

ABSTRACT WIENER SPACE, REVISITED

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ABSTRACT. This note contains some of my ruminations about L. Gross's theory of abstract Wiener space. None of the ideas introduced or conclusions drawn here is new. Instead, this is only my interpretation of a couple of the beautiful ideas and conclusions which appeared in Gross's seminal 1965 article [1].

1. The Basic Idea

Consider a separable, real Hilbert space H . When H is finite dimensional, the standard, Gaussian measure \mathcal{W}_H for H is the Borel measure given by

$$\mathcal{W}_H(dh) = (2\pi)^{-\frac{\dim(H)}{2}} e^{-\frac{\|h\|_H^2}{2}} \lambda_H(dh), \quad (1.1)$$

where λ_H denotes the Lebesgue measure (i.e., the translation invariant measure which assigns measure 1 to a unit cube in H). When H is infinite dimensional, the \mathcal{W}_H is also given by (1.1), only it fails to exist. The reason it fails to exist is well-known: if it did, then, for any orthonormal basis $\{h_m : m \geq 0\}$, the random variables $h \in H \mapsto X_m(h) = (h, h_m)_H$ would be independent, standard normal random variables and therefore, by the strong law of large numbers, $\|h\|^2 = \sum_{m=0}^{\infty} X_m(h)^2$ would be infinite for \mathcal{W}_H -almost every h .

Put another way, H is simply too small to accommodate \mathcal{W}_H . The idea introduced by Gross was to overcome this problem by completing H with respect to a more forgiving norm than $\|\cdot\|_H$ in such a way that the resulting Banach space would be large enough to house \mathcal{W}_H . To make this precise, he defined the triple $(H, \Theta, \mathcal{W}_H)$ to be an abstract Wiener space if Θ is a separable Banach space into which H is continuously embedded as a dense subset and \mathcal{W}_H is the Borel measure on Θ which has the "right" Fourier transform, the one which (1.1) predicts it should. That is, because H is continuously embedded as a dense subspace of Θ , its dual space Θ^* can be continuously embedded as a dense subspace of H . Namely, given $\lambda \in \Theta^*$, one can use the Riesz representation theorem for Hilbert space to determine $h_\lambda \in H$ by the relation $(h, h_\lambda)_H = \langle h, \lambda \rangle$, $h \in H$. Then, because (1.1) predicts that $h \rightsquigarrow (h, h_\lambda)_H$ should be a centered normal with variance $\|h_\lambda\|_H^2$, having the "right" Fourier transform means that

$$\widehat{\mathcal{W}}_H(\lambda) \equiv \int_{\Theta} e^{\sqrt{-1} \langle \theta, \lambda \rangle} \mathcal{W}_H(d\theta) = e^{-\frac{\|h_\lambda\|_H^2}{2}}. \quad (1.2)$$

2000 *Mathematics Subject Classification.* Primary 60G15, 60G17.

Key words and phrases. Abstract Wiener space.

Putting aside for moment the problem of constructing Θ , it is important to observe that, except in finite dimensions, there are a myriad ways of choosing Θ for the same H , and there seems to be no canonical choice. My own way of thinking about this situation is to interpret H as a scaffold onto which one has to put a coating before it is habitable. How thick to make the coat is a matter of taste.

2. Wiener Series

Of course, the reason why Gross chose the term “abstract Wiener space” is that N. Wiener’s construction of Brownian motion can be viewed as the original case in which a satisfactory Θ was found for a particular H . In Wiener’s case, H is the Hilbert space of absolutely continuous $h : [0, 1] \rightarrow \mathbb{R}$ such that $h(0) = 0$ and $\dot{h} \in L^2([0, 1]; \mathbb{R})$ with the norm $\|h\|_H = \|\dot{h}\|_{L^2([0, 1]; \mathbb{R})}$. One way to describe how Wiener went about one of his three constructions is to say that he chose an orthonormal basis $\{h_m : m \geq 0\}$ for this H and then considered the random series

$$\sum_{m=0}^{\infty} X_m h_m, \quad (2.1)$$

where $\{X_m : m \geq 0\}$ are independent, standard normal random variables. In terms to Gross’s idea, this is an entirely natural idea. Indeed, if \mathcal{W}_H lived on H , then

$$h = \sum_{m=0}^{\infty} (h, h_m)_H h_m,$$

and the random variables $h \rightsquigarrow (h, h_m)_H$ would be independent, standard normals.

