq -\frac{\lambda}{2}\left(1+\frac{C}{\lambda}\right)^{-1} \log I(\lambda). \qquad (2.3)\end{aligned}$$

Here

$$\begin{aligned}I(\lambda) &= \int_{P_{x,y}(M)} \exp\left\{\lambda(1+\frac{C}{\lambda})\left(\frac{1}{2}|b(1)|^2_{T_x(M)} - 2V(\gamma)\right)\right\} \\ &\quad \exp\left\{2(1+\frac{C}{\lambda})W(\gamma)\right\}\lambda^{-d/2} d\bar{\nu}_{\lambda,x,y}(\gamma)\end{aligned}$$

and $\bar{\nu}_{\lambda,x,y} = p(1/\lambda,x,y)\nu_{\lambda,x,y}$. Let

$$U(\gamma) = \frac{1}{2}E(\gamma) - \frac{1}{4}|b(1)|^2_{T_x(M)} + V(\gamma).$$

We assume that $U(\gamma)\geq 0$ for all γ and $\min U = 0$, $N=\{\gamma \mid U(\gamma)=0\}$ are finite set and the hessian of U at them are non-degenerate. These assumptions are standard in semi-classical analysis in finite dimensions. We note that $\lim_{\lambda\to\infty} I(\lambda)$ converges under these and certain additional integrability assumptions. See [16]. This implies that $\liminf_{\lambda\to\infty}\frac{E(\lambda,V)}{\lambda} > -\infty$. In the case where $M=\mathbb{R}^d$, $b(1)=y-x$ holds and the asymptotics of $I(\lambda)$ is a classical problem. We determined the value of $\lim_{\lambda\to\infty}\frac{E(\lambda,V)}{\lambda}$ under certain assumptions on V in the case where $M=\mathbb{R}^d$ in [3]. Now we consider the case where M is a compact connected Lie group G and the metric is bi-invariant under the action of G and x is the unit element e. Let Γ be the right invariant connection. That is, for a smooth curve $\gamma(t)$ ($\gamma(0)=e$), the parallel translation is given by $\tau(\gamma)_t^{-1}(h(t)) = (R_{\gamma(t)^{-1}})_* h(t)$, where $R_g h = hg$ ($g,h\in G$). Since this is a torsion skew symmetric connection, if the inequality (2.2) holds, then the above formal argument could be applied to this case. In the next section, we prove an LSI as in (2.2).

3. Refined Version of Gross's Logarithmic Sobolev Inequalities on Pinned Path Groups

Let G be a d-dimensional compact connected and simply connected Lie group. Let e be the unit element of G. We denote the Lie algebra of G by $\mathfrak{g}$ which is identified with T_eG. Actually, G is isomorphic to a Lie subgroup of n-dimensional unitary group $SU(n)$ and the Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{su}(n)$. In this case, e is the identity matrix and the Lie bracket is given by $[A,B]=AB-BA$. By this result, we may assume that G is a matrix group. That is $G\subset SU(n)\subset M(n,\mathbb{C})\cong\mathbb{C}^{n^2}$. $M(n,\mathbb{C})$ denotes the all $n\times n$-matrices whose elements are complex numbers. For $A,B\in M(n,\mathbb{C})$, let $(A,B)=$ real part of $\mathrm{tr}AB^*$. This defines an inner product on $M(n,\mathbb{C})$ and a bi-invariant Riemannian metric on G. We denote by dx the Riemannian measure (Haar measure). Then by the bi-invariance of the metric, we have

$$\int_G f(x)dx=\int_G f(gx)dx=\int_G f(xg)dx=\int_G f(x^{-1})dx. \tag{3.1}$$

For $A\in\mathfrak{g}$, $e^A,\exp A$ denotes the matrix exponential element as well as the exponential map in Riemannian geometry sense. Let $i(G)$ be the injectivity radius of G. That is,

$$i(G) \quad = \quad \sup\{r \mid \exp: B_r(0)\to G \text{ gives a local chart at } e\},$$

where $B_r(0)=\{v\in T_eG \mid |v|<r\}$. Let $P_e(G)$ be a set of continuous paths $\gamma(t)$ $(0\le t\le 1)$ with values in G with $\gamma(0)=e$. Let $\lambda>0$. There exists a probability measure which is called the Brownian motion measure ν_λ on $P_e(G)$ such that for $0=t_0<t_1<\cdots<t_n\le 1$,

$$\begin{aligned}&\nu_\lambda\left(\gamma_{t_1}\in A_1,\ldots,\gamma_{t_n}\in A_n\right)\\ &\quad=\int_{A_1}dx_1\cdots\int_{A_n}dx_n\prod_{i=1}^{n}p\left(\lambda^{-1}(t_i-t_{i-1}),x_{i-1},x_i\right),\end{aligned}$$

where $x_0=e$. Here $p(t,x,y)$ denotes the heat kernel of $e^{t\Delta/2}$. Let $P_{e,a}(G)$ be a subset of $P_e(G)$ such that $\gamma(1)=a$. Also we denote $P_{e,a,H^1}(G)=\{\gamma\in P_{e,a}(G)\mid E(\gamma):=\frac{1}{2}\int_0^1|\dot\gamma(t)|^2dt<\infty\}$. There exists a probability measure $\nu_{\lambda,a}$ which is called a Brownian bridge measure on $P_{e,a}(G)$ such that for $0=t_0<t_1<\cdots<t_n<t_{n+1}=1$

$$\begin{aligned}&\nu_{\lambda,a}\left(\gamma_{t_1}\in A_1,\ldots,\gamma_{t_n}\in A_n\right)\\ &\quad=p(\lambda^{-1},e,a)^{-1}\int_{A_1}dx_1\cdots\int_{A_n}dx_n\prod_{i=1}^{n+1}p\left(\lambda^{-1}(t_i-t_{i-1}),x_{i-1},x_i\right),\end{aligned}$$

where $x_0=e, x_{n+1}=a$. Let

$$\begin{aligned}H(\mathfrak{g}) &= \left\{h:[0,1]\to\mathfrak{g}\;\middle|\; h(0)=0, \|h\|_H^2:=\int_0^1|\dot h(t)|^2dt<\infty\right\}\\ H_0(\mathfrak{g}) &= \left\{h\in H(\mathfrak{g})\;\middle|\; h(1)=0\right\}.\end{aligned}$$

Below, we use the notation X to denote $P_{e,a}(G), P_e(G)$. Let $\mathfrak{F}C_b^\infty(X)$ be the set of smooth cylindrical functions on X. We define the H-derivative $(\nabla F)(\gamma)$ of $F(\gamma) \in \mathfrak{F}C_b^\infty(X)$ by the unique element of $H(\mathfrak{g})$ (or $H_0(\mathfrak{g})$) satisfying that

$$((\nabla F)(\gamma), h)_H := \lim_{\varepsilon \to 0} \frac{F\left(e^{\varepsilon h(\cdot)}\gamma(\cdot)\right) - F(\gamma)}{\varepsilon}.$$

for all $h \in H(\mathfrak{g})$ (or $h \in H_0(\mathfrak{g})$).

