# HEDGING CLAIMS WITH FEEDBACK JUMPS IN THE PRICE PROCESS

KISEOP LEE AND PHILIP PROTTER*

ABSTRACT. We study a hedging and pricing problem of a model where the price process of a risky asset has jumps with instantaneous feedback from the most recent asset price. We model these jumps with a doubly stochastic Poisson process with an intensity function depending on the current price. We find a closed form expression of the local risk minimization strategy using Föllmer and Schweizer decomposition and Feynman-Kac type integro-differential equation. The possibility that the jumps depend on the most recent price is new for this type of model.

## 1. Introduction

The original Black-Scholes paradigm of modelling an asset price process with a geometric Brownian motion is still widely used, although there is widespread dissatisfaction with it. Many alternatives have been proposed, including stochastic volatility models, general (non-linear) stochastic differential equations, and also replacing Brownian noise with noise coming from a Lévy process. An advantage of stochastic volatility models is the presence of 'heavy tails', an advantage shared by Levy noise models, and the Lévy noise models have the additional attribute of incorporating jumps into price process. All of these theories are well established and well known.

In this article, we propose a new type of price process, which incorporates jumps, but unlike other models the random jump times do not arise by a prior specified exogenous distributional hypotheses. In more simple traditional cases one can model the arrival times of the jumps of the noise via a point process $N$ as follows:

$$N_t = \sum_{i=1}^{\infty} 1_{\{t \geq T_i\}} \quad \text{where} \quad N_t - \int_0^t \lambda(s, \omega) ds = \text{martingale},$$

and the times $T_i$ are the arrival times, with $\lambda_s$ the arrival intensity. A standard feature of these models is that the arrival intensity process $\lambda$ is specified *a priori*, and is either non-random (as for example in the Poisson case, and more generally the Lévy case), or is a given stochastic process. A price model with this variety of

jumps can be written as the solution $S$ of the following type of equation:

$$dS_t = \sigma(S_t)dB_t + b(S_{t-})d(N_t - \int_0^t \lambda_s ds) + \mu(S_t)dt. \qquad (1.1)$$

In this article we propose a new way of modelling the arrival intensity of the jump times, one where the *instantaneous arrival intensity is a function of the past history of the asset price process up to that point.* This makes the jumps intrinsic to the pricing process and the past noise itself. Three examples where such a price process construction could be used, are to model (1) large trader behavior where a stop order kicks in at a certain level thereby causing a (perhaps small) jump in the asset price, or (2) a change in the stock price due to a creditor calling a loan, caused by the asset price falling below a certain level, or (3) aggregate behavior of many traders acting in concert due to a run-up or dramatic decline in an asset price, such as (again, for example) a sell-off on a Friday due to a rumor later proved false with a consequent repurchase on the following Monday. We can express this model in heuristic notation (to which we later give a rigorous meaning) as follows:

$$dS_t = \sigma(S_t)dB_t + b(S_{t-})d(N_t - \int_0^t \lambda(S_r; r \le s)ds) + \mu(S_t)dt.$$

We emphasize that the difference in the two types of equations is the instantaneous feedback loop present in the second model, which takes the jump arrival intensity $\lambda$ to be a functional of the past paths of the solution of the stochastic differential equation $S$, the model for the asset price process.

The mathematics involved of making sense of this idea is non trivial, and we will rely on earlier work of J.Jacod and P.Protter [17]. The inspiration to consider models of this type came form work of R.Frey [14] .

We will not only make sense of this idea for the price process, but we will find a closed form expression for the hedging strategy for a class of such asset price processes. We will also construct its minimal martingale measure in the Föllmer and Schweizer [11] sense.

On the hedging problem in incomplete markets, the local risk minimization and the mean-variance hedging have been two major quadratic approaches. The local risk minimization sacrifices the self-financing property, but its terminal value is the same as the payoff of a contingent claim. The mean-variance hedging, on the other hand, focuses on the self-financing property. Föllmer and Sondermann [12] studied the risk minimization when the asset price process is a martingale under the original measure, and later, Föllmer and Schweizer [11] and Schweizer [31] studied the local risk minimization for a general semimartingale case. Schweizer [32] provided the solution to the mean-variance hedging for general claims with continuous price processes.

While mean-variance hedging gives a control over the total risk, the local risk minimization often gives a simpler hedging strategy. (See Heath, Platen, and Schweizer [16], for example.) There have been many studies on the above quadratic criteria since they had been proposed. To name a few, Frey [14] studied a risk minimizing strategy when the price process is a pure jump process with a stochastic jump rate and a martingale under the original measure. Chan [7] found a local risk minimizing strategy when the price process is driven by general Lévy processes.

Lim [21] studied a closed form expression of the mean-variance hedging strategy in a specific jump diffusion model, using a backward stochastic differential equations and stochastic optimal control theory. Our suggested model has non-Lévy type jumps as introduced in Frey [14]. Another advantage of our model is that it allows asymmetric return distributions. We can obtain this asymmetry by controlling the jump size distribution $\nu$, as long as it has mean 0 and a finite second moment. This flexibility of the model gives us a better fit of real stock market data.

The outline for this paper is as follows. In Section 2.1, we introduce our feedback jump model and some technical assumptions. We discuss the Markov property of the model and the minimal martingale measure, dynamics of processes under the changed measure in Section 2.2. We construct a Feynman-Kac type integro-differential equation and show the representation property in Section 2.3. We have our main theorem, which gives us the hedging strategy in Section 2.4. We apply this result to liquidity modelling in Section 3, and we conclude in Section 4.

## 2. Hedging of Options in an Incomplete Market

**2.1. The Model.** We consider a market which consists of a risky asset and a riskless asset. For simplicity, we assume that the price of the riskless asset is always 1, which implies that the interest rate is 0. A portfolio $(\xi, \eta)$ is a vector process where $\xi_t$ is a unit amount of the risky asset at time $t$ and $\eta_t$ is a unit amount of the riskless asset at time $t$. Therefore, the value $V$ of a portfolio $(\xi, \eta)$ at time $t$ is given by $V_t = \int_0^t \xi_s dS_s + \eta_t$.

We are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ satisfying the usual conditions where $T$ is a fixed time. $\mathbb{P}$ represents the statistical or empirical probability measure. $(\mathcal{F}_t)_{0 \leq t \leq T}$ is a filtration which makes all the processes in the model adapted. We define our price process of a risky asset $S$ as a solution of the stochastic differential equation:

$$dS_t = f(S_{t-})dB_t + g(S_{t-})dR_t + h(S_{t-})dt, \tag{2.1}$$

$$0 \leq t \leq T,$$

where $B$ is a standard Brownian motion,

$$R_t = \sum_{n=1}^{N_t} U_n, \tag{2.2}$$

and $N$ is a doubly stochastic Poisson process with a bounded intensity function $\lambda(S_{t-})$, in other words,

$$N_t - \int_0^t \lambda(S_{s-})ds = \text{ a local martingale under } \mathbb{P}. \tag{2.3}$$

We notice that $N_t$ denotes the number of jumps up to time $t$, and $U_n$ denotes the size of $n$-th jump. For the details of doubly stochastic Poisson processes, readers can consult [4], [35]. Since the intensity of $N$ is a function of the left continuous version of the current stock price $S_{s-}$, the jump process $R$ gets an instantaneous feedback from the most recent stock price.

