PROPERTIES OF CERTAIN LÉVY AND GEOMETRIC LÉVY PROCESSES

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ABSTRACT. We study Lévy processes associated with the power-variance family of probability laws. Their path and structural properties as well as the exact asymptotics of the probabilities of large deviations are established. We use the techniques of Dolean-Meyer exponentials to introduce an additional class of Lévy processes. The ordinary exponentials of its members constitute the geometric Lévy processes which we utilize for describing the movements of equities. Thus, we consider a self-financing portfolio comprised of one bond and k equities assuming that the returns on all k equities belong to the latter class. We demonstrate that for the choice of constant Merton-type portfolio weights, the combined movement of k equities is governed by a geometric Lévy process which belongs to the same class. In the continuous case, we prove a converse of Merton's mutual fund theorem. We derive Pythagorean-type theorems for Sharpe measures emphasizing their relation to Merton-type weights and the additivity of shape parameter.

1. Introduction

In this paper, we investigate certain Lévy and geometric Lévy processes. The consideration of some of them is motivated by various topics of optimization of self-financing portfolios of securities. This goes back to Merton's papers [14]–[15]. However, our own interest in these topics is driven primarily by elegant mathematical properties of specific quantities which appear to be useful for illustrating the theory of optimization of the portfolio of investments. Thus, in contrast to numerous work on applications of Lévy and related processes to finance, we concentrate on the effect of the *Merton-type allocation of weights* on structural properties of the resulting cumulative price process and characterize it in terms of the additivity of the growth rate. In addition, we establish the *non-Cramér-type* behavior of the probabilities of large deviations of certain Hougaard processes.

Section 2 reviews basic facts on the *power-variance family* (or *PVF*) of probability laws. In particular, the *shape parameter* (defined by formula (2.9)) is of importance for the main topics of this paper.

We discover an interesting connection of the author's result on the additivity of shape parameter (see [23, Prop. 2.1]) with Merton's approach to the allocation of weights. In the case when $p = 0$, this connection is exhibited by Theorem 2.5. This result pertains to the case when the movements of equities follow *independent* geometric Brownian motions. The part (ii) of Theorem 2.5 can be regarded as the *converse* of the celebrated *Merton's*

mutual fund theorem. See also Theorem 4.2.ii that provides an analogue of the converse of Merton's theorem for a particular class of discontinuous geometric Lévy processes.

In Section 3, we introduce the class of *Hougaard stochastic processes*, which are generated by PVF. The results of that section are interesting in their own right. In addition, they provide the necessary background for those of Section 4. All the members of this class share some common properties. However, certain features of its *individual* members cover the whole spectrum of different possibilities for almost-sure path properties of Lévy processes (see Proposition 3.4). In the same section, we obtain and interpret *non-Cramér-type* results on large deviations for these processes (see Theorem 3.6).

In Section 4, we study the effect of *Merton-type* allocation of weights in the *discontinuous* setting. Some results of Section 4 illustrate the techniques of Dolean-Meyer exponentials.

Our main achievements are Theorems 2.5.ii, 3.6 and 4.2. At the same time, Theorem 3.5, Proposition 2.3.ii and Corollary 4.3 are also of a considerable interest.

In order to describe some of our assertions, we will now comment on certain relevant results which are widely used in both the theory and practice of selection of portfolios of investments. A relatively common method of the theory of portfolio optimization is to maximize the *expected Bernoulli logarithmic utility function.* This is partly explained by the fact that under this specific choice of the utility function, problems often become mathematically tractable. A comprehensive treatment of the topics pertinent to *logarithmic* utility can be found in Korn and Korn [12, pp. 212–213].

Let us summarize some basic terminology related to the *Sharpe portfolio performance measure* (or *Sharpe ratio*) SR_{port} for risk-adjusted return. Denote the Sharpe ratio for i^{th} asset of the portfolio by SR_i:

$$SR_i := (R_i - r)/\sigma_i. \tag{1.1}$$

Here, $r \geq 0$ stands for a *risk-free rate.* Throughout the paper, it is assumed to be *constant.* The quantities $R_i - r$ and σ_i represent the *expected excess return* and the *risk* (i.e., standard deviation) for i^{th} asset. By analogy to (1.1), set

$$SR_{port} := (R_{port} - r)/\sigma_{port}. \tag{1.2}$$

Hereinafter, $R_{port} - r$ and σ_{port} denote the expected excess return and risk for the entire portfolio, which is assumed to be comprised of *independent* assets. In the sequel, we will frequently denote R_i, R_{port} and σ_{port} by m_i, m and σ, respectively.

Merton [14, end of Sec. 4.6 – beginning of Sec. 4.7] solved this optimization problem for a portfolio comprised of one risk-free bond, for which the price process

$$S^{(b)}(t) = S^{(b)}(0) \cdot e^{r \cdot t}, \tag{1.3}$$

and one equity. He assumed that the price process for chaotic movements of a single share of the equity follows a *geometric Brownian motion*

$$S^{(e)}_{m,\sigma}(t) := S^{(e)}_{m,\sigma}(0) \cdot \exp\{(m - \sigma^2/2)) \cdot t + \sigma \cdot \mathcal{B}_t\}, \tag{1.4}$$

where constant $\sigma > 0$ and $\{\mathcal{B}_t,\ t \geq 0\}$ is the univariate Brownian motion. Let $m > r$. The price process (1.4) solves the following stochastic differential equation (or *SDE*):

$$dS^{(e)}_{m,\sigma}(t)/S^{(e)}_{m,\sigma}(t) = m \cdot dt + \sigma \cdot d\mathcal{B}_t \tag{1.5}$$

(see Elliott and Kopp [7, Sec. 7.1]). The parameters m and σ are interpreted as the *rate of exponential growth* and the *volatility* of this equity, respectively.

Merton [14] demonstrated that the expected Bernoulli logarithmic utility is optimized when the proportion of wealth invested in equity (1.4) at each time instant is *constant* (see (1.7) and (1.13)). To comment on this, we now introduce the *vector of weights*

$$W := (w_0, w_1, ..., w_k). \tag{1.6}$$

One should have $\sum_{i=0}^{k} w_i \equiv 1$. We allow some w_i's to be either negative or greater than 1. A *negative* value of w_0 can be attributed to *short-selling*. In the case when $k = 1$, Merton [14] proved that the following weight maximizes the logarithmic utility:

$$\widetilde{w}_1 := (m - r)/\sigma^2. \tag{1.7}$$

See also Korn and Korn [12, p. 213]. The securities are assumed to be traded *continuously*. In view of the choice of weight (1.7), one should equate

$$S_{m,\sigma}^{(e)}(t) \equiv \widetilde{w}_1 \cdot \mathcal{S}^{(W)}(t). \tag{1.8}$$

Hereinafter, $\mathcal{S}^{(W)}(t)$ denotes the *cumulative price process* for the portfolio considered. It is then straightforward to show that choosing the weight (1.7) for this equity results in the next (exponential) growth rate for such self-financing portfolio:

$$g(m, r, \sigma) := r + \frac{1}{2} \cdot \frac{(m - r)^2}{\sigma^2} . \tag{1.9}$$

By (1.1) and the comment below formula (1.5), the rightmost term in (1.9) is proportional to the square of the Sharpe ratio for the equity described by (1.4)–(1.5).

