

GENERIC FOCK QUANTUM MARKOV SEMIGROUPS WITH INSTANTANEOUS STATES

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ABSTRACT. We construct a generic quantum Markov semigroup with instantaneous states exploiting the invariance of the diagonal algebra and the explicit form of the action of the pre-generator on off-diagonal matrix elements. Our semigroup acts on a unital C^* -algebra and is strongly continuous on this algebra (Feller property). We discuss the generic hydrogen type atoms as an example.

1. Introduction

Generic quantum Markov semigroups arise in the stochastic limit of a generic system with Hamiltonian H_S coupled with a Boson reservoir. They are called “*generic*” because eigenvalues of pure point spectrum Hamiltonians of non-generic systems belong to a small subset (indeed, with measure 0) of a Euclidean space (see Accardi and Kozyrev [4] and the discussion in Sect. 2 here).

This class of quantum Markov semigroups is very interesting, not only because it is very big, but also for its rich structure arising from the investigations by Accardi, Hachicha and Ouerdiane [3], Accardi, Fagnola and Hachicha [2], Carbone, Fagnola and Hachicha [8]:

- a) they leave invariant the Abelian algebra generated by H_S (often called “diagonal algebra”),
- b) they admit a quite explicit representation formula as the sum of a classical Markov semigroup on this Abelian algebra and the conjugation with a contraction semigroup on the off-diagonal operators,
- c) the set of invariant states can be completely determined and the speed of convergence towards each invariant state can be computed,
- d) support projections of invariant states of irreducible generic semigroups belong to the diagonal algebra (decoherence), if irreducibility fails, as for instance for 0-temperature Boson reservoirs, invariant states may be pure (purification) with non-zero off diagonal part as other non generic semigroups (see e.g. the two-photon absorption semigroup [13]).

These properties, however, have been established under a regularity assumption on the structure of the form generator meaning that the mean sojourn time in each state (eigenvector of H_S) is strictly positive. This allows one to construct

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the semigroup by the minimal semigroup method (see Chebotarev and Fagnola [9]) generalising a well-known construction for time-continuous classical Markov chains.

When there are states with zero mean sojourn time, called *instantaneous states*, the construction of the semigroup is much more complicated even in the classical case. There are generalisations of the minimal semigroup method (see Chen and Renshaw [10], Gray, Pollet and Zhang [14]) for Markov chains with a single instantaneous state, however, it seems more convenient to change the method and try to construct a Feller semigroup on some smaller algebra (see Ethier and Kurtz [11] in the commutative, and Matsui [16] in the non-commutative case).

In this paper we construct the generic quantum Markov semigroup (with a finite number of instantaneous states) describing the evolution of a generic system interacting with a Boson, 0 temperature, gauge invariant reservoir. If the coupling has the dipole form with suitable form factors $g \in L^2(\mathbb{R}^d)$ instantaneous states may appear. This happens e.g. for Hamiltonians H_S like that of the hydrogen atom (see Sect. 3).

We show that, under suitable continuity assumptions (see **GS1**, **GS2**) on the transition rates of the underlying classical Markov process, it is possible to construct a quantum Markov semigroup, strongly continuous on a C^* -algebra \mathcal{A} , whose generator coincides with the form generator arising from the stochastic limit on a dense domain. Our assumptions hold for hydrogen type atoms and seem natural (and new) also from the point of view of classical Markov processes.

We start our investigations by describing generic quantum Markov semigroups in Sect. 2 and show that, when the system Hamiltonian H_S is has the same type of the hydrogen atom, then instantaneous states may appear (Sect. 3). In Sect. 4 we construct the associated classical Markov semigroup on the invariant Abelian algebra. The extension to the whole C^* -algebra \mathcal{A} is done in Sect. 5.

2. The Generic Fock QMS

Let S be the discrete system with the Hamiltonian

$$H_S = \sum_{\sigma \in V} \varepsilon_\sigma |\sigma\rangle \langle \sigma|, \quad (2.1)$$

where V is a finite or countable set, $(|\sigma\rangle)_{\sigma \in V}$ is an orthonormal basis of the complex separable Hilbert space \mathfrak{h} of the system.

The Hamiltonian H_S is called *generic* if the eigenspace associated with each eigenvalue ε_σ is one dimensional and one has $\varepsilon_\sigma - \varepsilon_{\sigma'} = \varepsilon_\tau - \varepsilon_{\tau'}$ for $\sigma \neq \sigma'$ if and only if $\sigma = \tau$ and $\sigma' = \tau'$.

The name “generic” is motivated by the fact that spectra of *non-generic* Hamiltonians lie in a set of measure 0 in $\mathbb{R}^{\dim(\mathfrak{h})}$. Indeed, denoting λ the product of $\dim(\mathfrak{h})$ probability measures absolutely continuous with respect to the Lebesgue measure on \mathbb{R} (or the Lebesgue measure itself if $\dim(\mathfrak{h}) < \infty$), it turns out that each set

$$L(\sigma, \sigma'; \tau, \tau') = \{ (\varepsilon_\sigma)_{\sigma \in V} \mid \varepsilon_\sigma - \varepsilon_{\sigma'} = \varepsilon_\tau - \varepsilon_{\tau'} \},$$

with $\sigma \neq \sigma', \tau \neq \tau'$ and $\sigma \neq \tau$, has λ measure equal to 0. Therefore the set

$$\cup_{\sigma \neq \sigma', \tau \neq \tau', \sigma \neq \tau} L(\sigma, \sigma'; \tau, \tau'),$$

containing non-generic spectra has λ measure 0 as a countable union of sets of measure 0.

Generic quantum Markov semigroups arise in the stochastic limit of a discrete system with generic free Hamiltonian H_S interacting with a mean zero, gauge invariant, Gaussian field (see Accardi and Kozyrev [4] and the book [5]). The interaction between the system and the field has the dipole type form $H_I = D \otimes A^+(g) + D^+ \otimes A(g)$ where D is a system satisfying the analyticity condition

$$\sum_{n \geq 1} \frac{|\langle \sigma', D^n \sigma \rangle|}{\Gamma(\theta n)} < \infty,$$

where Γ is the gamma Euler function, for all $\sigma, \sigma' \in V$ and some $\theta \in]0, 1[$. The operators $A^+(g), A(g)$ are the creation and annihilation operators on the Boson Fock space over a Hilbert space with test function (form factor) $g \in L^2(\mathbb{R}^d)$, $d \geq 3$.