Here, $U_n$'s are i.i.d. random variables with mean $0$ and a finite second moment $\sigma^2$ with density function $\nu(dx)$, and $f, g, h$ are bounded measurable Lipschitz functions. We need the following technical condition to guarantee an existence of the *unsigned* minimal martingale measure.

$$\frac{h(x)g(x)}{f(x)^2 + g(x)^2\lambda(x)\sigma^2} < 1 \tag{2.4}$$

for all $x$. Without this condition, the minimal martingale measure may not exist under the current model, since it becomes a signed measure. For more discussion on this issue, readers can consult [33].

It follows from the calculation of the compensator of the random measure $p^R(dt, dx)$ in Theorem 2.6 that $R$ is a local martingale, and hence by the definition of $S$, one can easily see that $S$ is a special semimartingale with the canonical decomposition $S = M + A$, where $M_t = \int_0^t (f(S_{s-})dB_s + g(S_{s-})dR_s)$ is the local martingale part, and $A_t = \int_0^t h(S_{s-})ds$ is the predictable finite variation part. Since $g$ is bounded, there exists a constant $K$ such that $|g(x)| < K$. Then, we get

$$[M, M]_T = \int_0^T f(S_{s-})^2 ds + \int_0^T g(S_{s-})^2 U_s^2 dN_s,$$

and

$$\|[M, M]_T^{\frac{1}{2}}\|_{L^2}^2 = E[M, M]_T$$
$$< \int_0^T E(f(S_{s-})^2)ds + \sigma^2 K^2 E(N_T) < \infty. \tag{2.5}$$

From (2.5) and

$$\|\int_0^T |h(S_{s-})|ds\|_{L^2}^2 = E(\int_0^T |h(S_{s-})|ds)^2 < \infty,$$

we notice that $S$ is a $\mathcal{H}^2$ semimartingale. Furthermore, $M$ is not only a local martingale, but also a square integrable martingale under $\mathbb{P}$. (For a discussion on sufficient conditions which make a local martingale a true martingale, see Section 2.6 of Protter [27].)

**2.2. The Markov Property and the Minimal Martingale Measure.** The existence of the SDE (2.1) can be showed by analogous steps in Theorem 3.2 of Jacod and Protter [17]. The Markov property of $S$ plays a key role in the pricing and hedging theory, since it often gives a much simpler form of value process $V$ of a portfolio $(\xi, \eta)$. Although $S$ has an intensity function depending on the solution itself, we can show that $S$ has a Markov property under $\mathbb{P}$. First we need a lemma. This is a modified version of the Theorem 3.1 of [17] and we omit the proof.

**Lemma 2.1.** *Suppose that $m$ is a Poisson random measure with compensator $\tilde{m}(dt, dx) = dt\, dx$. Then every solution process of the following equation is a solution process of (2.1)*

$$S_t = S_0 + \int_0^t f(S_s)dB_s + \int_0^t h(S_s)ds + \int_0^t \int_{\mathbb{R}} g(S_{s-})U_s 1_{\{0 \leq x \leq \lambda(S_{s-})\}} m(ds, dx),$$
$$\tag{2.6}$$

*and for $N_t$, one can take the process*

$$N_t = \int_0^t \int_{\mathbb{R}} 1_{\{0 \leq x \leq \lambda(S_{s-})\}} m(ds, dx). \tag{2.7}$$

Using the previous lemma and a standard iteration method, we can show the Markov property under the original measure $P$. Later, we can show the Markov property still holds under the minimal martingale measure, which will be introduced in Theorem 2.

**Theorem 2.2.** *Let $S_t$ be as in equation (2.1). Then $S_t$ is Markov under $\mathbb{P}$.*

*Proof.* Define $K_t = \int_0^t \int_{\mathbb{R}} x m(dx, ds)$ and $T'$ be an $\mathcal{F}_t$ - stopping time, $T' < \infty$ a.s. Define

$$\mathcal{G}^{T'} = \sigma\{K_{T'+u} - K_{T'}, B_{T'+u} - B_{T'}, u > 0\}.$$

Then $\mathcal{G}^{T'}$ is independent of $\mathcal{F}_{T'}$. For $u \geq 0$, define inductively

$$Y^0(x, T', u) = x,$$

$$Y^{n+1}(x, T', u) = x + \int_{T'}^{T'+u} f(Y^n(x, T', s)) dB_s + \int_{T'}^{T'+u} h(Y^n(x, T', s)) ds$$

$$+ \int_{T'}^{T'+u} \int_{\mathbb{R}} U_s g(Y^n(x, T', s)) 1_{\{0 \leq y \leq \lambda(Y^n(x, T', s))\}} m(dy, ds).$$

Then, by using induction and after some standard work, we can easily see that

$$E^x\{h(S(S_0, 0, T' + u)) | \bar{\mathcal{F}}_{T'}\} = E^x\{h(S(S_0, 0, T' + u)) | S(S_0, 0, T')\}.$$

for any bounded, Borel function $h$. □

Next, we find the minimal martingale measure. The minimal martingale measure can be used for local risk minimization, which will be explained in Section 2.4. We recall the definition of it for readers' convenience. For more discussion on the minimal martingale measure, readers can consult [11, 33]. A martingale measure $\mathbb{Q} \approx \mathbb{P}$ is called *minimal* if $\mathbb{Q} = \mathbb{P}$ on $\mathcal{A}_0$, and if any square-integrable $\mathbb{P}$-martingale $L$ that satisfies $\langle L, M \rangle = 0$ remains a martingale under $\mathbb{Q}$, where $M$ is the martingale part of $S$ in the canonical decomposition under $\mathbb{P}$.

**Theorem 2.3.** *Suppose that $f, g, h, \lambda$ satisfy (2.4). Let us define*

$$Z_t = 1 - \int_0^t \frac{Z_{s-} h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(S_{s-}) \sigma^2} (f(S_{s-}) dB_s + g(S_{s-}) dR_s).$$

*Then, $Z_t > 0$ and $E(Z_t) = 1$ for all $t \in (0, T]$. Furthermore, $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ is the unique minimal martingale measure of $S$.*

*Proof.* Suppose that there exists a minimal martingale measure and let us denote it by $\mathbb{P}^*$. Define $Z_t$ by

$$Z_t = E[\frac{d\mathbb{P}^*}{d\mathbb{P}} | \mathcal{F}_t]. \tag{2.8}$$

$M_t = \int_0^t (f(S_{s-}) dB_s + g(S_{s-}) dR_s)$ denotes the martingale part of the Doob-Meyer decomposition of $S_t$ and $A_t = \int_0^t h(S_{s-}) ds$ denotes its predictable part under $\mathbb{P}$.