In the same paper, Merton extended this method to the case when there are k (> 1) *risky* assets and one bond in the portfolio. To simplify his approach, suppose that in addition to a non-random price process (1.3) for a bond, we now have k *independent* processes $S_{m_i,\sigma_i}^{(e,i)}(t)$ which describe the random movements of equities. Suppose that they all admit representation (1.4) with their own m_i's $> r$ and σ_i's. Here, $1 \le i \le k$.

The above self-financing portfolio corresponds to the next *cumulative price process*:

$$\mathcal{S}^{(W)}(t) := \xi_0(t) \cdot S^{(b)}(t) + \sum_{i=1}^{k} \xi_i(t) \cdot S_{m_i,\sigma_i}^{(e,i)}(t). \tag{1.10}$$

It is important that $\forall$ fixed real $t \ge 0$, (dependent) r.v.'s $\xi_0(t), \xi_1(t), ..., \xi_k(t)$ hereinafter represent the amounts of shares of the corresponding securities which are held in the portfolio at time $t-$. In view of the above arguments, one should have that $\forall$ $1 \le i \le k$,

$$\xi_i(t) \cdot S_{m_i,\sigma_i}^{(e,i)}(t) \equiv w_i \cdot \mathcal{S}^{(W)}(t). \tag{1.11}$$

A subsequent combination of (1.10)–(1.11) with the independence of $S_{m_i,\sigma_i}^{(e,i)}(t)$'s implies that the cumulative price process $\mathcal{S}^{(W)}(t)$ solves SDE of the same type as (1.5):

$$d\mathcal{S}^{(W)}(t)/\mathcal{S}^{(W)}(t) \ \ (=: m \cdot dt + \sigma \cdot d\mathcal{B}_t)$$
$$= \left(w_0 \cdot r + \sum_{i=1}^{k} w_i \cdot m_i \right) \cdot dt + \left(\sum_{i=1}^{k} w_i \cdot \sigma_i \right) \cdot d\mathcal{B}_t. \tag{1.12}$$

Since the movements of all these assets are independent and follow their own geometric Brownian motions (1.4) with $m_i > r$, the *Merton's allocation of weights* and the (exponential) growth rate for the portfolio are straightforward generalizations of (1.7) and (1.9), respectively. Here, $1 \leq i \leq k$. By analogy to (1.7), $\forall \, 1 \leq i \leq k$ define *Merton's weights*:

$$\widetilde{w}_i := (m_i - r)/\sigma_i^2. \tag{1.13}$$

We attribute the following result to Merton, who proved a more general statement.

Theorem 1.1. *Let $\{\mathcal{S}^{(W)}(t), \ t \geq 0\}$ denote the cumulative price process of the self-financing portfolio of securities which is defined by (1.10). Then Merton's choice of weights $w_i := \widetilde{w}_i$, which is given by (1.13), implies the additivity of the quantity $g(m_1, ..., m_k, r, \sigma_1, ..., \sigma_k) - r$ that constitutes the optimal expected excess growth rate of the portfolio:*

$$\begin{aligned} g(m_1, ..., m_k, r, \sigma_1, ..., \sigma_k) - r &:= \frac{1}{2} \cdot \sum_{i=1}^{k} \frac{(m_i - r)^2}{\sigma_i^2} \\ &= \sum_{i=1}^{k} \big(g(m_i, r, \sigma_i) - r \big). \end{aligned} \tag{1.14}$$

In Section 4, we will explore the effect of Merton's allocation of weights on the closeness of a class of specific geometric Lévy processes, which are related to PVF.

Each univariate distribution considered in this paper is *infinitely divisible* and characterized by its *canonical triplet* (b, a, ν). Here, the *first characteristic* b and the *diffusion coefficient* $a \geq 0$ are some real-valued constants, whereas ν is a certain *Lévy measure* on $\mathbf{R}^1$. We utilize *truncation function* $h(x) := x \cdot \chi_{\{|x| \leq 1\}}$. Hereinafter, $\chi_{\mathcal{C}}$ denotes the *indicator* of Borel set $\mathcal{C}$, whereas $\chi_{\mathcal{C}}(u)$ is the *indicator function*.

Each infinitely divisible distribution on $\mathbf{R}^1$ generates a unique Lévy process, i.e., a real-valued stochastic process that starts from the origin and has independent and stationary increments. Assume that with probability 1, the trajectories of (generic) Lévy process $\mathcal{R}(t)$ belong to the *càdlàg* space $\mathbf{D}[0, \infty)$ of right-continuous functions which have limits from the left that is equipped with the Skorokhod topology. A classification of Lévy processes pertinent to their almost-sure path properties is presented in Definition 3.3.

We will utilize the following *Dolean-Meyer exponential* for univariate Lévy processes:

Definition 1.2. Given Lévy process $\{\mathcal{R}(t), \ t \geq 0\}$ with the generating triplet (b, a, ν), consider the càdlàg process $\{\mathcal{F}(t), \ t \geq 0\}$ with $\mathcal{F}(0) = 1$, which solves SDE (1.15):

$$d\mathcal{F}(t)/\mathcal{F}(t-) = d\mathcal{R}(t). \tag{1.15}$$

Hereinafter, process $\mathcal{F}(t)$ is referred to as the *Dolean-Meyer exponential* that corresponds to Lévy process $\mathcal{R}(t)$. It is denoted by $\mathcal{SE}\{\mathcal{R}(t)\}$.

An example is the Dolean-Meyer exponential $\mathcal{SE}\{mt + \sigma\mathcal{B}_t\}$ of the scaled Brownian motion with a drift. It coincides with geometric Brownian motion (1.4).

It is known that $\forall$ generating triplet (b, a, ν) that uniquely determines a specific Lévy process $\mathcal{R}(t)$, $\exists$ a unique solution to (1.15) (cf., e.,g., Cont and Tankov [5, Prop. 8.21]). In addition, their result contains a representation for $\mathcal{F}(t)$. It is also relevant that

$$\mathcal{F}(t) \overset{\text{a.s.}}{>} 0 \iff \nu((-\infty, 1]) = 0 \tag{1.16}$$

(cf., e.,g., Cont and Tankov [5, p. 286]). Under fulfillment of (1.16), $\mathcal{F}(t)$ coincides with the ordinary exponential of a different but uniquely determined Lévy process $\mathcal{L}(t)$:

$$\mathcal{F}(t) = \mathcal{SE}\{\mathcal{R}(t)\} \overset{\mathrm{d}}{=} \exp\{\mathcal{L}(t)\}. \tag{1.17}$$

We will call ordinary exponentials of Lévy processes the *geometric Lévy processes*. The generating triplets of Lévy processes $\mathcal{R}(t)$ and $\mathcal{L}(t)$ are in one-to-one correspondence (see Goll and Kallsen [9, Lm. A.8] or Cont and Tankov [5, Prop. 8.22]).