Starting from (2.1), the problem of constructing a Θ for Gross’s triple becomes that of finding a Banach space in which the series in (2.1) converges almost surely. To see that this is in fact exactly the same problem, note that if the series is almost surely convergent in Θ , then we can take $\mathcal{W}_H = F_* \gamma^{\mathbb{N}}$, where γ is the standard normal distribution on \mathbb{R} and $F : \mathbb{R}^{\mathbb{N}} \rightarrow \Theta$ is defined by

$$F(\mathbf{x}) = \begin{cases} \sum_{m=0}^{\infty} x_m h_m & \text{when the series converges,} \\ 0 & \text{otherwise,} \end{cases}$$

for $\mathbf{x} = (x_0, \dots, x_m, \dots) \in \mathbb{R}^{\mathbb{N}}$. Checking that this \mathcal{W}_H has the right Fourier transform is trivial. Namely, if $\lambda \in \Theta^*$, then, because the convergence is in Θ ,

$$\begin{aligned} \widehat{\mathcal{W}_H}(\lambda) &= \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{P}} \left[\exp \left(\sqrt{-1} \sum_{m=0}^n (h_m, h_\lambda)_H X_m \right) \right] \\ &= \exp \left(-\frac{1}{2} \sum_{m=0}^{\infty} (h_m, h_\lambda)_H^2 \right) \\ &= e^{-\frac{\|h_\lambda\|_H^2}{2}}. \end{aligned}$$

Conversely, if $(H, \Theta, \mathcal{W}_H)$ is an abstract Wiener space, then the series in (2.1) must be almost surely convergent in Θ . One way to show this is to take the following steps. First, using separability, check that the Borel field \mathcal{B}_Θ for Θ is the

smallest σ -algebra with respect to which all the maps $\theta \rightsquigarrow \langle \theta, \lambda \rangle$ are measurable. Second, introduce the Paley–Wiener map $h \in H \rightarrow \mathcal{I}(h) \in L^2(\mathcal{W}_H; \mathbb{R})$ which is obtained by extending to all of H the isometry given by $[\mathcal{I}(h_\lambda)](\theta) = \langle \theta, \lambda \rangle$ for $\lambda \in \Theta^*$. Third, note that the span of $\{\mathcal{I}(h_m) : m \geq 0\}$ is $L^2(\mathcal{W}; \mathbb{R})$ -dense in $\{\mathcal{I}(h) : h \in H\}$, and use this together with the first step to check that the Borel field \mathcal{B}_Θ is contained in the \mathcal{W}_H -completion of $\sigma(\bigcup_n^\infty \mathcal{F}_n)$, where \mathcal{F}_n is the σ -algebra generated by $\{\mathcal{I}(h_m) : 0 \leq m \leq n\}$. Fourth, use the fundamental property of Gaussian families to see that

$$\theta \rightsquigarrow S_n(\theta) \equiv \sum_{m=0}^n [\mathcal{I}(h_m)](\theta) h_m \tag{2.2}$$

is a \mathcal{W}_H -conditional expectation value of θ given \mathcal{F}_n . Finally, apply the Banach space version of the Marcinkewitz convergence theorem (Doob’s martingale convergence theorem for martingales of the form $\mathbb{E}^\mathbb{P}[X|\mathcal{F}_n]$) to conclude that $\theta = \lim_{n \rightarrow \infty} S_n(\theta)$ for \mathcal{W}_H -almost every θ .

3. Making Wiener’s Series Converge

If one takes a cavalier attitude toward the space at which one will arrive, the remarks in Section 2 make it easy to construct a Θ . For instance, refer to (2.1) and take Θ to be the completion of H with respect to the Hilbert norm

$$\|h\|_\Theta = \sqrt{\sum_{m=0}^\infty (1+m)^{-2} (h, h_m)_H^2}.$$

Because

$$\sum_{m=0}^\infty \frac{X_m^2}{(1+m)^2} < \infty \quad \text{almost surely,}$$

it is trivial to check that the series in (2.1) is almost surely convergent in Θ . Of course, the problem with this approach is that it ignores all subtle cancellation properties and therefore leads to less than optimal results. For example, consider Wiener’s case, and take, as he did,

$$h_0(t) = t \quad \text{and} \quad h_m(t) = \frac{2^{\frac{1}{2}} \sin(m\pi t)}{m\pi} \quad \text{for } m \geq 1.$$

It is then easy to identify the Θ to which the above procedure leads as $L^2([0, 1]; \mathbb{R})$, which is not the pathspace in which one wants Brownian paths to find themselves.