The form generator of the generic quantum Markov semigroup in the Fock, i.e. 0 temperature case, (see Accardi, Fagnola and Hachicha[2] and Accardi, Hachicha and Ouerdiane[3]) is

$$\mathcal{L}(x) = \frac{1}{2} \sum_{\sigma, \sigma' \in V, \varepsilon_{\sigma'} < \varepsilon_{\sigma}} (\gamma_{\sigma\sigma'} (2|\sigma\rangle\langle\sigma'|x|\sigma'\rangle\langle\sigma| - \{|\sigma\rangle\langle\sigma|, x\}) + 2i \xi_{\sigma\sigma'} [x, |\sigma\rangle\langle\sigma|]). \tag{2.2}$$

Looking at this form generator (disregarding for the moment the domain problems) we see immediately that the algebra of diagonal operators in the given basis determined by H_S is \mathcal{L} -invariant as well as the one-dimensional space generated by each off-diagonal rank-one operator $|\sigma\rangle\langle\tau|$ ($\sigma \neq \tau$). Moreover the restriction of \mathcal{L} to the algebra of diagonal operators is a classical Markov jump process with state space equal to the spectrum of the Hamiltonian H_S , and jump intensities $\gamma_{\sigma\sigma'}$.

It is well-known from the classical theory of Markov jump processes that the construction of the semigroup starting from a form generator with jump intensities $\gamma_{\sigma\sigma'}$ can be done by the standard minimal semigroup method (see [9], [12]) under the following summability condition

$$\sum_{\{\sigma' \in V \mid \varepsilon_{\sigma'} < \varepsilon_{\sigma}\}} \gamma_{\sigma\sigma'} < \infty \tag{2.3}$$

for all $\sigma \in V$. This condition means that the mean sojourn time of the Markov jump process in the state σ is strictly positive, i.e. the state σ is not instantaneous.

Indeed, in [2] we constructed the generic Fock (0-temperature) quantum Markov semigroup by this method and proved an explicit representation formula for the action of the semigroup. Moreover, assuming that H_S has a ground state, we computed the rate of convergence towards this state which is the invariant of the generic quantum Markov semigroup. The same problems were studied in [8] in the Gaussian gauge invariant (positive temperature) case.

We can find, however, special physical models where the above summability condition (2.3) does not hold and the semigroup must be constructed by another method. To show this we start by a more detailed analysis of the *generalised susceptivities* $\gamma_{\sigma\sigma'}$ and the *generalised Lamb shifts* $\xi_{\sigma,\sigma'}$ given by (see [4])

$$\gamma_{\sigma\sigma'} = 2\Re(g|g)_{\omega}^{-} |\langle\sigma', D\sigma\rangle|^2, \quad \xi_{\sigma\sigma'} = \Im(g|g)_{\omega}^{-} |\langle\sigma', D\sigma\rangle|^2, \quad (2.4)$$

with $\omega = \varepsilon_{\sigma} - \varepsilon_{\sigma'}$ (*Bohr frequency*) and

$$\Re(g|g)_{\omega}^{-} = \pi \int_{S(\omega)} |g(k)|^2 d_S k, \quad \Im(g|g)_{\omega}^{-} = -p.p. \int_{\mathbb{R}^d} \frac{|g(k)|^2}{\omega(k) - \omega} dk,$$

where $k \rightarrow \omega(k)$ is the dispersion function which is differentiable on $\mathbb{R}^d - \{0\}$ and strictly increasing along each half-line starting from 0, $d_S k$ denotes the surface integral on the sphere of center 0 and radius ω and *p.p.* denotes the principal part integral.

Reasonable cutoff functions g are rapidly decreasing at infinity therefore

$$\sup_{\omega > \omega_1} \Re(g|g)_{\omega}^{-} < \infty$$

for each fixed ω_1 . Thus, since $\sum_{\sigma'} |\langle\sigma', D\sigma\rangle|^2 = \|D\sigma\|^2$, the summability condition (2.3) on $\gamma_{\sigma\sigma'}$ can be violated only if, there exists a Bohr frequency ω with infinitely many smaller Bohr frequencies $\omega' \in]0, \omega[$. If the spectrum of H_S is bounded from below, this can happen only if it has at least an accumulation point.

The next section shows a class of models, based on hydrogen type atoms, in which the summability condition does not hold. These models motivate the following assumptions that will be in force throughout the paper:

HS *The set \mathbb{V} of eigenvalues of the system Hamiltonian H_S is compact and has a discrete set of accumulation points $(a_j)_{j \geq 1}$ belonging to \mathbb{V} that are left accumulation points, isolated from the right in \mathbb{V} and the series*

$$\mu_{\sigma} = \sum_{\sigma, \sigma' \in V, \varepsilon_{\sigma'} < \varepsilon_{\sigma}} \gamma_{\sigma\sigma'}, \quad \kappa_{\sigma} = \sum_{\sigma, \sigma' \in V, \varepsilon_{\sigma'} < \varepsilon_{\sigma}} \xi_{\sigma\sigma'}, \quad (2.5)$$

are absolutely convergent for all $\sigma \in V$ such that $\varepsilon_{\sigma} \notin \{a_1, \dots, a_n\}$.

Clearly a_j is a left accumulation point if it is the limit of an *increasing* sequence of eigenvalues of H_S but there is a smallest eigenvalue of H_S strictly bigger than a_j , unless a_j is the maximum eigenvalue of H_S . Denote by \mathbb{A} the set of accumulation points and note that, since it is discrete, it has no accumulation points. Note that, since all the a_j belong to \mathbb{V} , the spectrum of H_S is compact and bounded from above. This technical assumption could be removed by slight modifications of the construction in Section 4.

Under the above assumption it is clear that the form generator (2.2) is well-defined on the algebra generated by finite rank operators of the form $|\sigma'\rangle\langle\sigma|$ with $\varepsilon_{\sigma'}, \varepsilon_{\sigma} \in \mathbb{V} - \mathbb{A}$.

The divergence of the series defining μ_{a_j} means that a_j is an *instantaneous state* of the classical Markov process associated with the restriction of \mathcal{L} to the Abelian algebra of operators commuting with the system Hamiltonian H_S and the sojourn time of the process in a_j is zero.

3. Hydrogen Type Atoms

Hydrogen type atoms provide examples satisfying the previous hypothesis with appropriate conditions on D and the cutoff g .

The hydrogen atom Hamiltonian is the self-adjoint operator H on the Hilbert space $L^2(\mathbb{R}^3)$ given by

$$H = -\Delta - 2r^{-1}, \quad (3.1)$$

where $r = \|x\|$ with $x \in \mathbb{R}^3$. In spherical coordinates H is given by

$$H = -\frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} - \frac{2}{r}. \quad (3.2)$$

For each $n \geq 1$ there exist a unique radial solution of the equation

$$H\psi_n = -\frac{1}{n^2}\psi_n, \quad (3.3)$$

with unit norm in $L^2(\mathbb{R}^3)$ given by

$$\psi_n(r) = 2^{-1/2} n^{-5/2} e^{-r/2n} L_{n-1}^1(r/n), \quad (3.4)$$

where L_{n-1}^1 is the Laguerre polynomial (see [1]).