To conclude the Introduction, let us refer to monographs Applebaum [1], Cont and Tankov [5], Øksendal and Sulem [16], Schoutens [19] for more information.

2. The Power-variance Family and Converse of Merton's Theorem

Observe that properties of Hougaard processes (which are introduced in Section 3) are closely related to particular results on PVF. Certain assertions of Section 4 will also employ properties of members of PVF. Hence, it is now relevant to describe this family. See Vinogradov [23] for more detail.

Let us introduce the *power parameter* $p \in \Delta := \mathbf{R}^1 \setminus (0, 1)$. (This terminology is justified by formula (2.8).) Also, consider the *scaling parameter* $\lambda \in \Lambda := \mathbf{R}^1_+ := (0, \infty)$ and the *location parameter* μ. It is such that $\mu \in [0, \infty)$ if $p \in (-\infty, 0)$; $\mu \in \mathbf{R}^1$ in the case when $p = 0$; $\mu \in \mathbf{R}^1_+$ for $p \in [1, 2]$, and $\mu \in (0, \infty]$ if $p \in (2, \infty)$. Hereinafter, we refer to the values of parameters specified above as their *admissible values*. It is convenient to denote the above domains of the location parameter by Ω_p. Clearly, the boundary $\partial\Delta$ of Δ is the two-point set $\{0; 1\}$. For the corresponding values of p, define

$$\theta(p, \mu, \lambda) := \frac{1}{|1 - p|} \cdot \lambda \cdot \mu^{1-p}; \tag{2.1}$$

$$B_{p,\lambda} := \frac{|1 - p|^{(2-p)/(1-p)}}{|2 - p|} \cdot \lambda^{1/(p-1)} . \tag{2.2}$$

Definition 2.1. A generic member of PVF, which is hereinafter denoted by $Tw_p(\mu, \lambda)$ is characterized by arbitrary fixed admissible values of parameters p, μ and λ. It is described by virtue of its cumulant-generating function (or *c.g.f.*) $\zeta_{p,\mu,\lambda}(\cdot)$, which is given by formulas (2.3)–(2.7). Namely, in the case when $p \in (-\infty, 0]$, set

$$\zeta_{p,\mu,\lambda}(s) := B_{p,\lambda} \cdot \{(\theta(p, \mu, \lambda) + s)^{(2-p)/(1-p)} - \theta(p, \mu, \lambda)^{(2-p)/(1-p)}\}, \tag{2.3}$$

where $s \geq -\theta(p, \mu, \lambda)$ if $p \in (-\infty, 0)$ and $s \in \mathbf{R}^1$ if $p = 0$. Also,

$$\zeta_{1,\mu,\lambda}(s) := \mu \cdot \lambda \cdot (e^{s/\lambda} - 1) \tag{2.4}$$

with $s \in \mathbf{R}^1$. For $p \in (1, 2)$,

$$\zeta_{p,\mu,\lambda}(s) := B_{p,\lambda} \cdot \{(\theta(p, \mu, \lambda) - s)^{(2-p)/(1-p)} - \theta(p, \mu, \lambda)^{(2-p)/(1-p)}\} \tag{2.5}$$

with $s < \theta(p, \mu, \lambda)$, whereas

$$\zeta_{2,\mu,\lambda}(s) := -\lambda \cdot \log\{1 - s/\theta(2, \mu, \lambda)\}. \tag{2.6}$$

Here, $s < \theta(2, \mu, \lambda)$. For $p \in (2, +\infty)$ and $s \leq \theta(p, \mu, \lambda)$, define

$$\zeta_{p,\mu,\lambda}(s) := B_{p,\lambda} \cdot \{\theta(p, \mu, \lambda)^{(2-p)/(1-p)} - (\theta(p, \mu, \lambda) - s)^{(2-p)/(1-p)}\}. \tag{2.7}$$

The following formula is valid for every member of PVF:

$$\mathbf{Var}(Tw_p(\mu, \lambda)) = \lambda^{-1} \cdot \mathbf{E}(Tw_p(\mu, \lambda))^p = \lambda^{-1} \cdot \mu^p. \tag{2.8}$$

It is natural to interpret (2.8) as the *variance-to-mean relation of the power type*, since the variance is propotional to the mean raised to the power p. The coefficient λ is frequently termed the *scaling parameter*. The formula (2.8) justifies applying the name *the power-variance family* to this class. Set

$$\phi_p := \lambda \cdot \mu^{2-p}. \tag{2.9}$$

By (2.8), expression (2.9) represents the *reciprocal of the squared coefficient of variation* of r.v. $Tw_p(\mu, \lambda)$. The quantity ϕ_p is rather important, and it makes sense to call it the *shape parameter*. A justification of such terminology can be found in Vinogradov [23, p. 1010]. See also Lemma 2.2.ii, Theorem 3.6, formula (3.3) and Corollary 4.3.

The following assertion is due to Jørgensen [10, form. (4.7)] and Vinogradov [23, form. (1.1)].

Lemma 2.2. *Suppose that p, μ and λ attain their arbitrary fixed admissible values. Fix an arbitrary $w \in \mathbf{R}_+^1$. Then (i) r.v.*

$$\mathcal{T} := w \cdot Tw_p(\mu, \lambda) \stackrel{\mathrm{d}}{=} Tw_p(w \cdot \mu, w^{p-2} \cdot \lambda). \tag{2.10}$$

(ii) The shape parameter remains invariant with respect to multiplication by a positive constant:

$$\phi_p(\mathcal{T}) := \lambda_{\mathcal{T}} \cdot \mu_{\mathcal{T}}^{2-p} \equiv \phi_p. \tag{2.11}$$

In addition to the class of normal distributions $\{Tw_0(\mu, \lambda), \mu \in \Omega_0, \lambda \in \Lambda\}$, PVF contains two other subclasses of (extreme) *stable* distributions. The latter subclasses are characterized by the domains of parameters $\{p \in (-\infty, 0), \mu = 0, \lambda \in \Lambda\}$ and $\{p \in (2, \infty), \mu = \infty, \lambda \in \Lambda\}$. To describe them, we introduce the *index of stability*

$$\alpha \, (= \alpha(p)) := (2-p)/(1-p) \in (-\infty, 2]. \tag{2.12}$$

Given $\alpha \in (0, 1)$, consider a specific positive stable r.v. $\mathcal{Y}_\alpha$ characterized by its Laplace transform $\mathbf{E} \exp\{-v \cdot \mathcal{Y}_\alpha\} := \exp\{-v^\alpha\}$. Here, $v \in [0, \infty)$. Similar, given $\alpha \in (1, 2]$, we introduce a particular extreme stable r.v. $\mathcal{X}_\alpha$ with skewness parameter $\beta = -1$ by virtue of its moment-generating function $\mathbf{E} \exp\{v \cdot \mathcal{X}_\alpha\} := \exp\{v^\alpha\}$. Here, $v \in [0, \infty)$. Then it can be shown that $\forall p \in (-\infty, 0]$,

$$Tw_p(0, \lambda) \stackrel{\mathrm{d}}{=} B_{p,\lambda}^{(1-p)/(2-p)} \cdot \mathcal{X}_{\alpha(p)}, \tag{2.13}$$

where $B_{p,\lambda}$ and $\alpha(p)$ are defined by formulas (2.2) and (2.12), respectively. In addition, one ascertains that $\forall$ fixed $p \in (2, \infty)$,

$$Tw_p(\infty, \lambda) \stackrel{\mathrm{d}}{=} B_{p,\lambda}^{(1-p)/(2-p)} \cdot \mathcal{Y}_{\alpha(p)}. \tag{2.14}$$