We introduce necessary tools for our analysis. We define $\mathfrak{g}$-valued process

$$b(t, \gamma) \quad = \quad \int_0^t d\gamma(s) \circ \gamma(s)^{-1}.$$

Below, we may denote this simply by $b(t)$. The above integral is Stratonovich integral when $\gamma(t)$ is the Brownian motion. However $b(t)$ could be still defined under $\nu_{\lambda,a}$ by the quasi-sure analysis [17]. We collect basic results for $b(t)$. Below $\{\varepsilon_i\}_{i=1}^d$ denotes an orthonormal basis on $\mathfrak{g}$.

Lemma 3.1. *Below, we consider the stochastic processes under the law of* ν_λ.
(1) *The distribution of* $b(\cdot)$ *are the Brownian motion measure which satisfies*

$$E[((b(t), \varepsilon_i)(b(t), \varepsilon_j)] = \frac{1}{\lambda}\delta_{i,j} s \wedge t.$$

(2) *Let* $\mathrm{C} = \sum_{i=1}^d \varepsilon_i^2 \in M(n, \mathbb{C})$. *Then*

$$d\left(\gamma(t)^{-1}\right) = -\gamma(t)^{-1} db(t) + \frac{1}{2\lambda}\gamma(t)^{-1}\mathrm{C}dt.$$

The proof of this lemma is standard and we refer the proof to [13, 8]. Also we note that C is called the Casimir element which commutes any matrices in G.

In the lemma below, we use the notation in the Watanabe's distribution theory [18]. The reader may find the proof in [13].

Lemma 3.2. (1)

$$\nabla_h b(t) = h(t) + \int_0^t [h(s), db(s)] \tag{3.2}$$

(2)

$$b(t, e^{\cdot\xi}\gamma(\cdot)) = \int_0^t Ad\left(e^{s\xi}\right) db(s) + t\xi$$

(3)

$$\int_{P_e(G)} F(e^{\cdot\xi}\gamma(\cdot))G(b) \exp\left(-\lambda(b(1), \xi) - \frac{\lambda}{2}|\xi|^2\right) \delta_x(\gamma(1)) d\nu_\lambda(\gamma)$$
$$= \int_{P_e(G)} F(\gamma) G\left(\int_0^\cdot Ad(e^{-s\xi}) db(s) - \cdot\xi\right) \delta_{e^\xi x}(\gamma(1)) d\nu_\lambda(\gamma).$$

(4) *It holds that*

$$\int_{P_{e,a}(G)} \nabla_h F(\gamma) G(\gamma) d\nu_{\lambda,a}(\gamma)$$
$$= \int_{P_{e,a}(G)} F(\gamma)\left\{-\nabla_h G(\gamma) + \lambda(b,h)_H G(\gamma)\right\} d\nu_{\lambda,a}(\gamma). \tag{3.3}$$

We prove the following refined version of Gross's LSI. In the case where $a = e$, the theorem is proved by Getzler [10]. The proof is almost similar to Gross [13] and Getzler [10]. However, we give a proof of it since it may be not trivial that an LSI holds with the potential function similar to the case where $a = e$ and the precise form (the dependence on λ) is important in the forthcoming applications.

Theorem 3.3. *There exist constants $C_1, C_2 > 0$ such that for any sufficiently large $\lambda > 0$ and $f \in \mathfrak{F}C_b^\infty(P_{e,a}(G))$, it holds that*

$$\int_{P_{e,a}(G)} f^2(\gamma) \log\left(\frac{f^2(\gamma)}{\|f\|^2_{L^2(\nu_{\lambda,a})}}\right) d\nu_{\lambda,a}(\gamma) \leq \frac{2}{\lambda}\left(1 + \frac{C_1}{\lambda}\right) \mathcal{E}_{\lambda,V_{\lambda,a}}(f,f), \tag{3.4}$$

where

$$\begin{aligned}
\mathcal{E}_{\lambda,V_{\lambda,a}}(f,f) &= \int_{P_{e,a}(G)} |(\nabla f)(\gamma)|^2_{H_0} d\nu_{\lambda,a} \\
&\quad + \int_{P_{e,a}(G)} \lambda^2 V_{\lambda,a}(\gamma) f(\gamma)^2 d\nu_{\lambda,a}, \\
V_{\lambda,a}(\gamma) &= \frac{1}{4}\left\{|b(1)|^2 + \frac{2}{\lambda}\log\left(\lambda^{-d/2} p(1/\lambda, e, a)\right)\right\} \\
&\quad + \frac{C_2}{\lambda}\left\{1 + |b(1)|^2 + \left(\int_0^1 |b(s)| ds\right)^2\right\}.
\end{aligned}$$

Remark 3.4. If e and a are sufficiently close, then there exist $C_1, C_2 > 0$ such that for all sufficiently small t

$$C_1 t^{-d/2} e^{-d(e,a)^2/(2t)} \leq p(t,e,a) \leq C_2 t^{-d/2} e^{-d(e,a)^2/(2t)}.$$

Thus (3.4) is equivalent to the inequality which is obtained by replacing $V_{\lambda,a}(\gamma)$ by $\tilde{V}_{\lambda,a}(\gamma)$:

$$\tilde{V}_{\lambda,a}(\gamma) = \frac{1}{4}(|b(1)|^2 - d(e,a)^2) + \frac{C_2}{\lambda}\left\{1 + |b(1)|^2 + \left(\int_0^1 |b(s)| ds\right)^2\right\}.$$

We use the following lemmas to prove Theorem 3.3.

Lemma 3.5. *Let $\delta > 0$ and set $U_\delta(x) = \{y \in G \mid d(x,y) < \delta\}$. For a sufficiently small δ, there exists a smooth map $\log : U_\delta(e) \to \mathfrak{g}$ such that $\exp(\log y) = y$ for any $y \in U_\delta(e)$ and $\log e = 0$.*

Lemma 3.6. (1) *Assume that $d(x,e)$ is sufficiently small. Let*

$$\Psi(\varepsilon) = \log(x \exp(\varepsilon\xi)).$$

Then

$$\Psi'(0) = \xi + \frac{1}{2}[\log x, \xi] + A_{\log x}\xi, \tag{3.5}$$

where A_v is a linear map on $\mathfrak{g}$ such that $\|A_v\|_{op} \leq C|v|^2$.
(2) *For γ with $d(a,\gamma(1))$ to be sufficiently small,*

$$\nabla_h\left(\log(a\gamma(1)^{-1})\right) = -h(1) - \frac{1}{2}[\log(a\gamma(1)^{-1}), h(1)] + A_{\log(a\gamma(1)^{-1})}h(1).$$

Proof. In the calculation below, I denotes the identity matrix. We have

$$\begin{aligned}\Psi'(0) &= x\xi+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(x-I)^k x\xi(x-I)^{(n-k-1)}\\ &= \xi+\frac{1}{2}[x-I,\xi]+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(x-I)^{k+1}\xi(x-I)^{(n-k-1)}\\ &\quad+\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(x-I)^k\xi(x-I)^{(n-k-1)}.\end{aligned}$$