Under $\mathbb{P}^*$, the Doob-Meyer decomposition of $M_t$ is given by $M_t = S_t - A_t$. But the Girsanov-Meyer theorem (Section 3.8 of [27]) shows that $A_t$ also satisfies

$$-A_t = \int_0^t \frac{1}{Z_{s-}} d\langle M, Z\rangle_s.$$

Since $\langle M, Z\rangle \ll \langle M, M\rangle = \langle S, S\rangle$, there exists a predictable process $\alpha_t$ such that

$$A_t = \int_0^t \alpha_s d\langle S, S\rangle_s.$$

On the other hand, there exists some $\beta_t$ such that

$$Z_t = 1 + \int_0^t \beta_s dM_s + L_t,$$

where $L_t$ is a square integrable martingale under $\mathbb{P}$ orthogonal to $M$. Moreover,

$$-A_t = \int_0^t \frac{1}{Z_{s-}} d\langle M, Z\rangle_s = \int_0^t \frac{1}{Z_{s-}} \beta_s d\langle S, S\rangle_s$$

gives us the relation

$$\alpha_t = -\frac{\beta_t}{Z_{t-}}.$$

Since $\mathbb{P}^*$ is minimal and $L_t$ is a square integrable martingale under $\mathbb{P}$ orthogonal to $M$, $L$ is a martingale under $\mathbb{P}^*$. We observe that $\langle L, Z\rangle = 0$, since $L$ is both a $\mathbb{P}$ and a $\mathbb{P}^*$ martingale. (see the Lemma on p109, [27] for details.) Now we get

$$\langle L, L\rangle = \langle L, Z\rangle = 0,$$

and

$$Z_t = 1 + \int_0^t Z_s(-\alpha_s) dM_s, \tag{2.9}$$

where

$$\alpha_t = -\frac{\beta_t}{Z_{t-}}.$$

It remains to calculate $\alpha$. We can do it easily by

$$\alpha_s = \frac{dA_s}{d\langle S, S\rangle_s}$$
$$= \frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \sigma^2 \lambda(S_{s-})}. \tag{2.10}$$

Therefore, we get

$$Z_t = 1 - \int_0^t \frac{Z_{s-} h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(S_{s-}) \sigma^2} (f(S_{s-}) dB_s + g(S_{s-}) dR_s). \tag{2.11}$$

Since there is a unique solution of the equation (2.11), if there exists a minimal martingale measure, it is unique. We can easily check that (2.11) is also a sufficient condition. Let $M_t'$ be a $\mathbb{P}$ martingale such that $\langle M', M\rangle = 0$. What we should show is that $M_t'$ is $\mathbb{P}^*$ martingale, which is true since

$$\langle M', Z\rangle = \langle M', M\rangle = 0. \tag{2.12}$$

To show that $Z_t > 0$ and $E(Z_t) = 1$ for all $t$, notice that

$$Z_t = 1 + \int_0^t Z_{s-} dY_s,$$

and

$$Z_t = \exp(Y_t - \frac{1}{2}[Y,Y]_t^c) \prod_{0 < s \leq t} (1 + \Delta Y_s) \exp(-\Delta Y_s).$$

By the condition (2.4), we notice that $Z_t > 0$. By (2.11), $Z_t$ is a $\mathbb{P}$- local martingale. Since $Z_0 = 1$, it suffices to show that $E([Z,Z]_t) < \infty$ to get $E(Z_t) = 1$ for all $t$. For some constant $M$, we have

$$E([Z,Z]_t) < ME(\int_0^t Z_s^2 ds + \int_0^t Z_s^2 dN_s)$$

$$= ME(\int_0^t Z_s^2 ds + \int_0^t Z_s^2 \lambda(S_{s-}) ds)$$

$$< ME \int_0^t Z_s^2 ds$$

$$< M \int_0^t E(Z_s^2) ds$$

Let $Y_t = -\int_0^t \frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(S_{s-}) \sigma^2} (f(S_{s-}) dB_s + g(S_{s-}) dR_s)$. Since $f, g, h$ are bounded, and from direct calculation of expected values under normal and Poisson distributions, we get

$$E(Z_t^2) < E \exp(2Y_t) < N,$$

for some constant $N$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We study the new Brownian motion and compensated jump measure under the minimal martingale measure $\mathbb{Q}$ in next two theorems. The next theorem tells us on the new Brownian motion under $\mathbb{Q}$ is a shift of the old Brownian motion by some drift term.

**Theorem 2.4.** *Let $\mathbb{Q}$ be as in Theorem 2. Under $\mathbb{Q}$,*

$$\widetilde{B}_t = B_t + \int_0^t \frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(S_{s-}) \sigma^2} f(S_{s-}) ds$$

*is a Brownian Motion.*

*Proof.* By the Girsanov-Meyer theorem, we know that

$$N_t = B_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z, B \rangle_s$$

is a $\mathbb{Q}$ local martingale. However,

$$\langle Z, B \rangle_t = -\int_0^t \frac{Z_{s-} h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(S_{s-}) \sigma^2} f(S_{s-}) ds$$

which gives $N_t = \widetilde{B}_t$. Moreover,

$$\langle \widetilde{B}, \widetilde{B} \rangle_t = \langle B, B \rangle_t = t.$$

Therefore, by Lévy's theorem (see p.86, of [27]) , $\widetilde{B}$ is a $\mathbb{Q}$ Brownian motion.    $\square$

By $p^R(dt, dx)$ we denote the counting measure associated with the process $R_t$; $p^R(dt, dx)$ is a random measure on $[0, T] \times \mathbb{R}$ such that for functions $W : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$,

$$\int_0^t \int_\mathbb{R} W(\omega; s, x) p^R(ds, dx) = \sum_{n=1}^\infty W(\omega; T_n(\omega), U_n(\omega)) 1_{(T_n(\omega) \leq t)},$$

when $T_n$ is $n$-th jump time of $N_t$. In the next theorem, we study the change of the compensated measure of the random measure $p^R(dt, dx)$ of $R_t$ under the minimal martingale measure. To do this, we need a lemma. It is a part of Theorem 3.17 of Chapter III(page 157), Jacod and Shiryaev [18] and a version of Girsanov theorem for random measure. We need some definitions. $M_\mu^\mathbb{P}$ is the positive measure defined by $M_\mu^\mathbb{P}(W) = E(W * \mu_\infty)$ for all measurable nonnegative function $W$, where $W * \mu_\infty = \int_{[0, \infty)} \int_E W(\omega, s, x) \mu(\omega; ds, dx)$ where $\mu$ is a random measure. By the predictable $\sigma$-field $\tilde{\mathcal{P}}$, we mean a $\sigma$-field on $\Omega \times \mathbb{R}$ such that $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}$ where $\mathcal{B}$ is a Borel $\sigma$-field. There is a notion of conditional expectation relative to $M_\mu^\mathbb{P}$ with respect to the predictable $\sigma$-field $\tilde{\mathcal{P}}$: for every nonnegative measurable function $W$, let $W' = M_\mu^\mathbb{P}(W | \tilde{\mathcal{P}})$ denote the $M_\mu^\mathbb{P}$-a.s. unique $\tilde{\mathcal{P}}$-measurable function such that $M_\mu^\mathbb{P}(WU) = M_\mu^\mathbb{P}(W'U)$ for all nonnegative $\tilde{\mathcal{P}}$-measurable $U$. For more details and examples, we refer to Chapter 2 and 3 of Jacod and Shiryaev [18].