All the members of PVF are infinitely divisible (cf., e.g., Vinogradov [23, Prop. 1.2; errata and ref.'s]). Moreover, the law of each r.v. $Tw_p(\mu, \lambda)$ with $p \in \Delta \setminus \{0\}$ has *no Gaussian component*. In order to characterize their Lévy measures in the case when $p \in \Delta \setminus \partial\Delta$, it is convenient to introduce the following class of functions:

$$\upsilon_{p,\mu,\lambda}(x) = \frac{|1-p|^{1/(1-p)}}{|\Gamma(1/(p-1)|} \cdot \lambda^{1/(p-1)} \cdot |x|^{-(1+(2-p)/(1-p))} \cdot e^{-\theta(p,\mu,\lambda) \cdot |x|}. \tag{2.15}$$

Here, $x \in \mathbf{R}^1 \setminus \{0\}$, $\lambda \in \Lambda$, and $\theta(p, \mu, \lambda)$ is given by (2.1). For $p \in (-\infty, 0)$, we use the *analytic continuation* of function $\Gamma(1/(p-1))$.

All r.v.'s $Tw_p(\mu, \lambda)$ with $p < 0$ are *spectrally negative*. The Lévy measure of r.v. $Tw_p(\mu, \lambda)$ possesses the density with respect to Lebesgue measure such that $\forall\, x \in (-\infty, 0)$, it is equal to $\upsilon_{p,\mu,\lambda}(x)$. All r.v.'s $Tw_p(\mu, \lambda)$ with $p \geq 1$ are *spectrally positive*. $\forall\, p > 1$, the density of Lévy measure of $Tw_p(\mu, \lambda)$ with respect to Lebesgue measure is such that $\forall\, x \in (0, \infty)$, it coincides with $\upsilon_{p,\mu,\lambda}(x)$ (see (2.15)).

Let $\gamma(\delta, x)$ and $\Gamma(\delta, x)$ denote the *lower* and *upper incomplete gamma functions*, respectively. Then $\forall p \in (1, \infty)$, the first characteristic of the law of $Tw_p(\mu, \lambda)$ equals

$$b_{p,\mu,\lambda} = \mu \cdot \gamma(1/(p-1), \theta(p, \mu, \lambda))/\Gamma(1/(p-1)). \tag{2.16}$$

Similar, $\forall\, p \in (-\infty, 0)$, the first characteristic of the law of $Tw_p(\mu, \lambda)$ equals

$$b_{p,\mu,\lambda} = \mu \cdot \Gamma(1/(p-1), \theta(p, \mu, \lambda))/|\Gamma(1/(p-1))|.$$

For $p = 1$, the first characteristic of the law of $Tw_1(\mu, \lambda)$ is as follows:

$$b_{1,\mu,\lambda} = \begin{cases} \mu & \text{if } \lambda \geq 1; \\ 0 & \text{otherwise.} \end{cases} \tag{2.17}$$

Next, fix $p \in \Delta \setminus \partial\Delta$, and let all λ_i's, $1 \leq i \leq n$, belong to Λ. Suppose that $\forall\, 1 \leq i \leq n$, $\mu_i \in \Omega_p \setminus \{0\}$. Fix arbitrary $w_i \in \mathbf{R}_+^1$, where $1 \leq i \leq n$. Assume that

$$w_1^{-1} \cdot \lambda_1 \cdot \mu_1^{1-p} = ... = w_n^{-1} \cdot \lambda_n \cdot \mu_n^{1-p}, \tag{2.18}$$

and set

$$\mu := \sum_{i=1}^{n} w_i \cdot \mu_i. \tag{2.19}$$

Let $\lambda \in \Lambda$. Then it can be easily shown that

$$\lambda \cdot \mu^{1-p} = w_1^{-1} \cdot \lambda_1 \cdot \mu_1^{1-p} \tag{2.20}$$

if and only if

$$\lambda = \left(\sum_{i=1}^{n} w_i^{(p-2)/(p-1)} \cdot \lambda_i^{1/(p-1)} \right)^{p-1}. \tag{2.21}$$

Below we will employ the nth partial sum of the sequence $\{\mathcal{U}_i,\ 1 \leq i \leq n\}$ of independent r.v.'s with weights. $\forall\, 1 \leq i \leq n$, denote $\widetilde{\mathcal{U}}_i := w_i \cdot \mathcal{U}_i$. Set

$$\mathcal{S}_n := \sum_{i=1}^{n} w_i \cdot \mathcal{U}_i = \sum_{i=1}^{n} \widetilde{\mathcal{U}}_i. \tag{2.22}$$

We will refer to each component $\mathcal{U}_i$ of decomposition (2.22) as a *proper factor* if its distribution function (or *d.f.*) is not concentrated at a singleton. Members of analogous decompositions whose d.f.'s *are* concentrated at a singleton are called the *trivial factors*.

The next result pertains to the subclass $\{Tw_0(\mu, \lambda), \mu \in \Omega_0, \lambda \in \Lambda\}$ of normal laws.

Proposition 2.3. *Fix arbitrary $w_i \in \mathbf{R}_+^1$, where $1 \leq i \leq n$.*

(i) Consider independent r.v.'s $\{\mathcal{U}_i,\ 1 \leq i \leq n\}$ such that $\mathcal{U}_i \stackrel{\mathrm{d}}{=} Tw_0(\mu_i, \lambda_i)$, where $\forall\, 1 \leq i \leq n$, μ_i and λ_i $(=: 1/\sigma_i^2)$ are certain constants which belong to Ω_0 and Λ,

respectively. Then r.v. $\mathcal{S}_n$ *defined by (2.22) is such that* $\mathcal{S}_n \overset{\mathrm{d}}{=} Tw_0(\mu, \lambda)$ *with the value of* μ *given by (2.19) and* $\lambda := 1/\sigma^2$. *Here,*

$$\sigma^2 \left(= \mathbf{Var}(\mathcal{S}_n) \right) = \sum_{i=1}^{n} w_i^2 \cdot \sigma_i^2. \tag{2.23}$$

In addition, suppose that

$$w_1^{-1} \cdot \mu_1/\sigma_1^2 = \ldots = w_n^{-1} \cdot \mu_n/\sigma_n^2. \tag{2.24}$$