The operator H is not generic. Indeed, taking two distinct Pythagorean triples as, for instance $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$, multiplying them by 13^2 and 5^2 we find $39^2 + 52^2 = 25^2 + 60^2$. Finally, some obvious manipulations yield $(25.52.60)^{-2} - (39.52.60)^{-2} = (25.39.52)^{-2} - (25.39.60)^{-2}$. We know from L. Accardi (personal communication) that it is also possible to characterise all different pairs of eigenvalues in spectrum of the hydrogen Hamiltonian (3.1) having the same differences.

A small perturbation of this operator, however, leads to a generic Hamiltonian. Let $\tau \in]0, 1[$ be a transcendental number and, for $n \in V := \mathbb{N}^* \cup \{\infty\}$, we define

$$\varepsilon_n = -(n + \tau)^{-2}, \quad \varepsilon_\infty = 0. \quad (3.5)$$

Now, since τ can not be a solution to any polynomial equation with integer coefficients, it is clear that the operator H_S given by (2.1) with the above choice of the ε_n is generic.

If g is a square integrable radial function on \mathbb{R}^3 such that $g(r) = r^{-\theta/2}$ with $\theta < 3$, for $r = \|k\| < 1$, the dispersion function is $\omega(k) = |k|^2$ then, for any $n, m \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned} \gamma_{nm} &= 8\pi^2 \left(\frac{1}{(m + \tau)^2} - \frac{1}{(n + \tau)^2} \right)^{2-\theta} |\langle \psi_m, D\psi_n \rangle|^2, \quad 1 \leq m < n < \infty, \\ \gamma_{\infty m} &= 8\pi^2 \frac{1}{(m + \tau)^{2(2-\theta)}} |\langle \psi_m, D\psi_\infty \rangle|^2, \quad 1 \leq m < \infty. \end{aligned}$$

For any $n \in \mathbb{N}$ there are only a finite number of levels $m < n$. Therefore the series $\sum_{m < n} \gamma_{nm}$ defining μ_n is reduced to a finite sum. Suitable choices of the operator D yield generalised susceptibilities satisfying $\sum_m \gamma_{\infty m} = +\infty$. We can take, for

instance, a D such that

$$\begin{aligned} |\langle \psi_m, D\psi_n \rangle|^2 &= \left(\frac{1}{(m+\tau)^2} - \frac{1}{(n+\tau)^2} \right)^\alpha, & m < n < \infty, \\ |\langle \psi_m, D\psi_\infty \rangle|^2 &= \frac{1}{(m+\tau)^{2\alpha}}, & m < \infty. \end{aligned}$$

for some $\alpha > 1/2$ to ensure that $\sum_m |\langle \psi_m, D\psi_\infty \rangle|^2 = \|D\psi_\infty\|^2 < \infty$, to find

$$\mu_\infty = \sum_{n \geq 1} \gamma_{\infty n} = \infty \quad \text{for} \quad 4 < 2\alpha + 3 < 2\theta < 6.$$

This, however, is just an example where our results apply.

4. The QMS on the Diagonal Algebra

The construction of the Markov semigroup of a classical jump process with some instantaneous states is usually a difficult problem that has not a general solution (see the book [11] Ch.7 p. 376). Even the case of a *single* instantaneous state is also non-trivial (see the papers [10], [14]). Motivated by the discussion of hydrogen type atoms in Section 3, however, we give a solution to this problem for a class which, to the best of our knowledge, is new also for classical probability.

The natural definition of the generator L

$$(Lf)(\varepsilon_\sigma) = \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma))$$

might not may sense for $\varepsilon_\sigma = a_j$ if the function f is not continuous at a_j . Indeed, if $f(\varepsilon_{\sigma'}) > f(\varepsilon_\sigma) + c$ (c positive constant) in a left neighbourhood of a_j , then the series is divergent.

To circumvent this problem we start by restricting the candidate domain of the operator L to functions which are left continuous at points in \mathbb{A} . Therefore we work in the Banach space \mathcal{C} of complex-valued continuous functions on the set \mathbb{V} (endowed with the topology induced by the Euclidean topology of \mathbb{R}) of eigenvalues with norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{\sigma \in \mathbb{V}} |f(\varepsilon_\sigma)|.$$

We assume that the generalised susceptivities satisfy the following condition which is necessary for the generator L to be defined at least on indicator functions of isolated points of \mathbb{V} and functions constant in a left neighbourhood of accumulation points a_1, \dots, a_n (see Lemma 4.1 below).

GS1 For all $\varepsilon_\sigma \in \mathbb{V} - \mathbb{A}$ and $a_j \in \mathbb{A}$ with $a_j > \varepsilon_\sigma$ we have

$$\lim_{\varepsilon_\tau \rightarrow a_j^-} \gamma_{\tau\sigma} = \gamma_{a_j\sigma}. \quad (4.1)$$

For all accumulation point a_j , all ε_σ with $a_{j-1} < \varepsilon_\sigma < a_j$ (with the convention $a_{-1} = -\infty$), and all $a_k > a_j$ we have

$$\lim_{\varepsilon_\tau \rightarrow a_j^-} \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\tau\sigma'} = \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{a_j\sigma'}, \quad \lim_{\varepsilon_\tau \rightarrow a_k^-} \sum_{\varepsilon_\sigma \leq \varepsilon_{\sigma'} \leq a_j} \gamma_{\tau\sigma'} = \sum_{\varepsilon_\sigma \leq \varepsilon_{\sigma'} \leq a_j} \gamma_{a_k\sigma'}. \quad (4.2)$$

Remark. This assumption clearly holds for the generalised susceptivities of the hydrogen type atoms (Section 3). Indeed, (4.1) follows immediately. The first of (4.2) is a trivial consequence of (4.1) because in this model there is only a finite number of $\varepsilon_{\sigma'}$ smaller than a $\varepsilon_{\sigma} < 0$. The second one also follows immediately by monotone convergence since all the sequences $(\gamma_{nm})_{n \geq m}$ increase to $\gamma_{\infty m}$.

We shall prove that the operator $L : D(L) \rightarrow \mathcal{C}$

$$\begin{aligned} D(L) &= \left\{ f \in \ell^\infty(\mathbb{V}) \left| \exists \lim_{\varepsilon_{\sigma} \rightarrow a_j^-} \sum_{\varepsilon_{\sigma'} < \varepsilon_{\sigma}} \gamma_{\sigma\sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_{\sigma})) \forall j \right. \right\}, \\ (Lf)(\varepsilon_{\sigma}) &= \sum_{\varepsilon_{\sigma'} < \varepsilon_{\sigma}} \gamma_{\sigma\sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_{\sigma})), \quad \text{for } \varepsilon_{\sigma} \in \mathbb{V} - \mathbb{A}, \\ (Lf)(a_j) &= \lim_{\varepsilon_{\sigma} \rightarrow a_j^-} \sum_{\varepsilon_{\sigma'} < \varepsilon_{\sigma}} \gamma_{\sigma\sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_{\sigma})), \quad \text{for } a_j \in \mathbb{A}, \end{aligned}$$

generates a strongly continuous contraction semigroup on \mathcal{C} .