**Lemma 2.5.** *Assume that $\mathbb{Q} \ll \mathbb{P}$ and let $Z$ be the density process. Let $\mu = \mu(\omega; dt, dx)$ be an integer-valued random measure on $\mathbb{R} \times E$, and denote by $\upsilon = \upsilon(\omega; dt, dx)$ its $\mathbb{P}$-compensator of $\mu$. Let $Y$ be any nonnegative version of $M_\mu^\mathbb{P}(\frac{Z}{Z_-} 1_{\{Z_- > 0\}} | \tilde{\mathcal{P}})$ and $\upsilon'$ be a version of the $\mathbb{Q}$ compensator. Then $\upsilon'(\omega; dt, dx) = Y(\omega; t, x) \upsilon(\omega; dt, dx)$ $\mathbb{P} - a.s.$*

**Theorem 2.6.** *Let $\mathbb{Q}$ be as in Theorem 2. Under $\mathbb{Q}$, the compensated measure of $p^R(dt, dx)$ is given by*

$$q^*(dt, dx) = p^R(dt, dx) - (1 - \frac{h(S_{t-}) g(S_{t-})}{f(S_{t-})^2 + g(S_{t-})^2 \lambda(S_{t-}) \sigma^2} x) \lambda(S_{t-}) \nu(dx) dt.$$

*Proof.* Since $R_t = \sum_{n=1}^{N_t} U_n$, $N_t$ is a doubly stochastic Poisson process with intensity function $\lambda(S_{t-})$, and $U_n$ has a density $\nu(dx)$, the random measure $p^R(dt, dx)$ admits $(\mathbb{P}, \mathcal{F}_t)$ local characteristics $(\lambda(S_{t-}), \nu(dx))$ (see Definition 5, p236 of Brémaud [4] for a definition of local characteristics). Then, Corollary 15 on page 247 of Brémaud [4] tells us that the compensator of $p^R(dt, dx)$ under $\mathbb{P}$ is $\lambda(S_{t-}) \nu(dx) dt$ (p218-219 Frey [14] for details). We refer to Chapter 8 of Brémaud [4] for further discussion on properties of local characteristics. Using the notations of Lemma 2, $\mu(\omega; dt, dx) = p^R(\omega; dt, dx)$ and $\upsilon(\omega; dt, dx) = \lambda(S_{t-}) \nu(dx) dt$. By Lemma 2, to find $M_\mu^\mathbb{P}(\frac{Z}{Z_-} 1_{\{Z_- > 0\}} | \tilde{\mathcal{P}})$ is enough. Note that

$$Z_t = 1 + \int_0^t Z_{s-} dX_s,$$

where

$$X_t = - \int_0^t \frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(S_{s-})\sigma^2} (f(S_{s-})dB_s + g(S_{s-})dR_s).$$

Therefore,

$$\frac{Z_t}{Z_{t-}} = 1 + \Delta X_t = 1 - \frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(S_{s-})\sigma^2} g(S_{s-})\Delta R_s,$$

and

$$M_\mu^{\mathbb{P}}(\frac{Z}{Z_-} 1_{\{Z_->0\}}|\tilde{\mathcal{P}}) = 1 - \frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(S_{s-})\sigma^2} g(S_{s-}) M_\mu^{\mathbb{P}}(\Delta R_s|\tilde{\mathcal{P}}).$$

(2.13)

Observe that

$$M_\mu^{\mathbb{P}}(\Delta R_s) = E \int_0^\infty \int_{\mathbb{R}} x 1_{(\Delta R_s > 0, U_s = x)} p^R(\omega; ds, dx).$$

For any nonnegative $\tilde{\mathcal{P}}$-measurable $W(\omega; t, x)$, since $p^R(\omega; ds, dx)$ is zero unless $\Delta R_s > 0, U_s = x$, we get

$$\begin{aligned} M_\mu^{\mathbb{P}}(\Delta R_s W) &= E \int_0^\infty \int_{\mathbb{R}} x 1_{(\Delta R_s > 0, U_s = x)} W(\omega; s, x) p^R(\omega; ds, dx) \\ &= E \int_0^\infty \int_{\mathbb{R}} x W(\omega; s, x) p^R(\omega; ds, dx) \\ &= M_\mu^{\mathbb{P}}(xW). \end{aligned}$$

Since the function $(\omega; s, x) \to x$ is $\tilde{\mathcal{P}}$-measurable, we get

$$M_\mu^{\mathbb{P}}(\Delta R_s|\tilde{\mathcal{P}}) = x.$$

(2.14)

Therefore, by (2.13),(2.14) and Lemma 2, the compensator of $p^R$ under the minimal martingale measure $\mathbb{Q}$ is given by

$$(1 - \frac{g(S_{t-})h(S_{t-})}{f(S_{t-})^2 + g(S_{t-})^2 \lambda(S_{t-})\sigma^2} x)\lambda(S_{t-})\nu(dx)dt.$$

$\square$

In Section 2.2, we showed that $S_t$ is Markov under $\mathbb{P}$. But what we really need is the Markov property under the changed measure $\mathbb{Q}$, not under the original measure $\mathbb{P}$. This Markov property of $S$ under $\mathbb{Q}$ follows immediately from Theorem 3 and 4, following analogous steps of the proof of Theorem 1.

**2.3. The Integro-Differential Equation.** Let us define $V_t = E_{\mathbb{Q}}[H(S_T)|\mathcal{F}_t]$, where $H$ is a European style contingent claim. By the Markov property of $S$ under $\mathbb{Q}$,

$$V_t = E_{\mathbb{Q}}[H(S_T)|\mathcal{F}_t] = E_{\mathbb{Q}}[H(S_T)|S_t] = v(t, S_t),$$

for some function $v = v(t, x)$.

In order to use Itô's formula, we need $v$ to be $\mathcal{C}^{1,2}$. We can define $v$ by (using standard Markov process notation):

$$v(t, x) = E^x[H(S_{T-t})].$$

(2.15)

Then using Markov process theory (see, eg, [34]) we have

$$
\begin{aligned}
v(t,x) &= E^x[H(S_{T-t})] \quad \text{implies} \\
v(t,S_t) &= E^{S_t}[H(S_{T-t})] \\
&= E^x[H(S_{T-t}) \circ \theta_t | \mathcal{F}_t] \\
&= E^x[H(S_T)|S_t], \quad \text{which is intuitively equal to} \\
\text{``} &= E[H(S_T)|S_t = x]." 
\end{aligned}
$$

We now have the following result.

**Theorem 2.7.** *Let $S$ be as in equation (2.1), and we assume that the coefficients $f, g$ and $h$, and also $\lambda$, are all bounded and at least $b\mathcal{C}^2$, and that their second derivatives are Lipschitz continuous. We further assume that $H$ is bounded, and also in $b\mathcal{C}^2$. Then $v = v(t,x)$ as defined in equation (2.15) above, is $\mathcal{C}^{1,2}$.*

*Proof.* This theorem follows essentially from the theory of flows of stochastic differential equations, as presented (for example) in [27]. Nevertheless, because of the presence of the feedback term, we give its proof. Our candidate first derivative of $v$ in $x$ is given by

$$
D_t = 1 + \int_0^t f'(S_u^x)D_u dB_u + \int_0^t g'(S_{u-})D_{u-} dR_u + \int_0^t h'(S_u)D_u du. \quad (2.16)
$$

where $S_t^x$ denotes $S$ starting at the point $x$, at time $t$. Note that there is no problem with the existence of $D$: it is a well defined stochastic exponential of the semimartingale $S$. Since the coefficients are globally Lipschitz, we have that $x \mapsto S_t^x$ is $\mathcal{C}^2$ in $x$ by Theorem 40, on page 310, of [27].