Then $\mu/\sigma^2 = w_1^{-1} \cdot \mu_1/\sigma_1^2$, *and*

$$\mu^2/\sigma^2 = \sum_{i=1}^{n} \mu_i^2/\sigma_i^2. \tag{2.25}$$

(ii) Assume that r.v. $\mathcal{U} \overset{\mathrm{d}}{=} Tw_0(\mu, \lambda)$, *where* $\mu \in \Omega_0 = \mathbf{R}^1$ *and* $\lambda := \sigma^2 \in \Lambda$. *Consider independent r.v.'s* $\{\mathcal{U}_i, \ 1 \leq i \leq n\}$ *such that* $\mathcal{U} \overset{\mathrm{d}}{=} \sum_{i=1}^{n} \widetilde{\mathcal{U}}_i$ *(with* $\widetilde{\mathcal{U}}_i = w_i \cdot \mathcal{U}_i$*). Suppose that neither of r.v.'s* $\mathcal{U}_i$*'s is a trivial factor. Then*

(ii1) each $\mathcal{U}_i \overset{\mathrm{d}}{=} Tw_0(\mu_i, \lambda_i)$, *where* $\forall \ 1 \leq i \leq n$, μ_i *and* λ_i $(=: 1/\sigma_i^2)$ *are certain constants which belong to* Ω_0 *and* Λ, *respectively. The parameters* μ*'s and* σ*'s satisfy (2.19) and (2.23), respectively.*

(ii2) Suppose that all the conditions imposed in part (ii1) are met. In addition, assume that $\mu_1 \cdot \ldots \cdot \mu_n \neq 0$ *and equation (2.25) is fulfilled. Then*

$$\mu/\sigma^2 = w_1^{-1} \cdot \mu_1/\sigma_1^2 = \ldots = w_n^{-1} \cdot \mu_n/\sigma_n^2. \tag{2.26}$$

Proof. It is derived by combining Vinogradov [23, Prop. 2.1] with *Cramér's decomposition theorem* (cf., e.g., Cramér [6, Th. 19]). □

Proposition 2.3.ii stipulates the *converse* of Merton's theorem, which is given as Theorem 2.5.ii (Recall that Merton's Theorem 1.1 is presented in the Introduction and related to (1.7)–(1.14).) In addition, part (i) of Theorem 2.5 below, which refines Theorem 1.1, employs the following generalization of the weights given by (1.7) and (1.13).

Definition 2.4. Let $\kappa \in \mathbf{R}_+^1$ be a constant. $\forall \ 1 \leq i \leq k$, set

$$\widetilde{w}_i^{(\kappa)} := \kappa \cdot \widetilde{w}_i = \kappa \cdot (m_i - r)/\sigma_i^2, \tag{2.27}$$

where $\widetilde{w}_i$'s are given by (1.13). It is appropriate to term the coefficients $\widetilde{w}_i^{(\kappa)}$'s the *scalar multiple of Merton's weights*.

The following assertion employs the quantities SR_i and SR_{port}, which are defined by (1.1)–(1.2), respectively. In the context of the geometric Brownian motion, they can be written down as follows: $SR_i = (m_i - r)/\sigma_i$, and $SR_{port} = (m - r)/\sigma$. Here, m and σ are interpreted as the growth rate and the volatility of portfolio, respectively.

Theorem 2.5. *Let* $\{\mathcal{S}^{(W)}(t), \ t \geq 0\}$ *denote the cumulative price process of the self-financing portfolio of securities which is described by (1.10).*
(i) Suppose that $\kappa \in \mathbf{R}_+^1$ *is a constant. Then the scalar multiple of Merton's weights*

$w_i := \widetilde{w}_i^{(\kappa)}$, *which are given by (2.27), imply the validity of the Pythagorean theorem for Sharpe portfolio performance measure:*

$$SR_{port}^2 = \sum_{i=1}^{k} SR_i^2. \tag{2.28}$$

(ii) Given an arbitrary admissible allocation of weights $W = (w_0, w_1, ..., w_k)$, consider the cumulative price process $\mathcal{S}^{(W)}(t)$. It satisfies the leftmost equation in (1.12) constituting a geometric Brownian motion. Suppose that its parameters $m \in \mathbf{R}^1$ and $\sigma \in \mathbf{R}_+^1$ are known, $m > r$, and (2.28) holds. Then the values $(w_1, ..., w_k)$ must coincide with the specific scalar multiple of Merton's weights, such that

$$\kappa = \kappa(m, r, \sigma) := \sigma^2/(m - r). \tag{2.29}$$

Proof. (i) It is easily derived by applying Proposition 2.3.i.

(ii) The proof is obtained from Proposition 2.3.ii by setting $n = k$, $\mu = m - r$ and $\mu_i = m_i - r$. Here, $1 \leq i \leq k$. The assertion then follows from the fact that under these conditions, (2.25) implies (2.26). $\qquad \square$

Remark 2.6. **(i)** Due to properties of lognormal law, choosing the scalar multiple of Merton's weights given by (2.27) and (2.29) stipulates the next expected excess growth rate:

$$g_{\kappa(m,r,\sigma)}(m_1, ..., m_k, r, \sigma_1, ..., \sigma_k) - r$$
$$= (2 \cdot \kappa(m, r, \sigma) - \kappa(m, r, \sigma)^2) \cdot (g(m_1, ..., m_k, r, \sigma_1, ..., \sigma_k) - r) \tag{2.30}$$
$$= (\kappa(m, r, \sigma) - \kappa(m, r, \sigma)^2/2) \cdot SR_{port}^2$$

(compare to formulas (1.14) and (2.28)).

(ii) If $m - r = \sigma^2$ then Theorem 2.5.ii implies that the additivity of the expected excess growth rate of the portfolio necessitates Merton's allocation of weights (1.13).

3. Properties of Hougaard Processes

This section is devoted to a subfamily of Lévy processes which are constructed starting from the members of PVF. Hereinafter, we will refer to representatives of this subfamily as *Hougaard processes*. The above arguments on infinite divisibility of PVF justify that the next subfamily of Lévy processes is well defined. The class introduced via Definition 3.1 below will admit a natural decomposition into three *non-overlapping* subclasses. Namely, representatives of the distinct subclasses possess different path properties. At the same time, we demonstrate that certain properties are common for *all* Hougaard processes.

The following definition is an extension of Lee and Whitmore [13, Def. 2.1].

Definition 3.1. Hereinafter, we refer to an $\mathbf{R}^1$-valued Lévy process $\{X_{p,\mu,\lambda}(t), \ t \geq 0\}$ such that $X_{p,\mu,\lambda}(0) \stackrel{\text{a.s.}}{=} 0$ and

$$X_{p,\mu,\lambda}(1) \stackrel{\text{d}}{=} Tw_p(\mu, \lambda) \tag{3.1}$$

as the *Hougaard process* indexed by parameters p, μ and λ which take on their arbitrary fixed admissible values.

Some facts concerning the marginals of Hougaard processes are easily derived from the properties of the corresponding members of PVF given in the previous section.