As a first step we prove the following

Lemma 4.1. *Suppose that the hypothesis **GS1** holds. Then L is densely defined and closed.*

Proof. L is densely defined because $D(L)$ contains functions with finite support contained in $\mathbb{V} - \mathbb{A}$ and functions constant in a small left neighbourhood of points in \mathbb{A} . Indeed, this linear manifold is clearly dense in \mathcal{C} . Moreover, for all $\varepsilon_{\sigma} \in \mathbb{V}$ with $\varepsilon_{\sigma} \notin \mathbb{A}$, we have

$$(L1_{\{\varepsilon_{\sigma}\}})(\varepsilon_{\tau}) = \gamma_{\tau\sigma} \quad \text{if } \varepsilon_{\sigma} < \varepsilon_{\tau}, \quad (L1_{\{\varepsilon_{\sigma}\}})(\varepsilon_{\sigma}) = - \sum_{\varepsilon_{\sigma'} < \varepsilon_{\sigma}} \gamma_{\sigma\sigma'} = -\mu_{\sigma}$$

and $(L1_{\{\varepsilon_{\sigma}\}})(\varepsilon_{\tau}) = 0$ if $\varepsilon_{\sigma} > \varepsilon_{\tau}$. Therefore $1_{\{\varepsilon_{\sigma}\}}$ belongs to $D(L)$ by **GS1**. Also indicator functions f of intervals $[\varepsilon_{\sigma}, a_j]$ with, $a_{j-1} < \varepsilon_{\sigma} < a_j$, belong to $D(L)$ because, in this case, $(Lf)(\varepsilon_{\tau})$ vanishes if $\varepsilon_{\tau} < \varepsilon_{\sigma}$ and

$$(Lf)(\varepsilon_{\tau}) = \begin{cases} - \sum_{\varepsilon_{\sigma'} < \varepsilon_{\sigma}} \gamma_{\tau\sigma'}, & \text{if } \varepsilon_{\tau} \in [\varepsilon_{\sigma}, a_j], \\ \sum_{\varepsilon_{\sigma} \leq \varepsilon_{\sigma'} \leq a_j} \gamma_{\tau\sigma'}, & \text{if } \varepsilon_{\tau} > a_j. \end{cases}$$

It follows that f belongs to $D(L)$ by the hypothesis **GS1** on the generalised susceptivities.

We finally check that L is closed. Let $(f_m)_{m \geq 1}$ be a sequence in $D(L)$ such that $f_m \rightarrow f$ and $Lf_m \rightarrow g$ in norm in \mathcal{C} . For every $\delta > 0$ there exists m_{δ} such that, for any $m > m_{\delta}$

$$g(\varepsilon_{\sigma}) - \delta < \sum_{\varepsilon_{\sigma'} < \varepsilon_{\sigma}} \gamma_{\sigma\sigma'} (f_m(\varepsilon_{\sigma'}) - f_m(\varepsilon_{\sigma})) < g(\varepsilon_{\sigma}) + \delta,$$

for all $\varepsilon_{\sigma} \in \mathbb{V}$. Letting m to ∞ we find

$$g(\varepsilon_{\sigma}) - \delta < \sum_{\varepsilon_{\sigma'} < \varepsilon_{\sigma}} \gamma_{\sigma\sigma'} (f(\varepsilon_{\sigma}) - f(\varepsilon_{\sigma'})) < g(\varepsilon_{\sigma}) + \delta,$$

for all $\varepsilon_\sigma \in \mathbb{V} - \mathbb{A}$ because, for such ε_σ , convergence of the series $\sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} = \mu_\sigma < \infty$ allows us to exchange limit in m and summation by dominated convergence. Since g is continuous on \mathbb{V} , this also proves that the limit

$$\lim_{\varepsilon_\sigma \rightarrow a_j^-} \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma))$$

exists for all $j \in \{1, 2, \dots, n\}$. Thus, f belongs to $D(L)$ and by the arbitrariness of δ we have $Lf = g$. \square

Lemma 4.2. *The operator L is dissipative.*

Proof. Let $f \in D(L)$ fixed. Since the spectrum of H_S is compact there exists a $\varepsilon_\sigma \in \mathbb{V}$ such that $\|f\| = e^{i\theta_\sigma} f(\varepsilon_\sigma)$.

The functional $\eta : \mathcal{C} \rightarrow \mathbb{C}$ defined by

$$\eta(g) = e^{i\theta_\sigma} g(\varepsilon_\sigma)$$

is clearly tangent to f . Now, if $\varepsilon_\sigma \in \mathbb{V} - \mathbb{A}$, then

$$\begin{aligned} \Re \eta(Lf) &= \Re \left(e^{i\theta_\sigma} \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma)) \right) \\ &= \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} (\Re(e^{i\theta_\sigma} f(\varepsilon_{\sigma'})) - \|f\|) \leq 0. \end{aligned}$$

On the other hand, if $\varepsilon_\sigma = a_j$, with $j \in \{1, 2, \dots, n\}$ then, taking a sequence $(\varepsilon_{\sigma(p)})_{p \geq 1}$ in $\mathbb{V} - \mathbb{A}$ increasing to a_j such that $\Re(e^{i\theta_\sigma} f(\varepsilon_{\sigma(p)}))$ increases to $\|f\|_\infty$, we have

$$\begin{aligned} \Re \eta(Lf) &= \lim_{p \rightarrow \infty} \Re(e^{i\theta_\sigma} (Lf)(\varepsilon_{\sigma(p)})) \\ &= \lim_{p \rightarrow \infty} \Re \left(e^{i\theta_\sigma} \sum_{\varepsilon_{\sigma'} < \varepsilon_{\sigma(p)}} \gamma_{\sigma(p)\sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_{\sigma(p)})) \right) \\ &\leq \limsup_{p \rightarrow \infty} \sum_{\varepsilon_{\sigma'} < \varepsilon_{\sigma(p)}} \gamma_{\sigma(p)\sigma'} (\Re(e^{i\theta_\sigma} f(\varepsilon_{\sigma'})) - \|f\|_\infty). \end{aligned}$$

Thus, $\Re \eta(Lf) \leq 0$ and L is dissipative. \square

Lemma 4.3. *For all $f \in D(L)$ and $\lambda > 0$ we have*

$$\|\lambda f - Lf\| \geq \lambda \|f\|.$$

Proof. Let η be a tangent functional to f . Then

$$\|\lambda f - Lf\| \geq \Re \eta(\lambda f - Lf) = \lambda \|f\| - \Re \eta(Lf) \geq \lambda \|f\|$$

since $\Re \eta(Lf) \leq 0$. \square

We introduce now our last hypothesis on the generalised susceptibilities in order to check that the range of $\lambda - L$ is the whole \mathcal{C} .