Using the above, we now have by $H \in b\mathcal{C}^2$ and dominated convergence that

$$
\begin{aligned}
\frac{\partial}{\partial x} E[H(S_t^x)] &= E[\frac{\partial}{\partial x} H(S_t^x)] \\
&= E[H'(S_t^x)\frac{\partial}{\partial x} S_t^x] = E[H'(S_t^x)D_t^x]
\end{aligned}
$$

To repeat the argument to get $\mathcal{C}^2$ in $x$ we need to control $D_t^x$ in order to use dominated convergence, since we are assuming that $H'$ is bounded. However $D_t^x$ is the stochastic exponential of a semimartingale, so this follows by Theorem 55, page 326, of [27]. Therefore repeating the argument one more time yields that $x \mapsto v(t,x)$ is $\mathcal{C}^2$ in $x$. It remains to show that $t \mapsto v(t,x)$ is $\mathcal{C}^1$. To simplify notation, we take $x = 0$ and suppress the $x$ variable. Thus we want to show that $t \mapsto E_Q[H(S_t)]$ is $\mathcal{C}^1$, where $Q$ is a risk neutral measure for $S$, such that $S$ is Markov and a true martingale (and not only a local martingale); we have already constructed such a risk neutral measure $Q$. But this follows from Theorem 2.6 and our hypotheses on $h, g, f$ and $\lambda$, since we know the form of the compensation of $p^R(dt, dx)$. $\qquad \square$

Note that our hypotheses are not best possible in Theorem 2.7 in order for $v$ to be $\mathcal{C}^{1,2}$. For example, let us consider the linear function $H(x) = ax + b$. Then we get $v(t,x) = ax + b$ since $S_t$ is a martingale under $\mathbb{Q}$ and clearly it is $\mathcal{C}^{1,2}$.

When $v$ is $C^{1,2}$, by Itô's formula,

$$v(t, S_t) = v(0, S_0) + \int_0^t v_t(s, S_{s-})ds + \int_0^t f(S_{s-})v_x(s, S_{s-})d\widetilde{B}_s$$

$$+ \int_0^t \int_{\mathbb{R}} \{v(s, S_{s-}(1 + x\frac{g(S_{s-})}{S_{s-}})) - v(s, S_{s-})\}q^*(dx, ds)$$

$$+ \int_0^t v_x(s, S_{s-})h(S_{s-})ds + \frac{1}{2}\int_0^t v_{xx}(s, S_{s-})f(S_{s-})^2 ds$$

$$+ \int_0^t f(S_{s-})v_x(s, S_{s-})\frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2\lambda(S_{s-})\sigma^2}f(S_{s-})ds$$

$$- \int_0^t \int_{\mathbb{R}} \{v(s, S_{s-}(1 + x\frac{g(S_{s-})}{S_{s-}})) - v(s, S_{s-})\}$$

$$\times (1 - \frac{g(S_{s-})h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2\lambda(S_{s-})\sigma^2}x)\lambda(S_{s-})\nu(dx)ds. \tag{2.17}$$

Since $v(t, S_t)$ is a $\mathbb{Q}$ - martingale, the right side of (2.17) also should be a $\mathbb{Q}$ - martingale. Therefore, since a continuous process with finite variation can be a martingale only if it is constant, we need following conditions:

$$v_t(s, S_{s-}) + v_x(s, S_{s-})h(S_{s-}) + \frac{1}{2}v_{xx}(s, S_{s-})f(S_{s-})^2$$

$$- f(S_{s-})v_x(s, S_{s-})\frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2\lambda(S_{s-})\sigma^2}f(S_{s-})$$

$$+ \int_{\mathbb{R}} \{v(s, S_{s-}(1 + x\frac{g(S_{s-})}{S_{s-}})) - v(s, S_{s-})\}$$

$$\times (1 - \frac{g(S_{s-})h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2\lambda(S_{s-})\sigma^2}x)\lambda(S_{s-})\nu(dx) = 0 \tag{2.18}$$

for almost all $s$, a.s. and

$$v(T, S_T) = H(S_T). \tag{2.19}$$

This gives us an integro-differential equation which has a form of Feynman-Kac type differentiation operator plus an integration part, which arises from the jumps. We showed that when

$$v(t, x) = E_{\mathbb{Q}}[H(S_T)|S_t = x],$$

where $S$ is in (2.1) and $v$ is $C^{1,2}$, then $v$ is a solution of the integro-differential equation (2.18, 2.19).

We can show the opposite direction as well. If a solution of the integro-differential equation (2.18, 2.19) exists, it has a stochastic representation

$$v(t, x) = E_{\mathbb{Q}}^{t,x}(H(S_T)),$$

where $S$ is given by the solution of (2.1). To see this, we just do the same thing backward. From (2.17), Itô's formula applied to $v(T, S_T)$ gives us

$$v(T, S_T) = v(t, S_t) + \int_t^T v_t(s, S_{s-})ds + \int_t^T \mathcal{A}v(s, S_s)ds$$

$$+ \int_t^T f(S_{s-})v_x(s, S_{s-})d\widetilde{B}_s$$

$$+ \int_t^T \int_{\mathbb{R}} \{v(s, S_{s-}(1 + x\frac{g(S_{s-})}{S_{s-}})) - v(s, S_{s-})\}q^*(dx, ds).$$

By (2.21) and martingale properties of the last two terms, we get

$$v(t, x) = E_{\mathbb{Q}}^{t,x}(v(T, S_T)) = E_{\mathbb{Q}}^{t,x}(H(S_T)). \qquad (2.20)$$

Let us assume that $v$ is $C^{1,2}$ Then we have $v$ satisfies (2.18, 2.19) and any other solution of (2.18, 2.19) must have the stochastic representation $v(t, x) = E_{\mathbb{Q}}^{t,x}(H(S_T))$. Therefore, $v$ is the unique classical solution of the integro-differential equation (2.18, 2.19).

If we weaken the hypotheses of Theorem 2.7 so that we do not know that $v$ is $C^{1,2}$, then we do not know if the solution exists. But, we know that any solution of (2.18, 2.19) must have the stochastic representation $v(t, x) = E_{\mathbb{Q}}^{t,x}(H(S_T))$. Therefore, we have $v(t, x) = E_{\mathbb{Q}}^{t,x}(H(S_T))$ as an *a priori* estimator of the integro-differential equation (2.18, 2.19).

Note that we can denote the integro-differential equation (2.18, 2.19) using an infinitesimal generator of the Markov process $S$. Let $\mathcal{A}$ denote the infinitesimal generator of the Markov process $S_t$.

$$\mathcal{A}v(x) = \lim_{t \to 0} \frac{E_{\mathbb{Q}}^x\{v(S_t) - v(S_0)\}}{t}$$

$$= \lim_{t \to 0} \frac{E_{\mathbb{Q}}^x(v(S_t)) - v(x)}{t}$$

$$= \frac{d}{dt}E_{\mathbb{Q}}^x(v(S_t))|_{t=0}.$$

By Itô's formula applied to $v(S_t)$,

$$v(S_t) = v(S_0) + \int_0^t f(S_{s-})v_x(S_{s-})d\widetilde{B}_s$$

$$+ \int_0^t \int_{\mathbb{R}} \{v(S_{s-}(1 + x\frac{g(S_{s-})}{S_{s-}})) - v(S_{s-})\}q^*(dx, ds)$$

$$+ \int_0^t v_x(S_{s-})h(S_{s-})ds + \frac{1}{2}\int_0^t v_{xx}(S_{s-})f(S_{s-})^2 ds$$

$$+ \int_0^t f(S_{s-})v_x(S_{s-})\frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2\lambda(S_{s-})\sigma^2}f(S_{s-})ds$$

$$- \int_0^t \int_{\mathbb{R}} \{v(s, S_{s-}(1 + x\frac{g(S_{s-})}{S_{s-}})) - v(S_{s-})\}$$

$$\times (1 - \frac{g(S_{s-})h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2\lambda(S_{s-})\sigma^2}x)\lambda(S_{s-})\nu(dx)ds.$$