Note that $x\xi$ etc are matrix product in $GL(n,\mathbb{C})$. This follows from the Taylor expansion:

$$\begin{aligned}\Psi(\varepsilon) &= \log\big((I+(x-I)+\varepsilon x\xi+O(\varepsilon^2)\big)\\ &= \log x+\varepsilon\left(x\xi+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(x-I)^k x\xi(x-I)^{(n-k-1)}\right)+O(\varepsilon^2).\end{aligned}$$

Noting $\log x=x-I+O(|x-I|^2)$, we get (3.5). □

Lemma 3.7. *Let δ be a sufficiently small positive number. Let $S : P_{e,U_\delta(a)}(G)\to P_{e,a}(G)$ be the map which is defined by*

$$S(\gamma)(t)=\exp\big(t\log(a\gamma(1)^{-1})\big)\,\gamma(t)\quad (0\le t\le 1).$$

Here $P_{e,U_\delta(a)}(G)=\{\gamma\in P_e(G)\mid \gamma(1)\in U_\delta(a)\}$. Then, the following statements hold.

(1) *Let $T(\gamma)$ be the bounded linear operator from $H(\mathfrak{g})$ to $H_0(\mathfrak{g})$ such that*

$$(T(\gamma)h)(t)=\big\{\big(R_{S(\gamma)(t)}\big)_*\big\}^{-1}\frac{d}{d\varepsilon}\big(S(e^{\varepsilon h}\gamma)(t)\big)\Big|_{\varepsilon=0}.$$

R_* denotes the derivative of the right translation. Then for any $f\in\mathfrak{F}C_b^\infty(P_{e,a}(G))$ and $h\in H(\mathfrak{g})$, it holds that

$$(\nabla(f\circ S)(\gamma),h)_H=((\nabla f)(S\gamma),T(\gamma)h)_{H_0}. \tag{3.6}$$

Moreover $T(\gamma)$ can be written in the following form:

$$\begin{aligned}(T(\gamma)h)(t) &= h(t)-th(1)+\frac{t(t-1)}{2}\big[\log(a\gamma(1)^{-1}),h(1)\big]\\ &\quad+t[\log(a\gamma(1)^{-1}),h(t)-th(1)]+A_{\log(a\gamma(1)^{-1})}(t)h(t).\end{aligned}\tag{3.7}$$

$A_v(t)$ is a $GL(n,\mathbb{R})$-valued smooth function of $t\in[0,1]$ and $v\in\mathfrak{g}$ and satisfies that

$$\max_{0\le t\le 1}\left\{\|A_v(t)\|_{op}+\left\|\frac{\partial}{\partial t}A_v(t)\right\|_{op}\right\}\le C\|v\|_{\mathfrak{g}}^2$$

and $A_v(1)=0$ for all v.

(2) *For any $f\in\mathfrak{F}C_b^\infty(P_{e,a}(G))$, it holds that*

$$\begin{aligned}\|\nabla(f\circ S)(\gamma)\|_H^2 &\le (1+C\|\log(a\gamma(1)^{-1})\|_{\mathfrak{g}}^2)\|(\nabla f)(S\gamma)\|_{H_0}^2\\ &\quad+2\big(B_{\log(a\gamma(1)^{-1})}(\nabla f)(S\gamma),(\nabla f)(S\gamma)\big)_{H_0}.\end{aligned}$$

Here B_v is a bounded linear operator on $H_0(\mathfrak{g})$ such that $B_v h(t) = M^*([v, h(\cdot)])$ and M^* is the adjoint operator of M on $H_0(\mathfrak{g})$ which is given by $(Mh)(t) = th(t)$.

Proof. (1) (3.6) is the chain rule. We prove (3.7). For x which is sufficiently close to e, we denote $K_x\xi = \xi + \frac{1}{2}[\log x, \xi] + A_{\log x}\xi$, where this operator appeared in Lemma 3.6 (1). Let $k_\varepsilon(t) = S(e^{\varepsilon h}\gamma)(t)$. Then by (3.5),

$$\begin{aligned}
&k_\varepsilon(t) - k_0(t)\\
&\quad= \exp\Big\{t\left(\log\left(a\gamma(1)^{-1}\right) - \varepsilon K_{a\gamma(1)^{-1}}h(1) + O(\varepsilon^2)\right)\Big\}e^{\varepsilon h(t)}\gamma(t)\\
&\quad- \exp\left\{t\log(a\gamma(1)^{-1})\right\}\gamma(t)\\
&\quad= \Big\{\exp\left[t\left\{\log\left(a\gamma(1)^{-1}\right) - \varepsilon K_{a\gamma(1)^{-1}}h(1) + O(\varepsilon^2)\right\}\right]\\
&\qquad - \exp\left[t\log(a\gamma(1)^{-1})\right]\Big\}e^{\varepsilon h(t)}\gamma(t)\\
&\qquad + \exp\left(t\log(a\gamma(1)^{-1})\right)(e^{\varepsilon h(t)} - I)\gamma(t) := I_1(\varepsilon) + I_2(\varepsilon).
\end{aligned}$$

Below, we denote $[A, B]_+ = AB + BA$ and $A_v^{(i)}(t)$ denotes an $GL(n, \mathbb{R})$-valued smooth function of $0 \le t \le 1$ and $v \in \mathfrak{g}$ and satisfies that $\|A_v^{(i)}(t)\|_{op} \le C\|v\|_{\mathfrak{g}}^2$ for all $v \in \mathfrak{g}$. We have

$$\begin{aligned}
I_1(\varepsilon) \quad = \quad & -\Bigg[\varepsilon t K_{a\gamma(1)^{-1}}h(1) + \frac{\varepsilon}{2}t^2\left[\log(a\gamma(1)^{-1}), K_{a\gamma(1)^{-1}}h(1)\right]_+\\
& + \varepsilon\sum_{n=3}^{\infty}\frac{t^n}{n!}\left\{\sum_{k=0}^{n-1}\left(\log(a\gamma(1)^{-1})\right)^k K_{a\gamma(1)^{-1}}h(1)\left(\log(a\gamma(1)^{-1})\right)^{n-k-1}\right\}\\
& + O(\varepsilon^2)\Bigg]e^{\varepsilon h(t)}\gamma(t),
\end{aligned}$$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{I_1(\varepsilon)}{\varepsilon}\\
&= -\Bigg[tK_{a\gamma(1)^{-1}}h(1) + \frac{t^2}{2}\left[\log(a\gamma(1)^{-1}), K_{a\gamma(1)^{-1}}h(1)\right]_+\\
&\quad + \sum_{n=3}^{\infty}\frac{t^n}{n!}\left\{\sum_{k=0}^{n-1}\left(\log(a\gamma(1)^{-1})\right)^k K_{a\gamma(1)^{-1}}h(1)\left(\log(a\gamma(1)^{-1})\right)^{n-k-1}\right\}\Bigg]\gamma(t)\\
&= -\Big\{tK_{a\gamma(1)^{-1}}h(1) + \frac{t^2}{2}\left[\log(a\gamma(1)^{-1}), K_{a\gamma(1)^{-1}}h(1)\right]_+\\
&\quad + A^{(1)}_{\log(a\gamma(1)^{-1})}(t)h(1)\Big\}\gamma(t) =: J_1(t).
\end{aligned}$$