GS2 For all $\lambda > 0$ and all $a_j \in \mathbb{A}$, denoting τ^+ the index of smallest eigenvalue strictly bigger than a $\varepsilon_\tau \in \mathbb{V} - \mathbb{A}$ we have

$$\sum_{\varepsilon_\tau < a_j} \frac{\sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} |\gamma_{\tau^+ \sigma'} - \gamma_{\tau \sigma'}|}{\lambda + \sum_{\varepsilon_{\sigma'} < \varepsilon_{\tau^+}} \gamma_{\tau^+ \sigma'}} < \infty.$$

This condition is obviously fulfilled by the generalised susceptivities of the hydrogen type atom if $1/2 < \alpha < 1$, and $\theta = \alpha + 2$. Indeed, in this case $\gamma_{nm} = 1$ for all $n > m$. It could be checked also for other values of α and θ by complex but elementary computations.

Proposition 4.4. For all $\lambda > 0$ the range of the operator $\lambda - L$ is the whole \mathcal{C} .

Proof. We first show that $R(\lambda - L)$ is dense \mathcal{C} . Note that it contains constant functions because $(\lambda - L)1 = \lambda$. It suffices to prove that it contains functions with finite support not containing points in $\mathbb{A} = \{a_j \mid 1 \leq j \leq n\}$ and functions constant in a left neighbourhood of any a_j .

Step 1: g is the indicator function of a singleton $\{\varepsilon_\sigma\}$ with $\varepsilon_\sigma \notin \mathbb{A}$. The equation $\lambda f - Lf = g$ means

$$\lambda f(\varepsilon_\tau) - \sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{\tau \sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_\tau)) = \begin{cases} 1, & \text{if } \tau = \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $a_{j-1} < \varepsilon_\sigma < a_j$ for some j . The above equation is solved for $\varepsilon_\tau \leq \varepsilon_\sigma$ by putting $f(\varepsilon_\tau) = 0$ for all $\varepsilon_\tau < \varepsilon_\sigma$ and $f(\varepsilon_\sigma) = 1/(\lambda + \mu_\sigma)$. For any $\varepsilon_\tau \in]\varepsilon_\sigma, a_j[$, denoting ε_{τ^+} the smallest eigenvalue of H_S strictly bigger than ε_τ , define inductively

$$f(\varepsilon_{\tau^+}) = \frac{1}{\lambda + \mu_{\tau^+}} \sum_{\varepsilon_{\sigma'} < \varepsilon_{\tau^+}} \gamma_{\tau^+ \sigma'} f(\varepsilon_{\sigma'}).$$

Note that $f(\varepsilon_\tau) \geq 0$ for all $\varepsilon_\tau \in]\varepsilon_\sigma, a_j[$ and by recursion

$$f(\varepsilon_{\tau^+}) \leq \sum_{\varepsilon_{\sigma'} < \varepsilon_{\tau^+}} \frac{\gamma_{\tau^+ \sigma'} \max_{\varepsilon_\sigma \leq \varepsilon_{\sigma'} \leq \varepsilon_{\tau^+}} f(\varepsilon_{\sigma'})}{\lambda + \mu_{\tau^+}} = \frac{\mu_{\tau^+}}{(\lambda + \mu_\sigma)(\lambda + \mu_{\tau^+})} \leq \frac{1}{\lambda + \mu_\sigma}.$$

The difference $\lambda(f(\varepsilon_{\tau^+}) - f(\varepsilon_\tau))$ can be written in the form

$$\begin{aligned} & \sum_{\varepsilon_{\sigma'} < \varepsilon_{\tau^+}} \gamma_{\tau^+ \sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_{\tau^+})) - \sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{\tau \sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_\tau)) \\ &= -\gamma_{\tau^+ \tau} (f(\varepsilon_{\tau^+}) - f(\varepsilon_\tau)) - \sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{\tau^+ \sigma'} (f(\varepsilon_{\tau^+}) - f(\varepsilon_\tau)) \\ &+ \sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{\tau^+ \sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_\tau)) - \sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{\tau \sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_\tau)). \end{aligned} \quad (4.3)$$

It follows that

$$\left(\lambda + \sum_{\varepsilon_{\sigma'} < \varepsilon_{\tau^+}} \gamma_{\tau^+ \sigma'} \right) (f(\varepsilon_{\tau^+}) - f(\varepsilon_\tau)) = \sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} (\gamma_{\tau^+ \sigma'} - \gamma_{\tau \sigma'}) (f(\varepsilon_{\sigma'}) - f(\varepsilon_\tau))$$

and, for all $\varepsilon_\tau, \varepsilon_\eta$ with $\varepsilon_\sigma < \varepsilon_\tau < \varepsilon_\eta < a_j$ we have

$$\begin{aligned} |f(\varepsilon_\tau) - f(\varepsilon_\eta)| &\leq \sum_{\varepsilon_\tau \leq \varepsilon_w < \varepsilon_\eta} |f(\varepsilon_{w^+}) - f(\varepsilon_w)| \\ &\leq \frac{1}{\lambda} \sum_{\varepsilon_\tau \leq \varepsilon_w < \varepsilon_\eta} \frac{\sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} |\gamma_{\tau^+ \sigma'} - \gamma_{\tau \sigma'}|}{\lambda + \sum_{\varepsilon_{\sigma'} < \varepsilon_{\tau^+}} \gamma_{\tau^+ \sigma'}}. \end{aligned}$$

Condition **GS2** implies then that the limit $\lim_{\varepsilon_\tau \rightarrow a_j^-} f(\varepsilon_\tau)$ exists.

Letting a_j^+ denote the smallest eigenvalue ε_τ bigger than a_j the equation $\lambda f(a_j^+) - Lf(a_j^+) = 0$ yields

$$f(a_j^+) = \frac{1}{\lambda + \mu_{a_j^+}} \sum_{\varepsilon_{\sigma'} < a_j^+} \gamma_{a_j^+ \sigma'} f(\varepsilon_{\sigma'}).$$

We can now solve recursively $\lambda f(\varepsilon_\tau) - Lf(\varepsilon_\tau) = 0$ for all $\varepsilon_\tau \in]a_j, a_{j+1}[$ and show again by means of **GS2** that the limit $\lim_{\varepsilon_\tau \rightarrow a_{j+1}^-} f(\varepsilon_\tau)$ exists. Repeating the same argument for all the intervals $[a_j, a_{j+1}]$ we find an f belonging to the domain of L satisfying $\lambda f - Lf = 1_{\{\varepsilon_\sigma\}}$.