One might hope to improve matters by taking

$$\|h\|_\Theta = \sum_{m=0}^\infty (1+m)^{2(1-\alpha)} (h, h_m)_H^2.$$

As long as $\alpha \in (0, \frac{1}{2})$, there is no doubt that the Wiener series converges in the corresponding Θ . Moreover, in Wiener’s case with his choice of basis, one can identify to resulting Θ as the Sobolev space of L^2 -functions whose α th order derivative is in L^2 . Unfortunately, this is again not sufficient to get Brownian paths to be continuous, since Sobolev’s embedding theory for functions on $[0, 1]$ requires square integrable α th order derivatives for some $\alpha > \frac{1}{2}$. As we now know,

there are many ways to circumvent this difficulty. One way is to abandon Sobolev in favor of his student Besov. That is, for $p \in (1, \infty)$ and $\beta > 0$, define

$$\|h\|_{p,\beta} = \left[\iint_{[0,1]^2} \left(\frac{|h(t) - h(s)|}{|t - s|^\beta} \right)^p dsdt \right]^{\frac{1}{p}}.$$

By Doob's inequality for Banach space valued martingales

$$\mathbb{E}^P \left[\sup_{n \geq 0} \|S_n\|_{p,\beta}^p \right]^{\frac{1}{p}} \leq \frac{p}{p-1} \sup_{n \geq 0} \mathbb{E}^P \left[\|S_n\|_{p,\beta}^p \right]^{\frac{1}{p}},$$

where S_n denotes the n th partial sum of the series in (2.1). At the same time, because the $S_n(t) - S_n(s)$ is a centered Gaussian, if $h_{s,t}(\tau) = t \wedge \tau - s \wedge \tau$, then

$$\mathbb{E}^P \left[|S_n(t) - S_n(s)|^p \right] = K_p \mathbb{E}^P \left[|S_n(t) - S_n(s)|^2 \right]^{\frac{p}{2}} \leq K_p \left(\sum_{m=0}^{\infty} (h_{s,t}, h_m)_H^2 \right)^{\frac{p}{2}},$$

and

$$\sum_{m=0}^{\infty} (h_{s,t}, h_m)_H^2 = \|h_{s,t}\|_H^2 = |t - s|.$$

Thus,

$$\mathbb{E}^P \left[\|S_n\|_{p,\beta}^p \right] \leq K_p \iint_{[0,1]^2} |t - s|^{p(\frac{1}{2} - \beta)} dsdt \equiv K_{p,\beta}.$$

Since $K_{p,\beta} < \infty$ whenever $\beta < \frac{1}{2} + \frac{1}{p}$, it follows that, for each $\beta \in (0, \frac{1}{2})$ and $p \in (1, \infty)$,

$$\mathbb{E}^P \left[\sup_{n \geq 0} \|S_n\|_{p,\beta}^p \right] < \infty.$$

Knowing, as we already do, that $\{S_n : n \geq 0\}$ converges in $L^2([0, 1]; \mathbb{R})$ almost surely, it is now elementary to check that, for each $\beta \in (0, \frac{1}{2})$ and all $p \in (0, \infty)$, $\{S_n : t \geq 0\}$ is almost surely convergent in the Besov space $B_{p,\beta}$ obtained by completing H with respect to $\|\cdot\|_{p,\beta}$. Finally, Besov's embedding theorem says that the space of α -Hölder continuous functions is continuously embedded in $B_{p,\beta}$ whenever $\alpha < \beta - \frac{2}{p}$. Hence, this procedure proves that, for the classical Wiener case, one can take Θ to be any one of the α -Hölder spaces as long as $\alpha < \frac{1}{2}$.