Therefore

$$\begin{aligned}
&J_1(t)S(\gamma)(t)^{-1}\\
&\quad= -\Big\{th(1) - t^2h(1)\log(a\gamma(1)^{-1}) + \frac{1}{2}t\left[\log(a\gamma(1)^{-1}), h(1)\right]\\
&\qquad + \frac{t^2}{2}\left[\log(a\gamma(1)^{-1}), K_{a\gamma(1)^{-1}}h(1)\right]_+ + A^{(2)}_{\log(a\gamma(1)^{-1})}(t)h(1)\Big\}.
\end{aligned}$$

We consider the term $I_2(\varepsilon)$.

$$\lim_{\varepsilon\to 0}\frac{I_2(\varepsilon)}{\varepsilon}=\exp\left[t\log(a\gamma(1)^{-1})\right]h(t)\gamma(t)=:J_2(t).$$

Hence

$$\begin{aligned} J_2(t)S(\gamma)_t^{-1} &= Ad\left(\exp\left[t\log\left(a\gamma(1)^{-1}\right)\right]\right)h(t)\\ &= h(t)+t\left[\log(a\gamma(1)^{-1}),h(t)\right]+A^{(3)}_{\log(a\gamma(1)^{-1})}(t)h(t). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &(J_1(t)+J_2(t))\,(S(\gamma)(t))^{-1}\\ &\quad=h(t)-th(1)+\frac{t(t-1)}{2}\left[\log(a\gamma(1)^{-1}),h(1)\right]\\ &\quad+t[\log(a\gamma(1)^{-1}),h(t)-th(1)]+A^{(4)}_{\log(a\gamma(1)^{-1})}(t)h(t)+A^{(5)}_{\log(a\gamma(1)^{-1})}(t)h(1). \end{aligned}$$

Since $J_1(1)+J_2(1)=0$ for all h, $A_v^{(i)}(1)=0$ $(i=4,5)$ holds for all v and this implies (3.7) holds. We prove (2). Let $\{e_n\}$ be the complete orthonormal system on $H(\mathfrak{g})$ as follows: $e_n(t)=\varepsilon_n t$ $(1\le n\le d)$ and $e_n(1)=0$ for all $n\ge d+1$. Then it holds that for all $1\le n\le d$

$$|\left((\nabla f)(S\gamma),T(\gamma)e_n\right)|^2\le C\|\log(a\gamma(1)^{-1})\|_{\mathfrak{g}}^2|(\nabla f)(S\gamma)|^2_{H_0}.$$

Also for $n\ge d+1$,

$$\begin{aligned} |\left((\nabla f)(S\gamma),T(\gamma)e_n\right)|^2 &\le \left((\nabla f)(S\gamma),e_n\right)^2+\left((\nabla f)(S\gamma),[\tilde{e}_n,\log(a\gamma(1)^{-1})]\right)^2\\ &\quad+\left((\nabla f)(S\gamma),A_{\log(a\gamma(1)^{-1})}(\cdot)h(\cdot)\right)^2\\ &\quad+2\left((\nabla f)(S\gamma),e_n\right)\left((\nabla f)(S\gamma),[\tilde{e}_n,\log(a\gamma(1)^{-1})]\right)\\ &\quad+C\|(\nabla f)(S\gamma)\|^2_{H_0}\|\log(a\gamma(1)^{-1})\|^2_{\mathfrak{g}}, \end{aligned}$$

where $\tilde{e}_n(t)=te_n(t)$. Since $\{e_n\}_{n=d+1}^{\infty}$ is a complete orthonormal system of $H_0(\mathfrak{g})$,

$$\begin{aligned} &\sum_{n=d+1}^{\infty}\left((\nabla f)(S\gamma),e_n\right)\left((\nabla f)(S\gamma),[\tilde{e}_n,\log(a\gamma(1)^{-1})]\right)\\ &\qquad=\left(B_{\log(a\gamma(1)^{-1})}(\nabla f)(S\gamma),(\nabla f)(S\gamma)\right)_{H_0}. \end{aligned}$$

This completes the proof. □

Lemma 3.8. *For $f\in\mathfrak{F}C_b^{\infty}(P_{e,a}(G))$, let*

$$\begin{aligned} \hat{f}_\phi(\gamma) &= f\left(S(\gamma)\right)\Phi(\gamma)\bar{\phi}(\gamma),\\ \Phi(\gamma) &= \exp\left(-\frac{\lambda}{2}\left(\log(a\gamma(1)^{-1}),b(1)\right)-\frac{\lambda}{4}\left|\log\left(a\gamma(1)^{-1}\right)\right|^2\right),\\ \bar{\phi}(\gamma) &= \phi\left(\sqrt{\lambda}\log(a\gamma(1)^{-1})\right)\psi\left(\frac{d(a,\gamma(1))}{\delta}\right)D_\lambda^{-1}. \end{aligned}$$

Here ϕ is a non-negative smooth function on $\mathfrak{g}$ such that $\phi(v)=\phi(-v)$ for all v and $\phi(v)=0$ for v with $\|v\|\ge 1$. ψ is a smooth function with compact support on $\mathbb{R}$

and takes a value 1 *near* 0. δ *is a sufficiently small number such that* $\log(a\gamma(1)^{-1})$ *is well-defined. Also we set*

$$D_\lambda = \{p(1/\lambda, e, a)E_\lambda\}^{1/2}, \quad E_\lambda = \int_G \phi\left(\sqrt{\lambda}\log x\right)^2 \psi\left(\frac{d(e,x)}{\delta}\right)^2 dx.$$

Then
(1) $\lim_{\lambda\to\infty} \lambda^{d/2}E_\lambda$ *converges,*
(2)

$$\int_{P_e(G)} \hat{f}_\phi(\gamma)^2 d\nu_\lambda(\gamma) = \int_{P_{e,a}(G)} f(\gamma)^2 d\nu_{\lambda,a}(\gamma), \tag{3.8}$$