Step 2: g is the indicator function of an interval $1_{[\varepsilon_\sigma, a_j]}$ with $a_{j-1} < \varepsilon_\sigma < a_j$. The equation $\lambda f(\varepsilon_\tau) - Lf(\varepsilon_\tau) = g(\varepsilon_\tau)$ is satisfied for all $\varepsilon_\tau \leq a_j$ if we put $f(\varepsilon_\tau) = 0$ for all $\varepsilon_\tau < \varepsilon_\sigma$, $f(\varepsilon_\sigma) = 1/(\lambda + \mu_\sigma)$ and we solve recursively for $\varepsilon_\tau < a_j$. Now, for $\varepsilon_\sigma < \varepsilon_\tau < a_j$, we can write the difference $\lambda(f(\varepsilon_{\tau^+}) - f(\varepsilon_\tau))$ as in (4.3). It follows that the limit $\lim_{\varepsilon_\tau \rightarrow a_j^-} f(\varepsilon_\tau)$ exists by **GS2**. Finally we can conclude as in step 1 finding an f belonging to the domain of L satisfying $\lambda f - Lf = 1_{[\varepsilon_\sigma, a_j]}$.

We now complete the proof showing that $R(\lambda - L)$ is closed.

Let $(g_n)_{n \geq 1}$ be a sequence in $R(\lambda - L)$ converging to g a function in \mathcal{C} and let $(f_n)_{n \geq 1}$ be a sequence in $D(L)$ such that

$$\lambda f_n - Lf_n = g_n.$$

From Lemma 4.3 for all $n, m \in \mathbb{N}^*$ we have

$$\|g_n - g_m\| = \|(\lambda - L)(f_n - f_m)\| \geq \lambda \|f_n - f_m\|.$$

It follows that there exists a $f \in \mathcal{C}$ such that

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0.$$

Moreover, $Lf_n = \lambda f_n - g_n$ converges in \mathcal{C} thus (L is closed) $f \in D(L)$ and $\lambda f - Lf = g$. Therefore $g \in D(L)$. \square

Summing up we proved the following

Theorem 4.5. *The operator L generates a strongly continuous contraction semi-group $(T_t)_{t \geq 0}$ on \mathcal{C} .*

Proof. We did all the steps allowing us to apply Lumer-Phillips theorem [15] Th. 4.3 p. 14. Indeed, the operator L is closed and densely defined by Lemma 4.1, it is dissipative by Lemma 4.2 and the range of $\lambda - L$ is closed for all $\lambda > 0$ by Proposition 4.4. \square

We end this section showing that the operators T_t on \mathcal{C} are positive.

Theorem 4.6. *The semigroup $(T_t)_{t \geq 0}$ is Markov.*

Proof. Since the constant function 1 on the spectrum of H_S satisfies $T_t 1 = 1$, we must check that the operators T_t are positive.

This follows from the positive maximum principle. Indeed, all real-valued $f \in D(L)$ is a continuous function on the spectrum of H_S which is compact. Therefore it has a maximum at a point ε_σ and we find

$$(Lf)(\varepsilon_\sigma) = \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma)) \leq 0.$$

This proves positivity. \square

Remark. We could allow unbounded spectra by assuming some condition like **GS2** on eigenvalues at infinity.

5. Extension to Non-diagonal Operators

In this section we construct the generic Quantum Markov Semigroup with instantaneous states on a C^* -subalgebra \mathcal{A} of $\mathcal{B}(\mathfrak{h})$. The structure (2.2) of the form generator makes it natural the following definition of the action of the QMS on off-diagonal operators $|\sigma\rangle\langle\tau|$ with $\sigma, \tau \in \mathbb{V} - \mathbb{A}$.

Define the normal dissipative operator G on \mathfrak{h} by

$$G = \sum_{\sigma, \varepsilon_\sigma \in \mathbb{V} - \mathbb{A}} \left(-\frac{\mu_\sigma}{2} + i\kappa_\sigma \right) |\sigma\rangle\langle\sigma|.$$

Clearly we have

$$\mathcal{L}(|\sigma\rangle\langle\tau|) = G^*|\sigma\rangle\langle\tau| + |\sigma\rangle\langle\tau|G.$$

Denoting by $(P_t)_{t \geq 0}$ the strongly continuous contraction semigroup on \mathfrak{h} generated by G explicitly given by

$$P_t|\sigma\rangle = e^{-t(\mu_\sigma/2 - i\kappa_\sigma)}|\sigma\rangle, \quad \text{for } \varepsilon_\sigma \in \mathbb{V} - \mathbb{A}, \quad P_t|\sigma\rangle = |\sigma\rangle, \quad \text{for } \sigma \in \mathbb{A},$$

the action of the generic QMS $(\mathcal{T}_t)_{t \geq 0}$ on the off-diagonal operator $|\sigma\rangle\langle\tau|$ should be given, in a natural way, by

$$\mathcal{T}_t(|\sigma\rangle\langle\tau|) = P_t^*|\sigma\rangle\langle\tau|P_t = e^{-t((\mu_\sigma + \mu_\tau)/2 + i(\kappa_\tau - \kappa_\sigma))}|\sigma\rangle\langle\tau|,$$

for all σ, τ with $\varepsilon_\sigma, \varepsilon_\tau \in \mathbb{V} - \mathbb{A}$.

Let \mathcal{F} be the algebra generated by the operators $|\sigma\rangle\langle\tau|$ ($\varepsilon_\sigma, \varepsilon_\tau \in \mathbb{V} - \mathbb{A}$). Clearly the norm closure $\overline{\mathcal{F}}$ of \mathcal{F} is the C^* -algebra of all compact operators y on \mathfrak{h} such that $\langle a_j, ya_k \rangle = 0$ for all $a_j, a_k \in \mathbb{A}$ and the diagonal part $\sum_{\varepsilon_\sigma \in \mathbb{V} - \mathbb{A}} y_{\sigma\sigma} |\sigma\rangle\langle\sigma|$ of an operator y in this C^* -algebra defines a function $g(\varepsilon_\sigma) = y_{\sigma\sigma}$ in \mathcal{C} since $\lim_{\varepsilon_\sigma \rightarrow a_j^-} y_{\sigma\sigma} = 0$.

We identify functions $f \in \mathcal{C}$ with the corresponding multiplication (diagonal) operator and denote by \mathcal{O} the closed subspace of $\overline{\mathcal{F}}$ of operators with zero diagonal part (i.e. $\langle \sigma, x\sigma \rangle = 0$ for all $\sigma \in \mathbb{V}$).