Proposition 3.2. *(i)* $\forall$ *arbitrary fixed admissible values of parameters* p, μ *and* λ*, and* $\forall$ *fixed* $t \in \mathbf{R}_+^1$*, r.v.*

$$X_{p,\mu,\lambda}(t) \overset{\mathrm{d}}{=} Tw_p(\mu \cdot t, \lambda \cdot t^{p-1}). \tag{3.2}$$

(ii) The shape parameter $\phi_p^{(t)}$ *of r.v.* $X_{p,\mu,\lambda}(t)$ *is as follows:*

$$\phi_p^{(t)} := \lambda_t \cdot \mu_t^{2-p} = \lambda \cdot \mu^{2-p} \cdot t = \phi_p \cdot t. \tag{3.3}$$

Proof. (i) It is obtained by combining scaling relationship (2.10) with formula (3.1) and Jørgensen [10, form. (4.34)].
(ii) It follows from the combination of formulas (2.9) and (3.2). $\qquad\qquad\square$

By (3.3), the shape parameter of $X_{p,\mu,\lambda}(t)$ exhibits a *linear growth* with slope ϕ_p.

Let us describe some properties of the family of Hougaard processes. To this end, we will decompose it into non-overlapping classes, which possess distinct path properties. To clarify this, we review the following classification of Lévy processes which is necessitated by qualitative differences in the almost-sure behavior of their trajectories (see Sato [18, p. 6] for more detail). Given Lévy measure ν, we introduce the quantity

$$\mathcal{I}_\nu := \int_{|x| \leq 1} |x| \cdot \nu(dx).$$

Definition 3.3. Consider a Lévy process $\{\mathcal{R}_t, t \geq 0\}$, which corresponds to the generating triplet (b, a, ν). Then it is said to be of
(i) type **A** if $a = 0$ and $\nu(\mathbf{R}^1) < \infty$;
(ii) type **B** if $a = 0$, $\nu(\mathbf{R}^1) = \infty$, and $\mathcal{I}_\nu < \infty$;
(iii) type **C** if $a \neq 0$ or $\mathcal{I}_\nu = \infty$.

Proposition 3.4. *(i) Each Hougaard process with* $p \in (-\infty, 0]$ *is of type* **C***.*
(ii) $\forall p \in [1, 2)$*, Hougaard process* $X_{p,\mu,\lambda}(t)$ *is of type* **A***.*
(iii) Every Hougaard process characterized by $p \in [2, \infty)$ *is of type* **B***.*

Proof. It employs the above arguments combined with (2.15) and variants of Lévy formula given in Vinogradov [23, pp. 1013–1015]. $\qquad\qquad\square$

By (2.3), the class (3.1) includes the *scaled Brownian motion with a drift*

$$X_{0,\mu,\lambda}(t) := \mu \cdot t + \lambda^{-1/2} \cdot \mathcal{B}_t \tag{3.4}$$

that starts from the origin. The process $\{X_{1,\mu,1}(t), \ t \geq 0\}$ is the time-homogeneous Poisson process with intensity μ. A combination of this observation with (2.4) and (3.1) implies that $\forall \lambda \in \Lambda$, Hougaard process $\{X_{1,\mu,\lambda}(t), \ t \geq 0\}$ is a scaled Poisson process with intensity $\mu\lambda$ and the common magnitude of jumps equal to $1/\lambda$. In addition, $\{X_{2,\mu,\lambda}(t), \ t \geq 0\}$ is a specific gamma process. Of interest is the subclass of compound *Poisson-gamma* Hougaard processes. They correspond to $p \in (1, 2)$. A popular member of this subclass is obtained by setting $p = 3/2$ (see Vinogradov [25, p. 299]).

The Hougaard processes with the values of $p < 0$ and $p > 2$ are derived by an exponential tilting of extreme stable processes with the index of stability $\alpha(p) \in (1, 2)$ given by (2.12) and skewness parameter $\beta = -1$, and of positive stable processes with the index of stability $\alpha(p) \in (0, 1)$ given by (2.12) and skewness parameter $\beta = 1$, respectively. Thus, (2.12) implies that $\{X_{3,\mu,\lambda}(t), \ t \geq 0\}$ constitutes an *inverse Gaussian* (Lévy) process (compare to Chhikara and Folks [4, Sec. 11.3]). See also Vinogradov [22], [24].

The differences between two independent Hougaard processes with (not necessarily identical) values of p's belonging to $(-\infty, 0) \cup [2, \infty)$ are relevant for stochastic models for stock price movements. Thus, they emerged in Carr et al. [3]; Cont and Tankov [5, Sec. 4.5] under the names *CGMY* process and *tempered stable* process, respectively.

We regard the following result as the *local law of the iterated logarithm* (or *LLIL*).

Theorem 3.5. *Consider a subclass of Hougaard processes* $\{X_{p,\mu,\lambda}(t),\ t \geq 0\}$ *characterized by* $p \in (-\infty, 0] \cup (2, \infty)$. *Then*
(i) $\forall$ *fixed* $p \in (-\infty, 0]$ *and* $\forall$ *arbitrary fixed admissible values of parameters* μ *and* λ,

$$\mathbf{P}\left\{\limsup_{t \to 0} \frac{X_{p,\mu,\lambda}(t)}{t^{(1-p)/(2-p)}(\log\log 1/t)^{1/(2-p)}} = \left(\frac{(1-p)(2-p)}{\lambda}\right)^{\frac{1}{2-p}}\right\} = 1. \quad (3.5)$$

(ii) $\forall$ *fixed* $p \in (2, \infty)$ *and* $\forall$ *arbitrary fixed admissible values of parameters* μ *and* λ,

$$\mathbf{P}\left\{\liminf_{t \to 0} \frac{X_{p,\mu,\lambda}(t)}{t^{(1-p)/(2-p)}(\log\log 1/t)^{1/(2-p)}} = \left(\frac{(1-p)(2-p)}{\lambda}\right)^{\frac{1}{2-p}}\right\} = 1. \quad (3.6)$$

Proof. The local properties do not depend on a specific value of the exponential tilting parameter. Hence, set $\mu := 0$ when $p \in (-\infty, 0]$, or $\mu := \infty$ when $p \in (2, \infty)$.
(i) The rest follows from Sato [18, Th. 2.2] or Bertoin [2, Th. VIII.5.ii].
(ii) The remainder of proof relies on Sato [18, Th. 2.3] or Bertoin [2, Th. VIII.6.ii]. $\square$

Note that for $p \in (-\infty, 0) \cup (2, \infty)$, the law of $Tw_p(\mu, \lambda)$ is absolutely continuous. Its probability density $f_{p,\mu,\lambda}(x)$ is expressed in terms of the densities $f_{p,0,\lambda}(x)$ or $f_{p,\infty,\lambda}(x)$ of extreme stable r.v.'s $Tw_p(0, \lambda)$ or $Tw_p(\infty, \lambda)$, respectively, by the formulas given in Vinogradov [23, form. (1.8)–(1.9)]. The latter r.v.'s are defined by (2.13)–(2.14). We are interested in *non-Cramér-type* large deviations for Hougaard processes as $t \to \infty$.