(3) *For sufficiently large* $\lambda > 0$, *it holds that* $\nabla(\hat{f}_\phi)(\gamma)_t = I_1 + I_2 + I_3$, *where*

$$\begin{aligned}
I_1 &= T(\gamma)^*((\nabla f)(S(\gamma))_t \Phi(\gamma)\bar{\phi}(\gamma),\\
I_2 &= \frac{\lambda}{2} f(S(\gamma))\Big\{ t\Big(I + \frac{1}{2}\mathrm{ad}(\log(a\gamma(1)^{-1})) + Q^{(1)}_{\log(a\gamma(1)^{-1})}\Big)b(1)\\
&\quad -\mathrm{ad}\left(\log(a\gamma(1)^{-1})\right)\left(\int_0^t b(s)ds\right) + tQ^{(2)}_{\log(a\gamma(1)^{-1})}(\log(a\gamma(1)^{-1})\Big\}\Phi(\gamma)\bar{\phi}(\gamma),\\
I_3 &= f\left(S(\gamma)\right)\Phi(\gamma)D_\lambda^{-1}\Big\{\sqrt{\lambda}\psi\left(\frac{d(a,\gamma(1))}{\delta}\right)\\
&\quad \Big\{-t(I + \frac{1}{2}\mathrm{ad}(\log(a\gamma(1)^{-1})) + tQ^{(3)}_{\log(a\gamma(1)^{-1})}\Big\}(\nabla\phi)\left(\sqrt{\lambda}\log(a\gamma(1)^{-1})\right)\\
&\quad +t\phi\left(\sqrt{\lambda}\log(a\gamma(1)^{-1})\right)\delta^{-1}\psi'\left(\frac{d(a,\gamma(1))}{\delta}\right)\left(R_{\gamma(1)}\right)_*^{-1}\nabla_x d(a,x)\Big|_{x=\gamma(1)}\Big\},
\end{aligned}$$

where $Q^{(i)}_v$ *are linear maps satisfying that* $|Q^{(i)}_v| \le C|v|^2_{\mathfrak{g}}$.

Proof. (1) This is easily proved by that $\log : U_\delta(e) \to \mathfrak{g}$ is a smooth one to one invertible mapping and the image measure of log is equal to the Lebesgue measure on $\mathfrak{g}$ multiplied by a smooth positive function.
(2) By Lemma 3.2 (3),

$$\begin{aligned}
&\int_{P_e(G)} \hat{f}_\phi(\gamma)^2 d\nu_\lambda(\gamma)\\
&= \int_G dx \phi\left(\sqrt{\lambda}\log(ax^{-1})\right)^2 \psi\left(\frac{d(a,x)}{\delta}\right)^2 \int_{P_e(G)} \delta_x(\gamma(1)) f\left(\exp(\cdot\log(ax^{-1})\gamma(\cdot)\right)^2\\
&\quad \exp\left(-\lambda\left(\log(ax^{-1}), b(1)\right) - \frac{\lambda}{2}\left|\log\left(ax^{-1}\right)\right|^2\right) d\nu_\lambda(\gamma) D_\lambda^{-2}\\
&\quad = \int_G dx \int_{P_e(G)} \delta_a(\gamma(1)) f(\gamma(\cdot))^2 d\nu_\lambda(\gamma)\phi\left(\sqrt{\lambda}\log(ax^{-1})\right)^2 \psi\left(\frac{d(a,x)}{\delta}\right)^2 D_\lambda^{-2}\\
&\quad = \int_G dx\phi\left(\sqrt{\lambda}\log(ax^{-1})\right)^2 \psi\left(\frac{d(a,x)}{\delta}\right)^2 \int_{P_e(G)} \delta_a(\gamma(1)) f(\gamma(\cdot))^2 d\nu_\lambda(\gamma) D_\lambda^{-2}\\
&\quad = E_\lambda^{-2}\int_G\int_{P_{e,a}(G)} \phi\left(\sqrt{\lambda}\log(x)\right)^2 \psi\left(\frac{d(e,x)}{\delta}\right)^2 f(\gamma)^2 dx d\nu_{\lambda,a}(\gamma).
\end{aligned} \tag{3.9}$$

In the last equality we have used the invariance of the Riemannian volume (3.1).
(3) This follows from (3.2) and Lemma 3.6. □

We prove Theorem 3.3.

Proof. The proof is similar to Gross's inheritance method. Note that

$$\begin{aligned}&\int_{P_e(G)}\hat{f}_\phi(\gamma)^2\log\left(\hat{f}_\phi(\gamma)^2\right)d\nu_\lambda(\gamma)\\&=\int_{P_{e,a}}f(\gamma)^2\log f(\gamma)^2 d\nu_{\lambda,a}(\gamma)+\int_{P_e(G)}\hat{f}_\phi(\gamma)^2\log\left(\Phi^2(\gamma)\right)d\nu_\lambda(\gamma)\\&+\int_{P_e(G)}\hat{f}_\phi(\gamma)^2\log\left(\phi\left(\sqrt{\lambda}\log(a\gamma(1)^{-1})\right)^2\psi\left(\frac{d(a,\gamma(1))}{\delta}\right)^2\right)d\nu_\lambda(\gamma)\\&-2\log D_\lambda\int_{P_{e,a}(G)}f(\gamma)^2d\nu_{\lambda,a}(\gamma)\\&=J_1(\lambda)+J_2(\lambda)+J_3(\lambda)+J_4(\lambda).\end{aligned}$$

We estimate each terms. By the calculation similar to (3.8),

$$\begin{aligned}J_2(\lambda)&=\int_G D_\lambda^{-2}\phi\left(\sqrt{\lambda}\log x\right)^2\psi\left(\frac{d(e,x)}{\delta}\right)^2\bar{J}_2(\lambda,x)dx\\\bar{J}_2(\lambda,x)&=\int_{P_e(G)}f(\gamma)^2\left\{-\lambda(\log x,b(1))-\frac{\lambda}{2}|\log x|^2\right\}\delta_a(\gamma(1))d\nu_\lambda(\gamma).\end{aligned}$$

By $\log(x^{-1})=-\log x$, $\phi(v)=\phi(-v)$ and the bi-invariance of dx, we see that the integral of the term containing $(\log x, b(1))$ is zero. Thus, we have $J_2(\lambda)\ge -\frac{\lambda}{2}C\|f\|^2_{L^2(P_{e,a}(G))}$. By the similar calculation to (3.9) and the fact in (1), we have

$$\begin{aligned}&J_3(\lambda)\\&=E_\lambda^{-1}\int_G\phi\left(\sqrt{\lambda}\log x\right)^2\psi\left(\frac{d(e,x)}{\delta}\right)^2\log\left\{\phi\left(\sqrt{\lambda}\log x\right)^2\psi\left(\frac{d(e,x)}{\delta}\right)^2\right\}dx\\&\quad\cdot\int_{P_{e,a}(G)}f(\gamma)^2d\nu_{\lambda,a}(\gamma)\ge -C\int_{P_{e,a}(G)}f(\gamma)^2d\nu_{\lambda,a}(\gamma),\end{aligned}$$

where C is a positive constant. Again by the result in (1), there exists a positive constant C such that

$$J_4(\lambda)\ge-\log\left(C\lambda^{-d/2}p(1/\lambda,e,a)\right)\|f\|^2_{L^2(\nu_{\lambda,a})}.$$

Therefore, by Gross's LSI on $P_e(G)$ [13],

$$\begin{aligned}&\int_{P_{e,a}(G)}f^2\log\left(\frac{f^2}{\|f\|^2_{L^2(\nu_{\lambda,a})}}\right)d\nu_{\lambda,a}\\&\le\frac{2}{\lambda}\int_{P_e(G)}|(\nabla\hat{f}_\phi)(\gamma)|^2d\nu_\lambda(\gamma)+\left\{\log\left(C_1\lambda^{-d/2}p(1/\lambda,e,a)\right)+C_2\right\}\|f\|^2_{L^2(\nu_{\lambda,a})}.\end{aligned}$$

Next we estimate the integral of $|(\nabla \hat{f}_\phi)(\gamma)|^2$. By Lemma 3.8 (3), we need to estimate $|I_i|^2$ and the cross terms (I_i, I_j). In the calculation below, we denote $\phi_\lambda(x) = \phi\left(\sqrt{\lambda}\log x\right)^2 \psi\left(\frac{d(e,x)}{\delta}\right)^2$.