Of course, the preceding is not the most elementary route to the almost sure convergence result just derived. Because we know that such a result holds for all choices of bases once it holds for any one of them, it makes sense to look for a basis which makes the derivation particularly simple. Such a basis was found by P. Lévy, who was not thinking in terms of orthonormal bases, and by Z. Ciesielski, who was. The basis which they took was the Haar basis, the derivative of whose elements are the $L^2([0, 1]; \mathbb{R})$ -orthonormalization of the indicator functions of dyadic intervals.

That is, $\dot{h}_0 = 1$ and, if $m = 2^\ell + k$ for some $\ell \in \mathbb{N}$ and $0 \leq k < 2^\ell$,

$$\dot{h}_m(t) = 2^{\frac{\ell}{2}} \begin{cases} 1 & \text{when } 2^\ell t \in [k, \frac{2k+1}{2}), \\ -1 & \text{if } 2^\ell t \in [\frac{2k+1}{2}, k+1), \\ 0 & \text{otherwise.} \end{cases}$$

The advantage of this basis is that one can easily check that if $\|\cdot\|_u$ is the uniform norm on $C([0, 1]; \mathbb{R})$, then the associated S_n 's satisfy

$$\sup_{n > N} \|S_n - S_N\|_u \leq \sum_{\ell=L}^{\infty} 2^{-\frac{\ell}{2}} \max_{0 \leq k < 2^\ell} |X_{2^\ell+k}| \quad \text{for } N \geq 2^L.$$

Hence, since

$$\max_{0 \leq k < 2^\ell} |X_{2^\ell+k}| \leq \left(\sum_{2^\ell \leq m < 2^{\ell+1}} |X_m|^4 \right)^{\frac{1}{4}},$$

it is clear that

$$\lim_{N \rightarrow \infty} \mathbb{E}^{\mathbb{P}} \left[\sup_{n > N} \|S_n - S_N\|_u \right] = 0.$$

4. Some Properties of Abstract Wiener Spaces

In this concluding section I will discuss a few properties about abstract Wiener spaces, most of which are elementary applications of the Wiener series representation discussed in Section 2.

I begin with the renowned *Cameron–Martin formula*, the one which says that if $h \in H$ and

$$R_h(\theta) = \exp([\mathcal{I}(h)](\theta) - \frac{1}{2}\|h\|_H^2),$$

then the distribution of $\theta \rightsquigarrow \theta + h$ under \mathcal{W}_H is absolutely continuous with respect to \mathcal{W}_H and that R_h is the corresponding Radon–Nikodym derivative. From the standpoint of Wiener series, this observation comes down to the fact that if one translates the standard Gauss distribution γ on \mathbb{R} by $a \in \mathbb{R}$, then the translated measure is absolutely continuous with respect to γ and has Radon–Nikodym derivative $\exp(ax - \frac{1}{2}a^2)$. To see why this implies Cameron and Martin’s result, assume that $h \neq 0$, set $h_0 = \frac{h}{\|h\|_H}$, choose $\{h_m : m \geq 1\}$ so that $\{h_m : m \geq 0\}$ is an orthonormal basis in H , and observe that

$$\int_{\Theta} \Phi(\theta + h) \mathcal{W}_H(d\theta) = \int_{\mathbb{R}^{\mathbb{N}}} \Phi(S_\infty(\mathbf{x}) + h) \gamma^{\mathbb{N}}(d\mathbf{x}),$$

where $S_\infty(\mathbf{x})$ is the $\gamma^{\mathbb{N}}$ -almost sure limit of the partial sums in (2.1). Now apply the one-dimensional result with $a = \|h\|_H$ to the 0th coordinate, and check that

$$\int_{\mathbb{R}^{\mathbb{N}}} e^{\|h\|_H x_0 - \frac{1}{2}\|h\|_H^2} \Phi(S_\infty(\mathbf{x})) \gamma^{\mathbb{N}}(d\mathbf{x}) = \mathbb{E}^{\mathcal{W}_H} [R_h \Phi].$$