Then we take an expectation under $\mathbb{Q}$ and get

$$\frac{d}{dt}E_{\mathbb{Q}}^x(v(S_t))|_{t=0} = v'(x)h(x) + \frac{1}{2}v''(x)f(x)^2$$

$$- v'(x)f(x)^2\frac{h(x)}{f(x)^2 + g(x)^2\lambda(x)\sigma^2}$$

$$+ \int_{\mathbb{R}} \{v(x(1 + z\frac{g(x)}{x})) - v(x)\}$$

$$\times (1 - \frac{g(x)h(x)}{f(x)^2 + g(x)^2\lambda(x)\sigma^2}z)\lambda(x)\nu(dz)$$

$$= \mathcal{A}v(x).$$

Now, the previous integro-differential equation can be written in the form

$$-v_t = \mathcal{A}v, \tag{2.21}$$

where $v_t$ is a partial derivative and $v(T, x) = H(x)$.

**2.4. The Explicit Hedging Strategy.** For any portfolio $(\xi, \eta)$, the *cost process* $C$ is defined by $C_t = V_t - \int_0^t \xi_s dS_s$, $(0 \le t \le T)$. If $C$ is positive, the value of the portfolio is bigger than the cumulative gain from the portfolio, which means we have to inject some money to keep the portfolio. If it is negative, we can withdraw some money, since we have some overflows. If the portfolio is self-financing, we notice that $C$ is always constant, which means there is no cash-flow after the initial payment. By the local risk minimization strategy, we mean a portfolio whose cost process $C$ is a square integrable martingale orthogonal to $M$, where $M$ is the martingale part of $S$ under $\mathbb{P}$. Readers can consult [11, 31, 16] for more detailed discussion on the local risk minimization. Föllmer and Schweizer [11] suggested useful sufficient conditions for the existence of the local risk minimization strategy which are easier to calculate. They found these conditions when the price process is a continuous semimartingale. Although our model is not continuous, we can show that their result still works as long as the price process is $\mathcal{H}^2$ semimartingale.

Suppose that $H(S_T)$ is our contingent claim such as a European call option or put option, and $M$ denotes the martingale part of $S$ under $\mathbb{P}$. We also assume that $V_t = E_{\mathbb{Q}}[H(S_T)|\mathcal{G}_t]$ has a decomposition

$$V_t = V_0 + \int_0^t \xi_s^H dS_s + L_t, \tag{2.22}$$

where $L$ is a square-integrable $\mathbb{P}$-martingale that is orthogonal to $M$ under $\mathbb{P}$, in other words, $\langle L, M \rangle = 0$. For a given hedging strategy to be the local risk minimization strategy, its cost process should be a square-integrable martingale that is orthogonal to $M$ under $\mathbb{P}$. In other words, $\xi^H$ in the above decomposition (2.22) is, in fact, the local risk minimization strategy, since $L_t + V_0$ becomes the cost process and is a square-integrable $\mathbb{P}$-martingale that is orthogonal to $M$ under

$\mathbb{P}$. $\xi^H$ can be computed further in the following way. From $V_t = V_0 + \int_0^t \xi_s^H dS_s + L_t$, we get

$$\langle V, S \rangle_t = \int_0^t \xi_s^H d\langle S, S \rangle_s + \langle L, S \rangle_t.$$

Since $L$ is orthogonal to $M$ under $\mathbb{P}$,

$$\langle L, S \rangle_t = \langle L, M \rangle_t + \langle L, A \rangle_t = 0.$$

Therefore, we have $\langle V, S \rangle_t = \int_0^t \xi_s^H d\langle S, S \rangle_s$ and thus,

$$\xi_t^H = \frac{d\langle V, S \rangle_t}{d\langle S, S \rangle_t}, \tag{2.23}$$

where the quadratic variations are calculated under $\mathbb{P}$. For more details, we refer Föllmer and Schweizer [11], Chan [7]. Thus, as long as we know the existence of the decomposition of $V$, we can calculate the closed form local risk minimization strategy from the equation (2.23).

Notice that the quadratic variations are calculated under $\mathbb{P}$. In Föllmer and Schweizer [11], we did not need to specify a measure under which the bracket processes, $\langle \cdot, \cdot \rangle$, are calculated, since the bracket processes are invariant under a change of measure if processes are continuous semimartingales. If processes are discontinuous, in general, the bracket processes are different if we calculated them under different measures. For the definition of the bracket processes when processes are discontinuous semimartingales and properties, readers can consult Section 3.5 of [27].

First, we assume the decomposition (2.22) exists. Later, in Theorem 6, we can show that this decomposition always exists under our model. The following is the main theorem of our paper.

**Theorem 2.8.** *If the decomposition of $V_t = E_{\mathbb{Q}}[H(S_T)|\mathcal{F}_t]$ exists, the locally risk minimizing strategy is given by*

$$\xi_t^H = \frac{j(t, S_{t-})}{f(S_{t-})^2 + g(S_{t-})^2 \sigma^2 \lambda(S_{t-})}, \tag{2.24}$$

*where*

$$j(t, S_{t-}) = \lambda(S_t) \int_{\mathbb{R}} \{v(t, S_{t-}(1 + x\frac{g(S_{t-})}{S_{t-}})) - v(t, S_{t-})\} x \nu(dx) g(S_{t-})$$
$$+ v_x(t, S_{t-}) f(S_{t-})^2.$$

*Proof.* Recall that the optimal strategy is given by

$$\xi_t^H = \frac{d\langle V, S \rangle_t}{d\langle S, S \rangle_t}.$$

We get

$$d\langle R, R \rangle_t = \sigma^2 \lambda(S_{t-}) dt, \tag{2.25}$$

and

$$d\langle S, S \rangle_t = f(S_{t-})^2 dt + g(S_{t-})^2 \sigma^2 \lambda(S_{t-}) dt. \tag{2.26}$$

Observe that from the equation (2.17), then we have

$$V_t = v(t, S_t) = v(0, S_0) + \int_0^t v_t(s, S_{s-})ds + \int_0^t f(S_{s-})v_x(s, S_{s-})d\widetilde{B}_s$$
$$+ \int_0^t \int_{\mathbb{R}} \{v(s, S_{s-}(1 + x\frac{g(S_{s-})}{S_{s-}})) - v(s, S_{s-})\}q^*(dx, ds).$$

From this, we can calculate $d\langle V, R\rangle$ and $d\langle V, B\rangle$.

$$d\langle V, R\rangle_t = dt\lambda(S_{t-}) \int_{\mathbb{R}} \{v(t, S_{t-}(1 + x\frac{g(S_{t-})}{S_{t-}}) - v(t, S_{t-})\}x\nu(dx),$$

$$d\langle V, B\rangle_t = v_x(t, S_{t-})f(S_{t-})dt.$$

From (2.26) and

$$d\langle V, S\rangle_t = g(S_{t-})d\langle V, R\rangle_t + f(S_{t-})d\langle V, B\rangle_t,$$

we get the result. □

In Theorem 5, we assumed the existence of decomposition of $V_t = E_{\mathbb{Q}}[H(S_T)|\mathcal{F}_t]$. To guarantee the existence of an optimal strategy, we need to show the existence of the decomposition of $V_t = E_{\mathbb{Q}}[H(S_T)|\mathcal{F}_t]$. We can construct the decomposition in the following way.