(1) I_1: By the similar calculation to (3.9), we have

$$\begin{aligned}\int_{P_e(G)} |I_1|^2 d\nu_\lambda(\gamma) &\leq \left(1+\frac{C}{\lambda}\right)\int_{P_{e,a}(G)} |(\nabla f)(\gamma)|^2 d\nu_{\lambda,a}(\gamma)\\ &\quad + \int_G dx \int_{P_{e,a}(G)} 2\phi_\lambda(x)\left(B_{\log x}\left(\nabla f\right)(\gamma), \left(\nabla f\right)(\gamma)\right) d\nu_{\lambda,a}(\gamma).\end{aligned}$$

Since $B_{\log(x^{-1})} = -B_{\log(x)}$ and $\phi_\lambda(x) = \phi_\lambda(x^{-1})$, the second integral is 0.

(2) I_2: By Lemma 3.2 (1) and (2),

$$\int_{P_e(G)} |I_2|^2 d\nu_\lambda(\gamma) = \frac{\lambda^2}{4E_\lambda}\int_{P_{e,a}(G)}\int_G I_2(x,\gamma)\phi_\lambda(x) f(\gamma)^2 dx d\nu_{\lambda,a}(\gamma).$$

$$\begin{aligned}I_2(x,\gamma) &= \int_0^1 \Bigg| b(1) + \int_0^1 \left(Ad\left(e^{-s\log x}\right) - I\right) db(s) - 2\log x\\ &\quad + \frac{1}{2}\mathrm{ad}(\log x)\left(\int_0^1 Ad\left(e^{-u\log x}\right) db(u)\right) + Q^{(1)}_{\log x}(\log x)\\ &\quad - Q^{(2)}_{\log x}\left(\int_0^1 Ad(e^{-s\log x}) db(s)\right)\\ &\quad - \mathrm{ad}(\log x)\left(\int_0^t Ad(e^{-u\log x}) db(u) - t\log x\right)\Bigg|^2 dt.\end{aligned}$$

Then

$$I_2(x,\gamma) \leq |b(1)|^2 + C|\log x|^2\left\{1 + |b(1)|^2 + \left(\int_0^1 |b(s)|ds\right)^2\right\} + \tilde{A}_{\log x}(\gamma),$$

where $v \to \tilde{A}_v(\gamma)$ is a linear function. By the similar reason to I_1,

$$\begin{aligned}&\frac{\lambda^2}{4E_\lambda}\int_G I_2(x,\gamma)\phi_\lambda(x)dx\\ &\qquad \leq \frac{\lambda^2}{4}|b(1)|^2 + C\lambda\left\{1 + |b(1)|^2 + \left(\int_0^1 |b(s)|ds\right)^2\right\}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_{P_e(G)} |I_2|^2 d\nu_\lambda(\gamma) &\leq \int_{P_{e,a}(G)} \frac{\lambda^2}{4}|b(1)|^2 f(\gamma)^2 d\nu_{\lambda,a}(\gamma)\\ &\quad + \lambda C\int_{P_{e,a}(G)}\left\{1 + |b(1)|^2 + \left(\int_0^1 |b(s)|ds\right)^2\right\} f(\gamma)^2 d\nu_{\lambda,a}(\gamma).\end{aligned}$$

(3) I_3: We have $|I_3|^2 \leq C\lambda \|f\|^2_{L^2(P_{e,a}(G),\nu_{\lambda,a})}$.

(4) The cross term (I_2, I_3).

Since $\phi^2(\xi)=\phi^2(-\xi)$, we have $(\nabla\phi^2)(\xi)=-(\nabla\phi^2)(-\xi)$. By the invariance of the Haar measure,

$$\int_G (\nabla\phi^2)\left(\sqrt{\lambda}\log x\right)\psi\left(\frac{d(e,x)}{\delta}\right)^2 dx$$
$$=\int_G (\nabla\phi^2)\left(-\sqrt{\lambda}\log x\right)\psi\left(\frac{d(e,x)}{\delta}\right)^2 dx.$$

Therefore, this integral value is 0. Thus, the integral of the term containing $\left(b(1),\nabla\phi\left(\sqrt{\lambda}\log(a\gamma(1)^{-1})\right)\right)$ is 0. Consequently,

$$\int_{P_{e,a}(G)}(I_2,I_3)d\nu_{\lambda,a}(\gamma)$$
$$\le C\lambda\int_{P_{e,a}(G)}\left\{1+|b(1)|^2+\left(\int_0^1|b(s)|ds\right)^2\right\}f(\gamma)^2 d\nu_{\lambda,a}(\gamma).$$

(5) The cross terms: $(I_1,I_2),(I_1,I_3)$:

To estimate these terms, we use the integration by parts formula (3.3) and the fact that $T(\gamma)\eta=A_{\log(a\gamma(1)^{-1})}(t)\eta(t)$ for $\eta(t)=t\xi$. Here A_v is the operator in (3.7). The calculations are similar to the previous terms. So we omit the details. Thus, we have

$$\int_{P_{e,a}(G)}(I_1,I_i)d\nu_{\lambda,a}(\gamma)$$
$$\le C\lambda\int_{P_{e,a}(G)}\left\{1+|b(1)|^2+\left(\int_0^1|b(s)|ds\right)^2\right\}f(\gamma)^2 d\nu_{\lambda,a}(\gamma).$$

Combining all the above estimates, we have

$$\int_{P_{e,a}(G)} f^2(\gamma)\log\left(\frac{f^2(\gamma)}{\|f\|^2_{L^2(\nu_{\lambda,a})}}\right)d\nu_{\lambda,a}(\gamma)$$
$$\le\frac{2}{\lambda}\left(1+\frac{C_1}{\lambda}\right)$$
$$\times\left\{\int_{P_{e,a}(G)}|(\nabla f)(\gamma)|^2 d\nu_{\lambda,a}(\gamma)+\int_{P_{e,a}(G)}\frac{\lambda^2}{4}|b(1)|^2 f(\gamma)^2 d\nu_{\lambda,a}(\gamma)\right\}$$
$$+\left(\log(C_2\lambda^{-d/2}p(1/\lambda,e,a))\right)\|f\|^2_{L^2(P_{e,a}(G))}$$
$$+\int_{P_{e,a}(G)}2C_3\left\{1+|b(1)|^2+\left(\int_0^1|b(s)|ds\right)^2\right\}f(\gamma)^2 d\nu_{\lambda,a}(\gamma)$$

which completes the proof. □

4. Exponential Integrability

It is crucial to check on which domains $e^{\frac{\lambda}{2}|b(1)|^2}$ is integrable to use the lower bound estimate (2.3). To this end, we introduce the following:

Definition 4.1. For $\gamma, \eta \in P_{e,a}(G)$, define $d(\gamma,\eta) = \max_{0\le t\le 1} d(\gamma(t),\eta(t))$. We denote

$$\begin{aligned}\Omega_L &= \left\{\gamma \in P_{e,a}(G) \mid \sqrt{E(\gamma)} \le L\right\},\\ B_\varepsilon(\eta) &= \{\gamma \in P_{e,a}(G) \mid d(\gamma,\eta) < \varepsilon\},\\ \Omega_{L,\varepsilon} &= \{\gamma \in P_{e,a}(G) \mid \text{there exists } \eta \in \Omega_L \text{ such that } \gamma \in B_\varepsilon(\eta)\}.\end{aligned}$$

Also we define for $0 < \alpha < 1$,

$$\|\gamma\|_\alpha = \sup_{0\le s,t\le 1} \frac{d(\gamma(t),\gamma(s))}{|t-s|^\alpha}.$$

Lemma 4.2. (1) *For any $\varepsilon > 0$, it holds that $\lim_{L\to\infty} \nu_{\lambda,a}(\Omega_{L,\varepsilon}) = 1$.*
(2) *We denote $l(\gamma) = \int_0^1 |\dot\gamma(t)|dt$. Take a finite energy path $\gamma_0 \in P_{e,a,H^1}(G)$. Let $p > 0$. There exist $C_i > 0$ $(1 \le i \le 4)$ and $m \in \mathbb{N}$ such that, for $\varepsilon < (18pC_1)^{-1/2} =: \varepsilon_p$, it holds that*

$$\begin{aligned}&\int_{B_\varepsilon(\gamma_0)} \exp\left(\frac{p\lambda}{2}|b(1)|^2\right) d\nu_{\lambda,a}(\gamma)\\ &\qquad \le \frac{C_2\lambda^{m+(d/2)}}{\sqrt{1-18p\varepsilon^2C_1}} \exp\left\{\frac{3p\lambda}{2}\Big(|b(1,\gamma_0)|^2 + C_3 + C_4\varepsilon^2 l(\gamma_0)^2\Big)\right\}.\end{aligned}$$

The constants C_i and m are independent of p, ε, γ_0.
(3) *Let $p > 0$ and set $\varepsilon < \varepsilon_p/2$. There exist positive numbers C_i $(i = 5,6)$ such that the following inequality holds. C_5 depends on L and ε but C_6 is independent of L.*

$$\int_{\Omega_{L,\varepsilon}} \exp\left(\frac{p\lambda}{2}|b(1)|^2\right) d\nu_{\lambda,a}(\gamma) \le \frac{C_5\lambda^{m+(d/2)}}{\sqrt{1-72p\varepsilon^2C_1}} e^{p\lambda C_6(1+L^2)}. \tag{4.1}$$

Also for any M and p, we have $\int_{\{\|\gamma\|_\alpha<M\}} e^{\frac{p\lambda}{2}|b(1)|^2} d\nu_{\lambda,a}(\gamma) < \infty$.

Proof. (1) Note that for $0 < \alpha < 1/2$, $\nu_{\lambda,a}(\|\gamma\|_\alpha < \infty) = 1$ holds. Take a positive number M. Note that if $d(x,y) \le i(G)$ $(x, y \in G)$, then x and y are joined by the unique minimal geodesic. Take a positive integer N such that $MN^{-\alpha} < i(G)$. Suppose that γ satisfies that $\|\gamma\|_\alpha < M$. Let $t_k = k/N$. Since $d(\gamma(t_{k+1}),\gamma(t_k)) \le MN^{-\alpha}$, $\log\left(\gamma(t_{k+1})\gamma(t_k)^{-1}\right)$ is well-defined. Let γ_N be the piecewise geodesic path such that $\gamma_N(t) = \gamma(t_k)$ for $t = t_k$ $(0 \le k \le N)$ and $\gamma_N(t) = \exp\left(\frac{t-t_k}{t_{k+1}-t_k}\log\left(\gamma(t_{k+1})\gamma(t_k)^{-1}\right)\right)\gamma(t_k)$ for $t_k < t < t_{k+1}$. Then for $t_k \le t \le t_{k+1}$,

$$\begin{aligned}d(\gamma_N(t),\gamma(t)) &\le d(\gamma_N(t),\gamma_N(t_k)) + d(\gamma_N(t_k),\gamma(t_k)) + d(\gamma(t_k),\gamma(t))\\ &\le 2MN^{-\alpha}.\end{aligned}$$

Furthermore, we take N such that $2MN^{-\alpha} < \varepsilon$, that is, $N > \max\left(\frac{M}{i(G)}, \frac{2M}{\varepsilon}\right)^{1/\alpha}$. Finally note that $E(\gamma_N)^{1/2} \le MN^{1-\alpha}$. Hence for $L > M\max\left(\frac{1}{i(G)}, \frac{2}{\varepsilon}\right)^{\frac{1}{\alpha}-1}$, it holds that $\{\gamma \mid \|\gamma\|_\alpha < M\} \subset \Omega_{L,\varepsilon}$. This proves (1).

(2) We consider $b(1)$ in $B_\varepsilon(\gamma_0)$. Note that the calculation below should be understood as the calculation under ν_λ at first. After that it can be extended to pinned measure case too by the quasi-sure analysis.

By the Itô formula,

$$\begin{aligned} b(1) &= \int_0^1 d\gamma(t)\circ\gamma(t)^{-1} \\ &= \int_0^1 \dot\gamma_0(t)\gamma(t)^{-1}dt + \int_0^1 d\left(\gamma-\gamma_0\right)(t)\circ\gamma(t)^{-1} \\ &= \int_0^1 \dot\gamma_0(t)\gamma(t)^{-1}dt - \int_0^1 \left(\gamma(t)-\gamma_0(t)\right)\circ d\left(\gamma(t)^{-1}\right). \end{aligned}$$

By Lemma 3.1,

$$\begin{aligned} b(1) &= \int_0^1 \dot\gamma_0(t)\gamma(t)^{-1}dt + \int_0^1 (\gamma(t)-\gamma_0(t))\gamma(t)^{-1}db(t) \\ &\quad -\frac{1}{2\lambda}\int_0^1 \left(\gamma(t)-\gamma_0(t)\right)\gamma(t)^{-1}\mathrm{C}dt \\ &= b(1,\gamma_0) + \int_0^1 \dot\gamma_0(t)\left(\gamma(t)^{-1}-\gamma_0(t)^{-1}\right)dt \\ &\quad -\frac{1}{2\lambda}\int_0^1 \left(\gamma(t)-\gamma_0(t)\right)\gamma(t)^{-1}\mathrm{C}dt + M(1,\gamma), \end{aligned}$$

where $M(t,\gamma)=\int_0^t(\gamma(s)-\gamma_0(s))\gamma(s)^{-1}db(s)$.