The next observation is that if $(H, \Theta, \mathcal{W}_H)$ is an abstract Wiener space, then *the support of \mathcal{W}_H is the whole of Θ* . To see this, first observe that, because H is dense in Θ , one need only check that balls centered at elements of H have positive \mathcal{W}_H -measure. Second, use the Cameron–Martin formula to check that, for any $\epsilon \in H$, $\mathcal{W}_H(B_\Theta(h, r)) > 0$ if $\mathcal{W}_H(B_\Theta(0, r)) > 0$. Hence, all that remains is to

show that $\mathcal{W}_H(\|\theta\|_\Theta < r) > 0$ for all $r > 0$. To this end, choose an orthonormal basis $\{h_m : m \geq 0\}$ for H , and remember that (cf. (2.2)), \mathcal{W}_H -almost surely, $S_n(\theta) \rightarrow \theta$ in Θ . Moreover, as an application of the fundamental property of Gaussian families, one can easily check that θ is \mathcal{W}_H -independent of $\theta - S_n(\theta)$. Hence,

$$\mathcal{W}_H(\|\theta\|_\Theta < r) \geq \mathcal{W}_H(\|\theta - S_n(\theta)\|_\Theta < \frac{r}{2})\mathcal{W}_H(\|S_n(\theta)\|_\Theta < \frac{r}{2}).$$

By taking n large enough, the first factor on the right can be made positive. At the same time,

$$\|S_n(\theta)\|_\Theta \leq C \left(\sum_{m=0}^n [\mathcal{I}(h_m)](\theta)^2 \right)^{\frac{1}{2}},$$

where $C < \infty$ is the bound on the map taking H into Θ . Hence, the second factor dominates the γ^{n+1} -measure of the ball $B_{\mathbb{R}^{n+1}}(0, \frac{r}{2C})$, which is positive for all n 's.

The next property of an abstract Wiener space is one about which I do not feel completely comfortable. The property is the converse to the Cameron–Martin formula: *the translate of \mathcal{W}_H by a $\varphi \in \Theta \setminus H$ is singular to \mathcal{W}_H* . To explain this property, let $\varphi \in \Theta$ be given, and use $T_\varphi\mathcal{W}_H$ to denote the translate of Wiener measure by φ . Next, for $\lambda \in \Theta^*$ with $\|h_\lambda\|_H = 1$, let \mathcal{F}_λ be the σ -algebra generated by $\theta \rightsquigarrow \langle \theta, \lambda \rangle$ and set

$$r_\lambda(\theta) = \exp(\langle \varphi, \lambda \rangle \langle \theta, \lambda \rangle - \frac{1}{2} \langle \varphi, \lambda \rangle^2).$$

Then, proceeding as in the proof of the Cameron–Martin formula, one can easily check that r_λ is the Radon–Nikodym derivative of $T_\varphi\mathcal{W}_H \upharpoonright \mathcal{F}_\lambda$ with respect to $\mathcal{W}_H \upharpoonright \mathcal{F}_\lambda$. Hence, if $T_\varphi\mathcal{W}_H$ is not singular to \mathcal{W}_H and if R is the Radon–Nikodym derivative of its absolutely continuous part, then

$$r_\lambda \geq \mathbb{E}^{\mathcal{W}_H} [R \mid \mathcal{F}_\lambda] \geq \mathbb{E}^{\mathcal{W}_H} [R^{\frac{1}{2}} \mid \mathcal{F}_\lambda]^2,$$

and so

$$\exp\left(-\frac{\langle \varphi, \lambda \rangle^2}{8}\right) = \mathbb{E}^{\mathcal{W}_H} [r_\lambda^{\frac{1}{2}}] \geq \alpha \equiv \mathbb{E}^{\mathcal{W}_H} [R^{\frac{1}{2}}] \in (0, 1].$$