**Theorem 2.9.** *Let $M$ be the martingale part of $S_t$ and $\xi_s^H$ be as in Theorem 5. Then, $V_t = E_{\mathbb{Q}}[H(S_T)|\mathcal{F}_t]$ has a decomposition*

$$V_t = V_0 + \int_0^t \xi_s^H dS_s + L_t,$$

*where $L_t$ is a square integrable $\mathbb{P}$ martingale such that $\langle L, M\rangle_t = 0$ under $\mathbb{P}$. In other words, there exists an optimal strategy.*

*Proof.* The theorem follows from the next two lemmas (Lemma 2.10 and Lemma 2.11). □

**Lemma 2.10.** $L_t = V_t - \int_0^t \xi_s^H dS_s$ *is a square integrable $\mathbb{P}$-martingale.*

*Proof.* Since $H_t$ and $\int_0^t \xi_s^H dS_s$ are $\mathbb{Q}$- local martingales, $L_t$ is a $\mathbb{Q}$ - local martingale. Let $L_t = M_t' + A_t'$ be the canonical decomposition of $L_t$ under $\mathbb{P}$ where $M_t'$ is the local martingale part and $A_t'$ is the predictable part. By the Girsanov-Meyer theorem,

$$M_t' - \int_0^t \frac{1}{Z_{s-}}d\langle Z, M'\rangle_s = L_t, \tag{2.27}$$

where $Z_t$ is the density process as in Theorem 2. By the uniqueness of the decomposition, there is only one $M'_t$ which satisfies (2.27). On the other hand,

$$\int_0^t \frac{1}{Z_{s-}} d\langle Z, L\rangle_t \tag{2.28}$$

$$= \int_0^t \frac{1}{Z_{s-}} (d\langle Z, V\rangle_s - \xi_s^H d\langle Z, S\rangle_s)$$

$$= \int_0^t (\frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2\lambda(S_{s-})\sigma^2})^2 (d\langle S, V\rangle_s - \xi_s^H d\langle S, S\rangle_s)$$

$$= 0,$$

which implies $M' = L$, i.e. $L$ is a $\mathbb{P}$ local martingale. To show that $L$ is a square integrable $\mathbb{P}$ martingale, it is enough to show that $E\langle L, L\rangle_T < \infty$. Notice that

$$\langle L, L\rangle_T = \langle V - \int_0^{\cdot} \xi_s^H dS_s, V - \int_0^{\cdot} \xi_s^H dS_s\rangle_T$$

$$= \langle V, V\rangle_T - 2\int_0^T \xi_s^H d\langle S, V\rangle_s + \int_0^T (\xi_s^H)^2 d\langle S, S\rangle_s$$

$$= \langle V, V\rangle_T - \int_0^T (\xi_s^H)^2 d\langle S, S\rangle_s$$

$$= \langle V, V\rangle_T - \int_0^T (\xi_s^H)^2 (f(S_{s-})^2 + +g(S_{s-})^2\sigma^2\lambda(S_{s-}))ds,$$

and

$$E\langle L, L\rangle_T = EV_T^2 - \int_0^T E((\xi_s^H)^2 (f(S_{s-})^2 + g(S_{s-})^2\sigma^2\lambda(S_{s-})))ds < \infty.$$

$$\square$$

**Lemma 2.11.** $\langle L, M\rangle_t = 0$ *under* $\mathbb{P}$.

*Proof.*

$$\langle L, M\rangle_t = \langle V_{\cdot} - \int_0^{\cdot} \xi_s^H dS_s, \int_0^{\cdot} (f(S_{s-})dB_s + g(S_{s-})dR_s)\rangle_t$$

$$= \langle V_{\cdot}, \int_0^{\cdot} (f(S_{s-})dB_s + g(S_{s-})dR_s)\rangle_t$$

$$- \int_0^t \xi_s^H f(S_{s-})d\langle S, B\rangle_s - \int_0^t \xi_s^H g(S_{s-})d\langle S, R\rangle_s$$

$$= \int_0^t f(S_{s-})d\langle V, B\rangle_s + \int_0^t g(S_{s-})d\langle V, R\rangle_s$$

$$- \int_0^t \xi_s^H f(S_{s-})d\langle S, B\rangle_s - \int_0^t \xi_s^H g(S_{s-})d\langle S, R\rangle_s$$

$$= \langle S, V\rangle_t - \int_0^t \xi_s^H f(S_{s-})d\langle S, B\rangle_s - \int_0^t \xi_s^H g(S_{s-})d\langle S, R\rangle_s. \tag{2.29}$$

Notice that

$$f(S_{s-})d\langle S, B\rangle_s + g(S_{s-})d\langle S, R\rangle_s = f(S_{s-})^2 ds + g(S_{s-})^2 \sigma^2 \lambda(S_{s-})ds$$
$$= d\langle S, S\rangle_s.$$

Therefore,

$$\int_0^t \xi_s^H (f(S_{s-})d\langle S, B\rangle_s + g(S_{s-})d\langle S, R\rangle_s) = \int_0^t d\langle S, V\rangle_s = \langle S, V\rangle_t \qquad (2.30)$$

Now, from (2.29) and (2.30), we get

$$\langle L, M\rangle_t = \langle S, V\rangle_t - \langle S, V\rangle_t = 0.$$

$\square$

## 3. Application to Liquidity Modelling

We discuss a possible application of this type of price process model to the topic of liquidity. In recent liquidity research (see for example [1], [5], [6], [19], [20], [30]) one uses the notion of a **supply curve**, which is a mathematical model of a limit book for the purchase of relatively liquid stocks. This means as demand increases very rapidly, one must "climb" the ladder of the limit order book to fill the orders, as demand temporarily exceeds ready supply. Therefore one sees small jumps upward during this period, the size of the jump depending on the price and the structure of the order book. In empirical/modelling work by Marcel Blais in his PhD thesis [2], for liquid stocks the supply curve takes the form:

$$S(t, x) = M_t^+ x + b(t), \quad x \geq 0$$
$$S(t, x) = M_t^- x + a(t), \quad x < 0$$

where $t$ is time, $x$ is the size of an initiated trade ($x > 0$ is a buy, and $x < 0$ is a sell, both of $x$ shares of stock), and $S(t, x)$ is the price paid or received *per share* if the order is of size $x$. $M_t^+$ and $M_t^-$ represent the slopes of what turned out to be, as shown by fitting the data to cubic splines, a piecewise linear supply curve. Actually, in many cases for liquid stocks, it turned out that $M_t^+ = M_t^-$; We then simplify notation by writing $M_t$; that is, the supply curve is linear, with a random and time varying slope (although Blais' study indicates that the variation of $M_t$ is small, and continuous in $t$). Also, often $a(0) = b(0)$, which means that the bid-ask spread is negligible. In this case, it might be reasonable to represent the price process as the solution to the following SDE:

$$dS_t = \sigma(S_t)dB_t + b(S_{t-}, M_s)d(N_t - \int_0^t \lambda(S_{s-}, M_s)ds) + \mu(S_t)dt, \qquad (3.1)$$

which is a variant of equation (1.1). We see in equation (3.1) that the intensity of jumps changes with the price, which reflects the climb up the order book, and the typical structure that as one climbs, the offered prices become more sparse and more largely spaced; and the coefficient $b$ depending on $M$ represents the slope of the supply curve affecting the size of the small jump changes in price, with a steeper slope representing an increased inelasticity, and thus one would expect $b(s, m)$ to be increasing in $m$.