Noting that $|A^{-1}-B^{-1}|\le|A^{-1}||A-B||B^{-1}|$, we have

$$|b(1)|\le|b(1,\gamma_0)|+\frac{C_1\varepsilon}{2\lambda}+C_2\varepsilon l(\gamma_0)+|M(1,\gamma)|.$$

Therefore,

$$|b(1)|^2 \le 3\left(|b(1,\gamma_0)|^2+C^2\varepsilon^2\left(1+l(\gamma_0)\right)^2+|M(1,\gamma)|^2\right).$$

Hence using the lower bound estimate on $p(t,e,a)$,

$$\begin{aligned} &\int_{B_\varepsilon(\gamma_0)} e^{\frac{p\lambda}{2}|b(1)|^2}d\nu_{\lambda,a}(\gamma) \\ &\le \exp\left[\frac{3p\lambda}{2}\left(|b(1,\gamma_0)|^2+C\varepsilon^2\left(1+l(\gamma_0)\right)^2\right)\right] \\ &\times p(1/\lambda,e,a)^{-1}J_\lambda \\ &\le C\lambda^{d/2}\exp\left[\frac{3p\lambda}{2}\left(|b(1,\gamma_0)|^2+C'\left(1+\varepsilon l(\gamma_0)\right)^2\right)\right]J_\lambda, \end{aligned} \tag{4.2}$$

where

$$J_\lambda = \int_{B_\varepsilon(\gamma_0)}\Psi_\lambda(b)\delta_a(\gamma(1,b))d\mu_\lambda(b),$$

and

$$\Psi_\lambda(b) = \exp\left(\lambda p\left|\int_0^1 \varphi\Big(\gamma(t,b)-\gamma_0(t)\Big)\gamma(t,b)^{-1}db(t)\right|^2\right).$$

Here μ_λ is the Brownian motion measure of b. φ is a smooth function on $M(n,\mathbb{C})$ such that $\varphi(x)=x\rho(x)$ and $\rho(x)=1$ for $\|x\|\le 3\varepsilon$, $\rho(x)=0$ for $\|x\|\ge 4\varepsilon$. Here we use very rough estimate:

$$\begin{aligned} J_\lambda &\le \int_B \Psi_\lambda(b)\delta_a(\gamma(1,b))d\mu_\lambda(b) \\ &\le \frac{C\lambda^m}{\sqrt{1-18p\varepsilon^2 C}}. \end{aligned}$$

This estimate follows from the integration by parts formula and an estimate on exponential martingale. The natural number m depends on how many times we apply the integration by parts formula on Wiener space.
(3) Since Ω_L is a compact subset in $P_{e,a}(G)$, there exists a finite set of smooth curves $\{\gamma_i\}_{i=1}^N\subset\Omega_L$ such that $\Omega_L\subset\cup_{i=1}^N B_\varepsilon(\gamma_i)$. This implies $\Omega_{L,\varepsilon}\subset\cup_{i=1}^N B_{2\varepsilon}(\gamma_i)$. Hence

$$\int_{\Omega_{L,\varepsilon}} e^{\frac{p\lambda}{2}|b(1)|^2}d\nu_{\lambda,a}(\gamma)\le\sum_{i=1}^N\int_{B_{2\varepsilon}(\gamma_i)} e^{\frac{p\lambda}{2}|b(1)|^2}d\nu_{\lambda,a}(\gamma).$$

For each integral on the right-hand side can be estimated as in (4.2) by replacing γ_0 by γ_i. Note that N depends on L and ε. This and (2) proves (4.1). The last statement follows from (4.1) and the fact which we proved in the proof of (1). □

Remark 4.3. In general, $\int_{P_{e,a}(G)} e^{\lambda\frac{|b(1)|^2}{2}}d\nu_{\lambda,a}(\gamma)=+\infty$. Actually, we can prove a stronger statement: The integral of $e^{\lambda|b(1)|^2/2}$ in a neighborhood of countably many geodesics diverges. Let us prove this in the case where $G=SU(n)$. Let v_0 be an element of $\mathfrak{g}$ such that $l_0(t)=e^{tv_0}$ is a geodesic between e and a. Then there exists $g\in G$ such that $gv_0g^{-1}=D[\sqrt{-1}\eta_1,\dots,\sqrt{-1}\eta_n]$, where $D[a_1,\dots,a_n]$ denotes the diagonal matrix whose (i,i)-element is a_i. Take $v_i=g^{-1}D[(\eta_1+2\pi k_1^{(i)})\sqrt{-1},\dots,(\eta_n+2\pi k_n^{(i)})\sqrt{-1}]g$, where $(k_j^{(i)})_{1\le j\le n}\in\mathbb{Z}^n$ denotes the distinct points of integer lattice satisfying $\sum_j k_j^{(i)}=0$. Let $l_i(t)=e^{tv_i}$. Then $\{l_i\}_{i=1}^\infty$ are distinct geodesics joining e and a. Let $p>0$. Take a sufficiently small positive number ε. By Lemma 4.2 (2), it holds that

$$\int_{B_\varepsilon(l_i)}\exp\left(\frac{p\lambda}{2}|b(1)|^2\right)d\nu_{\lambda,a}(\gamma)<\infty.$$

Also we have $B_\varepsilon(l_i)\cap B_\varepsilon(l_j)=\emptyset$ for sufficiently small ε since

$$B_\varepsilon(l_i)=\{e^{t(v_i-v_j)}\gamma(t)\mid\gamma\in B_\varepsilon(l_j)\}$$

for any i, j. By Lemma 3.2 (2), we can prove that

$$\begin{aligned}
&\int_{B_\varepsilon(l_i)} \exp\left(\frac{p\lambda}{2}|b(1)|^2\right) d\nu_{\lambda,a}(\gamma) \\
&= \int_{B_\varepsilon(l_0)} \exp\Biggl\{\frac{\lambda}{2}\Biggl(\left|\int_0^1 Ad(e^{s(v_i-v_0)})db(s)\right|^2 \\
&\quad +(p-1)\left|\int_0^1 Ad(e^{s(v_i-v_0)})db(s)+v_i-v_0\right|^2\Biggr)\Biggr\}d\nu_{\lambda,a}(\gamma).
\end{aligned}$$

This shows that

$$\int_{B_\varepsilon(l_i)} \exp\left(\frac{\lambda}{2}|b(1)|^2\right) d\nu_{\lambda,a}(\gamma) \geq \nu_{\lambda,a}\left(B_\varepsilon(l_0)\right).$$

Thus it holds that

$$\int_{P_{e,a}(G)} \exp\left(\frac{\lambda}{2}|b(1)|^2\right) d\nu_{\lambda,a}(\gamma) > \int_{\cup_{i=1}^\infty B_\varepsilon(l_i)} \exp\left(\frac{\lambda}{2}|b(1)|^2\right) d\nu_{\lambda,a}(\gamma) = +\infty.$$

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