Proposition 5.1. *The operators x on \mathfrak{h} that can be decomposed as $f + y$ with $f \in \mathcal{C}$ and $y \in \mathcal{O}$ form a C^* -subalgebra \mathcal{A} of $\mathcal{B}(\mathfrak{h})$.*

Proof. The linear space of operators decomposable as above clearly form a *-subalgebra of $\mathcal{B}(\mathfrak{h})$. Indeed, \mathcal{C} is a *-subalgebra of $\mathcal{B}(\mathfrak{h})$, the product of $f \in \mathcal{C}$ and $y \in \mathcal{O}$ belongs to \mathcal{O} and the product of $y, z \in \mathcal{O}$ can be written as the sum of an $f \in \mathcal{C}$ vanishing at each a_j and an element of \mathcal{O} .

The proof will be complete if we show that it is norm closed. To this end note that

$$\|f + y\| = \sup_{\|\varphi\| \leq 1, \|\psi\| \leq 1} |\langle \varphi, (f + y)\psi \rangle| \geq \sup_{\sigma \in V} |\langle \sigma, (f + y)\sigma \rangle| = \sup_{\sigma \in V} |\langle \sigma, f\sigma \rangle| = \|f\|.$$

Let $(f_n + y_n)_{n \geq 1}$ be a norm convergent sequence. By the above inequality, for all n, m , we have

$$\|f_n - f_m\| \leq \|f_n + y_n - (f_m + y_m)\|.$$

It follows that both sequences $(f_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ are Cauchy in $\mathcal{B}(\mathfrak{h})$, converge in norm respectively to $f \in \mathcal{C}$ and $y \in \mathcal{O}$ and

$$\lim_{n \rightarrow \infty} f_n + y_n = f + y$$

belongs to \mathcal{A} . □

Definition 5.2. For all $t \geq 0$ let \mathcal{T}_t be the linear map on \mathcal{A}

$$\mathcal{T}_t(a) = T_t f + P_t^* y P_t$$

if $a = f + y$ with $f \in \mathcal{C}$ and $y \in \mathcal{O}$.

The decomposition $a = f + y$ being unique, the definition is not ambiguous.

The semigroup property is easily checked. Indeed, $P_t^* y P_t \in \mathcal{O}$ for all $y \in \mathcal{O}$, and the semigroup property of $(T_t)_{t \geq 0}$, $(P_t)_{t \geq 0}$ entail

$$\mathcal{T}_{t+s} a = T_{t+s} f + P_{t+s}^* y P_{t+s} = T_t T_s f + P_t^* (P_s^* y P_s) P_t = \mathcal{T}_t \mathcal{T}_s a.$$

In order to prove that the maps \mathcal{T}_t are completely positive we start by the following estimate

Lemma 5.3. For all non-negative $f \in \mathcal{C}$, $t \geq 0$ and all $\varepsilon_\sigma \in \mathbb{V} - \mathbb{A}$ we have

$$(T_t f)(\varepsilon_\sigma) \geq e^{-t\mu_\sigma} f(\varepsilon_\sigma).$$

Proof. Denote by M_μ the non-densely defined multiplication operator $(M_\mu g)(\varepsilon_\sigma) = \mu_\sigma g(\varepsilon_\sigma)$. Clearly M_μ is not defined on functions with support at points $a_j \in \mathbb{A}$ because $\mu_{a_j} = +\infty$. Nevertheless, for all $\lambda > 0$, we can define everywhere the contraction $(\lambda + M_\mu)^{-1}$ by

$$((\lambda + M_\mu)^{-1} g)(\varepsilon_\sigma) = (\lambda + \mu_\sigma)^{-1} g(\varepsilon_\sigma)$$

with the understanding $(\lambda + \mu_{a_j})^{-1} = 0$. We can easily check that $(\lambda + M_\mu)^{-1}$ is continuous at points $a_j \in \mathbb{A}$. In fact, since $\mu_{a_j} = +\infty$, for all $r > 0$ there exists a finite set $F \subseteq V$ such that

$$\sum_{\sigma' \in F, \varepsilon_{\sigma'} < a_j} \gamma_{a_j \sigma'} > 2r.$$

By the hypothesis **GS1** (4.1), there exists a ε_σ such that $|\gamma_{a_j\sigma'} - \gamma_{\tau\sigma'}| < r/\text{card}(F)$ for all $\varepsilon_\tau \in]\varepsilon_\sigma, a_j[$. It follows that, for all such ε_τ

$$\begin{aligned} \sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{\tau\sigma'} &\geq \sum_{\sigma' \in F, \varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{\tau\sigma'} \\ &= \sum_{\sigma' \in F, \varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{a_j\sigma'} + \sum_{\sigma' \in F, \varepsilon_{\sigma'} < \varepsilon_\tau} (\gamma_{\tau\sigma'} - \gamma_{a_j\sigma'}) \\ &\geq \sum_{\sigma' \in F, \varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{a_j\sigma'} - r > r. \end{aligned}$$

It follows that

$$\liminf_{\varepsilon_\tau \rightarrow a_j^-} \mu_\tau = \liminf_{\varepsilon_\tau \rightarrow a_j^-} \sum_{\varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{\tau\sigma'} \geq -r + \liminf_{\varepsilon_\tau \rightarrow a_j^-} \sum_{\sigma' \in F, \varepsilon_{\sigma'} < \varepsilon_\tau} \gamma_{a_j\sigma'} = r.$$

Since r is arbitrary, we have $\lim_{\varepsilon_\tau \rightarrow a_j^-} \mu_\tau = +\infty = \mu_{a_j}$.

An inspection at the proof of Proposition 4.4 reveals that for all g which is either the indicator function of a singleton $\varepsilon_\sigma \notin \mathbb{A}$ or the indicator function of an interval $[\varepsilon_\sigma, a_j]$ with $a_{j-1} < \varepsilon_\sigma < a_j$ and all $\lambda > 0$ we have

$$(\lambda - L)^{-1} \geq (\lambda + M_\mu)^{-1}g.$$

By density of the linear span of such functions in \mathcal{C} , the above inequality holds for all non-negative $g \in \mathcal{C}$. Since $(\lambda + M_\mu)^{-1}g$ also belongs to \mathcal{C} , and $(\lambda - L)^{-1}$ is positive by Theorem 4.6 iterating, we have

$$(\lambda - L)^{-m}g \geq (\lambda + M_\mu)^{-m}g,$$

for all non-negative $g \in \mathcal{C}$ and $m \geq 1$. Putting $\lambda = m/t$ and letting m tend to infinity, we find the conclusion by the Trotter-Kato formula. \square

Proposition 5.4. $(\mathcal{T}_t)_{t \geq 0}$ is a semigroup of completely positive maps on \mathcal{A} .