Since this means that $|\langle \varphi, \lambda \rangle| \leq \sqrt{-8 \log \alpha} \|h_\lambda\|_H$ for all $\lambda \in \Theta^*$, it follows that φ must be in H . In conjunction with the result of Cameron and Martin, this means that $T_\varphi\mathcal{W}_H \ll \mathcal{W}_H$ or $T_\varphi\mathcal{W}_H \perp \mathcal{W}_H$ according to whether φ is or is not an element of H . So far so good. What bothers me is that there is another approach to this problem. Namely, choose $\{\lambda_m : m \geq 0\} \subseteq \Theta^*$ so that $\{h_{\lambda_m} : m \geq 0\}$ forms an orthonormal basis in H . Then, as an application of the Wiener series representation of \mathcal{W}_H and Kakutani's theorem about absolute continuity of product measures, one can show that $T_\varphi\mathcal{W}_H \ll \mathcal{W}_H$ or $T_\varphi\mathcal{W}_H \perp \mathcal{W}_H$ according to whether $\sum_{m=0}^{\infty} \langle \varphi, \lambda_m \rangle^2$ converges or diverges. Combining these two, we arrive at the conclusion that, for any $\varphi \in \Theta$, $\varphi \in H$ if and only if $\sum_{m=0}^{\infty} \langle \varphi, \lambda_m \rangle^2 < \infty$. Although this latter conclusion seems reasonable, I do not think that the analogous statement holds for every separable Hilbert space H which is continuously embedded as a dense subset of a Banach space Θ , but it does hold if H and Θ are components of an abstract Wiener triple.

I close with a remark which seems potentially useful, even though I have not found any particular use for it. It is based on the fact that, up to isometry, all

infinite dimensional, separable, real Hilbert space are the same. If one investigates what implication this fact has for abstract Wiener spaces, one finds that it leads to a rigid relationship between the family of abstract Wiener spaces associated with different Hilbert spaces. To be precise, let F be a linear isometry from H^1 onto H^2 , and let Θ^1 be a Banach space for which $(H^1, \Theta^1, \mathcal{W}_{H^1})$ is an abstract Wiener space. Then there is a Banach space Θ^2 and a linear isometry \tilde{F} from Θ^1 onto Θ^2 such that $\tilde{F} \upharpoonright H = F$ and $(H^2, \Theta^2, \tilde{F}_* \mathcal{W}_{H^1})$ is an abstract Wiener space. Like many such abstract results, this one is easier to prove than to state. To prove it, define $\|h^2\|_{\Theta^2} = \|F^{-1}h^2\|_{\Theta^1}$, and let Θ^2 be the completion of H^2 with respect to $\|\cdot\|_{\Theta^2}$. Trivially, F is an isometry from H^1 onto H^2 when H^1 is given the norm $\|\cdot\|_{\Theta^1}$ and H^2 the norm $\|\cdot\|_{\Theta^2}$. Hence, F admits a unique extension \tilde{F} as an isometry from Θ^1 onto Θ^2 . Moreover, if $\mu = \tilde{F}_* \mathcal{W}_{H^1}$, then

$$\hat{\mu}(\lambda^2) = \widehat{\mathcal{W}_{H^1}}(\tilde{F}^\top \lambda^2) = \exp(-\frac{1}{2}\|h_{\tilde{F}^\top \lambda^2}^1\|_{H^1}^2),$$

where \tilde{F}^\top is the adjoint map from the dual space of Θ^2 to the dual of Θ^1 . Finally, it is easy to check that $h_{\tilde{F}^\top \lambda^2}^1 = F^{-1}h_{\lambda^2}^2$, which, since F is an isometry, completes the proof that $(H^2, \Theta^2, \tilde{F}_* \mathcal{W}_{H^1})$ is an abstract Wiener space.

The reason for my thinking that this remark might be useful is that it allows one to choose one Hilbert space, for instance, the classical one, and use the family of abstract Wiener spaces associated with this one to calibrate the abstract Wiener spaces for any Hilbert space. For example, if one believes that $C_0 \equiv \{\theta \in C([0, 1]; \mathbb{R}) : \theta(0) = 0\}$ is *better* than $L^2([0, 1]; \mathbb{R})$ in the classical case, then one ought to believe that for any Hilbert space the Θ corresponding to C_0 is better than the one corresponding to $L^2([0, 1]; \mathbb{R})$. Unfortunately, except for inclusion properties, the meaning of *better* here is not entirely clear and may not be of any significance. In this connection, I have been wondering about the following question, to which one of my readers may already know the answer. Clearly, the family of abstract Wiener spaces for a given Hilbert H space forms a net under inclusion. Is it true that the limit of this net is H itself and, if so, in what sense?

Acknowledgment. I am grateful to Linan Chen for several conversations during which many of these ideas became clearer to me.

References

1. Gross, L.: Abstract Wiener spaces, in: *Proc. 5th Berkeley Symp. Math. Stat. and Probab.* **2**, part 1 (1965) 31–42, University of California Press, Berkeley.

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