Indeed, one can imagine other applications of this type of modelling, when changes in small jumps are sensitive to price, both for their frequency and their size, a phenomenon which happens often in financial modelling.

## 4. Conclusion

This paper introduces a method to find an explicit form of the local risk minimization strategy in general class of models with jumps, extending the results of Frey [14] and Chan [7]. It gives flexibility of models, by changing an intensity function $\lambda$ or a jump distribution $\nu(dx)$. It can also be applied to the deterministic intensity case, simply choosing $\lambda$ independent of the price process. The case when the price process is driven by a Lévy process, which was studied by Chan [7], also follows using a Lévy decomposition and a specific choice of jumps. The most original aspect of our model is an instantaneous feedback of the current price. It would be ideal if we allow an intensity depending on the whole path of the stock history, which is $\lambda(S_s, 0 \leq s \leq t)$ instead of $\lambda(S_t)$, but we defer this to future research. One problem of $\lambda(S_s, 0 \leq s \leq t)$ is the loss of the Markov property, which plays a key role in our approach. This result may be extended to more general jump diffusion cases. But in general this method may fail since one cannot in general guarantee the existence of the decomposition (2.22) and failure of the martingale representation theorem. Our model also allows asymmetric return distributions, which is another advantage. The stochastic representation of the Feynman-Kac type integro-differential equation naturally suggests to us a Monte Carlo method to obtain a solution, but this too will be addressed in the future.

## References

1. Acerbi, C. and Scandolo, G.: Liquidity Risk Theory and Coherent Measures of Risk, December 18, 2007, Available at SSRN: http://ssrn.com/abstract=1048322
2. Blais, M.: Liquidity and Modelling the Stochastic Supply Curve for a Stock Price, Ph. D. dissertation, Cornell University, 2005
3. Black, F., Scholes, M.: The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* **81** (1973) 637–639.
4. Brémaud,P.: *Point Processes*, Springer-Verlag, New York, 1981
5. Çetin, U., Jarrow, R., and Protter, P.: Liquidity Risk and Arbitrage Pricing Theory, *Finance and Stochastics* **8** (2004) 311–341.
6. Çetin, U., Jarrow, R., Protter, P., and Warachka, M.: Pricing Options in an Extended Black Scholes Economy with Illiquidity: Theory and Empirical Evidence, *Review of Financial Studies* **19** (2006) 493–529.
7. Chan, T.: Pricing Contingent Claims on Stocks Driven by Levy Process, *Annals of Applied Probability* **9** (1999) 504–528.
8. Ethier, T. and Kurtz, T.: *Markov Processes*, Wiley Series in Probability and Mathematical Statistics,1986
9. Fama, E.: Mandelbrot and the stable Paretian hypothesis, *Journal of Business* **36** (1963) 420–429.
10. Fama, E.: The behavior of stock market prices, *Journal of Business* **38** (1963) 34–105.
11. Föllmer, H. and Schweizer, M.: *Hedging of Contingent Claims under Incomplete Information*, Applied Stochastic Analysis, Stochastic Monographs, Vol.5, Goldon and Breach, 389–414, 1991
12. Föllmer, H. and Sondermann, D.: Hedging of Non-Redundant Contingent Claims, in *Contributions to Mathematical Economics*, North-Holland, 205–223, 1990

13. Fouque, J-P., Papanicolaou, G., and Sircar, R.: *Derivatives in Financial Markets with Stochastic Volatilty* Cambridge University Press, 2004

14. Frey, R.: Risk-Minimization with Incomplete Information in a Model for High-Frequency Data, *Mathematical Finance* **10** (2000) 215–225.

15. El Karoui, N., Peng, S., and Quenez, M. C.: Backward Stochastic Differential Equations in Finance, *Mathematical Finance* **7** (1997) 1–71.

16. Heath, D., Platen, E., Schweizer, M.: A Comparison of Two Quadratic Approaches to Hedging in Incomplete Markets, *Mathematical Finance* **11** (2001) 385–413.

17. Jacod, J. and Protter, P.: Quelques Remarques sur un Nouveau Type d'Equations Différentielles Stochastiques, *Séminaire de Probabilités XVI*, Springer Lect. Notes in Math, 1980

18. Jacod, J. and Shiryaev, N.: *Limit Theorems for Stochastic Process*, Springer-Verlag, Heidelberg, 1987

19. Jarrow, R. and Protter, P.: Liquidity Risk and Risk Measure Computation, *Review of Futures Markets*, 14, 27–39, 2005.

20. Jarrow, R. and Protter, P.: Liquidity Risk and Option Pricing Theory, in *Handbook in Operation Research and Management Science: Financial Engineering*, 15, 727762, J. Birge and V. Linetsky, eds., North Holland, 2007.

21. Lim, A. E. B.: Hedging Default Risk in an Incomplete Market, *Mathematics of Finance*, G. Yin and Q. Zhang, eds., AMS, Providence, 231–245, 2004

22. Mandelbrot, B.: New methods in statistical econometrics, *Journal of Political Economy* **71** (1963) 421–440.

23. Mandelbrot, B.: The variation of certain speculative prices, *Journal of Business* **26** (1963) 394–419.

24. Mandelbrot, B.: The variation of some other speculative prices, *Journal of Business* **40** (1967) 393–413.

25. Merton, R. C.: Option Pricing When Underlying Stock Returns are Discontinuous, *Journal of Financial Economics* **3** (1976) 125–144.

26. Morgan, I. G.: Stock Prices and Heteroscedasticity, *Journal of Business* **49** 1976) 496–508.

27. Protter, P.: *Stochastic Integration and Differential Equations*, Springer-Verlag,Heidelberg, 2004

28. Protter, P.: A Partial Introduction to Financial Asset pricing Theory, *Stochastic Process and Their Applications* **91** (2001) 169–203.

29. Richardson, M., Smith, T.: A Direct Test of the Mixture of Distribution Hypothesis: Measuring the Daily Flow of Information, *Journal of Financial and Quantitative Analysis* **29** (1994) 101–116.

30. Rogers, C. and Singh, Surbjeet: "Modelling liquidity and its effects on price" preprint, 2006

31. Schweizer, M.: Option Hedging for Semimartingales, *Stochastic Processes and Their Applications* **37** (1991) 339–363.

32. Schweizer, M.: Mean-Variance Hedging in Incomplete Markets, *Annals of Applied Probability* **2** (1992) 171–179.

33. Schweizer, M.: On the Minimal Martingale Measure and the Fllmer–Schweizer Decomposition, *Stochastic Analysis and Applications* **13** (1995) 573–599.

34. Sharpe, M.: *General Theory of Markov Processes*, Academic Press, 1988.

35. Wong, E.: *Stochastic Processes in Engineering System*, Springer-Verlag, New York, 1985

Kiseop Lee: Department of Mathematics, University of Louisville, Louisville, KY 40292, USA
  *E-mail address*: `kiseop.lee@louisville.edu`

Philip Protter: School of OR&IE, Cornell University, Ithaca, NY 14853, USA
  *E-mail address*: `pep4@cornell.edu`