Proof. Call $(\mathcal{P}_t)_{t \geq 0}$ the semigroup of completely positive maps on $\mathcal{B}(\mathfrak{h})$ given by $\mathcal{P}_t(x) = P_t^* x P_t$. For all $n > 0$ let \mathcal{M}_n be the algebra of $n \times n$ matrices and let $\mathcal{T}_t^{(n)}$ (resp. $T_t^{(n)}$, $\mathcal{P}_t^{(n)}$) be the linear maps on $\mathcal{A} \otimes \mathcal{M}_n$ (resp. $\mathcal{C} \otimes \mathcal{M}_n$, $\mathcal{B}(\mathfrak{h}) \otimes \mathcal{M}_n$) given by $\mathcal{T}_t \otimes I_n$ (resp. $T_t \otimes I_n$, $\mathcal{P}_t \otimes I_n$) where I_n denotes the identity map on \mathcal{M}_n . Note that $\mathcal{P}_t(\mathcal{C}) \subseteq \mathcal{C}$.

The map $T_t^{(n)} - \mathcal{P}_t$ is positive because $T_t - \mathcal{P}_t$ is a positive map on an Abelian algebra by Lemma 5.3. Note that, if $a \in \mathcal{A}$ decomposed as $f + y$ is positive, then it is also positive $\tilde{f} + y$ where $\tilde{f} = \sum_{\varepsilon_\sigma \notin \mathbb{A}} f(\varepsilon_\sigma) |\sigma\rangle \langle \sigma|$.

The identity $\mathcal{A} \otimes \mathcal{M}_n = (\mathcal{C} \otimes \mathcal{M}_n) \oplus (\mathcal{O} \otimes \mathcal{M}_n)$ implies then

$$\begin{aligned} \mathcal{T}_t^{(n)}(a) &= T_t^{(n)}(f) + \mathcal{P}_t^{(n)}(y) \\ &\geq \mathcal{P}_t^{(n)}(\tilde{f}) + \mathcal{P}_t^{(n)}(y) = \mathcal{P}_t^{(n)}(\tilde{f} + y) \geq 0 \end{aligned}$$

because the maps $\mathcal{P}_t^{(n)}$, explicitly written in the Kraus' form, are completely positive. \square

Proposition 5.5. The semigroup $(\mathcal{T}_t)_{t \geq 0}$ is strongly continuous on \mathcal{A} .

Proof. Decomposing $a \in \mathcal{A}$ as $f + y$ with $f \in \mathcal{C}$ and $y \in \mathcal{O}$, we can write

$$\|\mathcal{T}_t(a) - a\| \leq \|\mathcal{T}_t(f) - f\| + \|P_t^* y P_t - y\|.$$

The first term vanishes as t goes to 0 because the semigroup $(\mathcal{T}_t)_{t \geq 0}$ is strongly continuous. We now check this happens also for second term by an approximation argument from [6], [7]. Suppose first that $y = |u\rangle\langle v| \in \mathcal{F}$. In this case we have

$$\begin{aligned} \|P_t^* y P_t - y\| &\leq \| |P_t^* u\rangle\langle P_t^* v| - |u\rangle\langle v| \| \\ &\leq \| |P_t^* u\rangle\langle P_t^* v| - |u\rangle\langle P_t^* v| \| + \| |u\rangle\langle P_t^* v| - |u\rangle\langle v| \| \\ &= \| |P_t^* u - u\rangle\langle P_t^* v| \| + \| |u\rangle\langle P_t^* v - v| \| \\ &\leq \|v\| \cdot \|P_t^* u - u\| + \|u\| \cdot \|P_t^* v - v\| \end{aligned}$$

and $P_t^* y P_t - y$ tends to 0 in norm as t goes to 0. The same conclusion holds for an arbitrary element of $\overline{\mathcal{F}}$, in particular for a $y \in \mathcal{O}$, by a 3ε argument because finite rank operators in \mathcal{F} are norm dense in \mathcal{F} . \square

We summarize the previous results by the following

Theorem 5.6. *There exists a strongly continuous quantum Markov semigroup $(\mathcal{T}_t)_{t \geq 0}$ on the C^* -algebra \mathcal{A} whose generator \mathcal{L} defined by*

$$\begin{aligned} D(\mathcal{L}) &= \left\{ a \in \mathcal{A} \mid \exists \text{norm} - \lim_{t \rightarrow 0^+} \frac{\mathcal{T}_t(a) - a}{t} \right\}, \\ \mathcal{L}(a) &= \text{norm} - \lim_{t \rightarrow 0^+} \frac{\mathcal{T}_t(a) - a}{t} \end{aligned}$$

coincides with the form generator (2.2) on $a = f + y$ with f either indicator function of a $\varepsilon_\sigma \in \mathbb{V} - \mathbb{A}$ or indicator function of an interval $[\varepsilon_\sigma, a_j] \cap \mathbb{V}$ (with $a_{j-1} < \varepsilon_\sigma < a_j$) and y is in the linear span of rank-one operators $|\sigma\rangle\langle\tau|$ with $\sigma \neq \tau$ and $\varepsilon_\sigma, \varepsilon_\tau \in \mathbb{V} - \mathbb{A}$.

It is worth noticing here that $(\mathcal{T}_t)_{t \geq 0}$ is Feller because it is strongly continuous on a C^* -algebra.

It could be shown easily that an essential domain for \mathcal{L} is given by those $a = f + y$ with f belonging to an essential domain for the generator of the associated classical Markov process L and y in the linear span of rank-one operators $|\sigma\rangle\langle\tau|$ with $\sigma \neq \tau$ and $\varepsilon_\sigma, \varepsilon_\tau \in \mathbb{V} - \mathbb{A}$.

Remark. Note that \mathcal{A} does not separate normal states on $\mathcal{B}(\mathfrak{h})$. Indeed, defining $\rho_\pm = |(\sigma \pm \iota)/\sqrt{2}\rangle\langle(\sigma \pm \iota)/\sqrt{2}|$, for any σ, ι with $\varepsilon_\iota \in \mathbb{A}$ and $\varepsilon_\sigma \in \mathbb{V} - \mathbb{A}$ one has $\text{tr}(\rho_+ a) = \text{tr}(\rho_- a)$ for all $a \in \mathcal{A}$. This is not a serious problem since normal states whose support overlaps with instantaneous state have 0 mean lifetime.

We do not see, however, how to enlarge \mathcal{A} by constructing the QMS with local algebra methods (see e.g. Matsui [16]).

After constructing the QMS the natural question about its properties as, for instance, invariant states and convergence towards invariant states arises. The instantaneous state here decays with infinite speed and has life-time 0. Therefore this does not change too much the qualitative behaviour of the dynamics described in [3].

In particular we expect that the ground state of H_S is the unique normal invariant state and finitely supported states (as well as low tail states) converge at an exponential speed e^{-gt} with g given essentially by the infimum of the μ_σ on the support of the initial state.

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