Theorem 3.6. *Fix a member of the subclass of Hougaard processes* $\{X_{p,\mu,\lambda}(t),\ t \geq 0\}$ *characterized by* $p \in (-\infty, 0) \cup (2, \infty)$, $\mu \in \mathbf{R}_+^1$ *and* $\lambda \in \Lambda$. *Suppose that* $t \to +\infty$, *and* y_t *is such that* $y_t/t^{(1-p)/(2-p)} \to -\infty$ *if* $p \in (-\infty, 0)$ *or* $y_t/t^{(1-p)/(2-p)} \to +\infty$ *if* $p \in (2, \infty)$. *Then (i)* $\forall$ *fixed* $p \in (-\infty, 0)$,

$$\mathbf{P}\{X_{p,\mu,\lambda}(t) < y_t\} \sim t \cdot \mathbf{P}\{X_{p,\mu,\lambda}(1) < y_t\} \cdot \exp\{(t-1) \cdot \phi_p/(p-2)\}. \quad (3.7)$$

(ii) $\forall$ *fixed* $p \in (2, \infty)$,

$$\mathbf{P}\{X_{p,\mu,\lambda}(t) > y_t\} \sim t \cdot \mathbf{P}\{X_{p,\mu,\lambda}(1) > y_t\} \cdot \exp\{(t-1) \cdot \phi_p/(p-2)\}. \quad (3.8)$$

Proof. By (3.2), the density $\mathbf{q}_{p,\mu,\lambda,t}(\cdot)$ of r.v. $X_{p,\mu,\lambda}(t)$ is as follows:

$$\mathbf{q}_{p,\mu,\lambda,t}(x) \equiv f_{p,\mu t,\lambda t^{p-1}}(x). \quad (3.9)$$

For $p \in (-\infty, 0) \cup (2, \infty)$, denote

$$f_{p,0/\infty,\lambda}(x) \equiv \begin{cases} f_{p,0,\lambda}(x) & \text{if } p < 0; \\ f_{p,\infty,\lambda}(x) & \text{if } p > 2. \end{cases} \quad (3.10)$$

Similar to (3.10), we will utilize the notation involving $0/\infty$ for density $\mathbf{q}_{p,\mu,\lambda,t}(x)$. A combination of (3.9)–(3.10) with the scaling property of stable processes ascertains that

$$\mathbf{q}_{p,0/\infty,\lambda,t}(x) \equiv t^{-(1-p)/(2-p)} \cdot f_{p,0/\infty,\lambda}(x/t^{(1-p)/(2-p)}). \quad (3.11)$$

Next, combine (2.2) and (3.2) with the asymptotic expansions in negative powers of $|y|$ of (extreme) stable densities (which can be found in Zolotarev [26]) to yield the following.

Define ρ $(= \rho(p,\mu,\lambda)) := -sign\ (p) \cdot \theta(p,\mu,\lambda)$ and fix $M \in \mathbf{N}$. Then $\forall$ fixed $t \in \mathbf{R}_+^1$, and either $\forall$ *fixed* $p \in (-\infty, 0)$ and as $y_t \to -\infty$, or $\forall$ *fixed* $p \in (2, \infty)$ and as $y_t \to +\infty$,

$$
\mathbf{q}_{p,\mu,\lambda,t}(y_t) = e^{t \cdot \phi_p / (p-2)} \frac{e^{\rho y_t}}{y_t} \cdot \sum_{n=1}^{M} \frac{(-1)^n t^n \left(B_{p,\lambda} |y_t|^{-(2-p)/(1-p)} \right)^n}{n! \cdot \Gamma\left(\frac{2-p}{p-1} \cdot n \right)} \tag{3.12}
$$
$$
+ o\left(t^M e^{t \cdot \phi_p / (p-2)} e^{\rho \cdot y_t} |y_t|^{-(1+M \cdot (2-p)/(1-p))} \right).
$$

The same scaling argument implies that in the case when $t \to +\infty$, representation (3.12) remains valid in the ranges of deviations $y_t / t^{(1-p)/(2-p)} \to -\infty$ if $p \in (-\infty, 0)$ or $y_t / t^{(1-p)/(2-p)} \to +\infty$ if $p \in (2, \infty)$.

It is straightforward to check that for $t = 1$ and $\rho = 0$, the principal first term of expansion (3.12) coincides with Lévy density of the law of $Tw_p(\mu, \lambda)$ given by (2.15).

It is well known that in the '*non-tilted*' case of extreme stable processes per se (whose marginal densities are given by (3.11)), the probabilities of large deviations occur mainly due to a *finite number of big jumps* (cf., e.g., Godovan'chuk [8, Ex. 3]). Hence, assume that $\mu \in \mathbf{R}_+^1$. Fix $\{p \in (-\infty, 0), \mu \in \mathbf{R}_+^1, \lambda \in \Lambda\}$. In view of Vinogradov [20, Ch. 5], a *negative* value of p stipulates the existence of a critical point (hereinafter denoted by u_0) in the formation of large deviations. The constant u_0 is given by the first derivative of c.g.f. $\zeta_{p,\mu,\lambda}(s)$ at the threshold of its domain. By (2.3), $u_0 \equiv 0 \ \forall p < 0$. For such p, one also has to determine the value of the Legendre transform $\mathcal{H}_{p,\mu,\lambda}(u)$ of $\zeta_{p,\mu,\lambda}(s)$ at 0 (see Vinogradov [20, p. 24]). By (2.3), $\mathcal{H}_{p,\mu,\lambda}(0) = \phi_p / (2 - p)$ (compare to (3.7)).

The rest is obtained by combining (3.11)–(3.12) with Laplace's method and follows along the same lines as those of Vinogradov [20, Sec. 5.1].

The case when $p > 2$ is considered by following along the same lines. $\qquad \square$

Remark 3.7. **(i)** For $p \in (-\infty, 0)$ and $y_t \in [\epsilon \cdot t, (\mu - \epsilon) \cdot t]$, or $p \in (2, \infty)$ and $y_t \sim Const \cdot t$, the exact asymptotics of the probabilities of the corresponding large deviations are given by Cramér's formula (compare to Vinogradov [20, form. (5.1.12)]).
(ii) For $p \in (-\infty, 0)$, the critical point $u_0 \equiv 0$ serves as a threshold between two different mechanisms of formation of the probabilities of large deviations (compare to Vinogradov [20, Sec.'s 5.1, 5.3], [21], [22, p. 154], [24, pp. 104–105]).
(iii) The exact asymptotics of the corresponding tails of d.f. of r.v. $X_{p,\mu,\lambda}(1)$ (which emerge in formulas (3.7)–(3.8)) are obtained from (3.11)–(3.12) by an application of Laplace's method (compare to Vinogradov [20, pp. 171–175]).

4. Property of a Class of Geometric Lévy Processes Related to Merton's Weights

Here, we first illustrate a general result on Dolean-Meyer exponentials (see Lemma 4.1). Subsequently, we will apply it to a problem of financial mathematics, which generalizes Theorem 2.5 to the discontinuous setting (see Theorem 4.2 and Corollary 4.3).

It turns out that for a portfolio of securities whose movements are *independent* of each other and satisfy SDE's (4.1), Merton-type allocation of weights leads to the same structure of the cumulative price process as those of individual assets (see Theorem 4.2). However, Merton-type weights do not maximize the expected logarithmic utility in the discontinuous setting. In the case when the logarithmic returns on equities are driven by discontinuous Lévy processes, this problem was solved by Kallsen [11]. Note that

Kallsen's weights differ from those by Merton. It is plausible that under some constraints, Merton's selection of weights can be close to optimal in the discontinuous case.

In this section, suppose that there is one risk-free bond whose price process is non-random and given by (1.3). In addition, there are k equities satisfying the next SDE's:

$$dS^{(i)}_{p,m_i,\lambda_i,r}(t)/S^{(i)}_{p,m_i,\lambda_i,r}(t-) = r \cdot dt + dX^{(i)}_{p,m_i-r,\lambda_i}(t), \tag{4.1}$$

respectively (compare to (1.15)). Here, $1 \le i \le k$, all m_i's $> r$, all λ_i's $\in \Lambda$, and all the Hougaard processes $X^{(i)}_{p,m_i-r,\lambda_i}(\cdot)$ (described in Definition 3.1) are assumed to be independent. Also, it is natural to suppose that the risk-free rate r is non-negative. However, it is for the purposes of Lemma 4.1 and the proof of Corollary 4.3 that for now, we drop this constraint assuming that constant $r \in \mathbf{R}^1$.

By Cont and Tankov [5, Prop. 8.21], $\forall p \in \Delta \; \exists$ a unique solution to (4.1). Combine (1.16) with the fact that $\forall \, p < 0$, Hougaard processes are *spectrally negative* with Lévy density given by (2.15). This implies that $\forall \, p < 0$, the solution to (4.1) may be negative. (By (2.15), the downward jumps are *unbounded* from below.) Hence, the solution to (4.1) may constitute the price process for an equity only for $p \in \{0\} \cup [1, \infty)$. The case $p = 0$, corresponds to the geometric Brownian motion. Hence, we may assume that $p \in [1, \infty)$. The solution $S_{p,\mu,\lambda,r}(t)$ to (4.1) can be represented in terms of the ordinary exponential (4.2) of a certain Lévy process, which is denoted by $\mathcal{L}_{p,\mu,\lambda,r}(t)$ (compare to (1.17)). The next assertion contains the representations for members of the generating triplet of the latter process. Recall that $\upsilon_{p,\mu,\lambda}(\cdot)$ is defined by (2.15).

Lemma 4.1. *Fix* $r \in \mathbf{R}^1$, $p \in [1, \infty)$, $\mu \in \mathbf{R}^1_+$ *and* $\lambda \in \Lambda$. *Then the Lévy process* $\mathcal{L}_{p,\mu,\lambda,r}(t)$ *which satisfies*

$$S_{p,\mu,\lambda,r}(t) := \mathcal{SE}\{rt + X_{p,\mu,\lambda}(t)\} = \exp\{\mathcal{L}_{p,\mu,\lambda,r}(t)\} \tag{4.2}$$

has no Gaussian component. In addition,
(i1) If $p = 1$ *then the Lévy measure* $\nu_{\mathcal{L}_{1,\mu,\lambda,r}}$ *of process* $\mathcal{L}_{1,\mu,\lambda,r}(t)$ *is concentrated at the singleton* $\{\log(1 + 1/\lambda)\}$ *such that*

$$\nu_{\mathcal{L}_{1,\mu,\lambda,r}}(\{\log(1 + 1/\lambda)\}) = \mu \cdot \lambda. \tag{4.3}$$

The first characteristic $b_{\mathcal{L}_{1,\mu,\lambda,r}}$ *of the generating triplet of process* $\mathcal{L}_{1,\mu,\lambda,r}(t)$ *equals*

$$b_{\mathcal{L}_{1,\mu,\lambda,r}} = \begin{cases} r & \text{if } \lambda < 1/e; \\ r + \log(1 + 1/\lambda) & \text{if } 1/e \le \lambda < 1; \\ r + \mu + \log(1 + 1/\lambda) - 1/\lambda & \text{if } \lambda \ge 1. \end{cases} \tag{4.4}$$

(i2) If $p > 1$ *then the Lévy measure* $\nu_{\mathcal{L}_{p,\mu,\lambda,r}}$ *is concentrated on* $\mathbf{R}^1_+$, *where it possesses the following density:*

$$\pi_{\mathcal{L}_{p,\mu,\lambda,r}}(x) = \frac{(p-1)^{1/(1-p)}}{\Gamma(1/(p-1))} \cdot \lambda^{1/(p-1)} \cdot \exp\{\theta(p,\mu,\lambda)\}$$
$$\cdot (e^x - 1)^{-(1+(2-p)/(1-p))} \cdot \exp\{x - \theta(p,\mu,\lambda) \cdot e^x\}. \tag{4.5}$$

The first characteristic $b_{\mathcal{L}_{p,\mu,\lambda,r}}$ *of the generating triplet of process* $\mathcal{L}_{p,\mu,\lambda,r}(t)$ *equals*

$$b_{\mathcal{L}_{p,\mu,\lambda,r}} = r + \mu \cdot \gamma(1/(p-1), \theta(p,\mu,\lambda))/\Gamma(1/(p-1))$$
$$+ \int_0^1 (\log(1+x) - x) \cdot \upsilon_{p,\mu,\lambda}(x) \cdot dx + \int_1^e \log(1+x) \cdot \upsilon_{p,\mu,\lambda}(x) \cdot dx. \tag{4.6}$$

Proof. The derivation of (4.3)–(4.6) starting from (4.2) is straightforward. To this end, combinine Cont and Tankov [5, Prop. 8.22] with (2.16)–(2.17) and (4.1). □

The rate $g_p(\mu, r, \lambda)$ of exponential growth (or decay) of $S_{p,\mu,\lambda,r}(t)$ is obtained from the fact that by Cont and Tankov [5, Prop. 8.23], $\exp\{\mathcal{L}_{p,\mu,\lambda,-\mu}(t)\}$ constitutes an exponential martingale:

$$\mathbf{E}S_{p,\mu,\lambda,r}(t) = \mathbf{E}\exp\{\mathcal{L}_{p,\mu,\lambda,r}(t)\} = e^{(\mu+r)\cdot t}. \tag{4.7}$$

Next, consider Hougaard process $X_{p,m_i-r,\lambda_i}(t)$ that emerges in SDE (4.1). By analogy to Definition 2.4, $\forall$ fixed $\kappa \in \mathbf{R}^1_+$ and $\forall\, 1 \leq i \leq k$, define Merton-type weights

$$\widehat{w}_i^{(\kappa)} := \kappa \cdot \lambda_i \cdot (m_i - r)^{1-p} \equiv \kappa \cdot \mathbf{E}X_{p,m_i-r,\lambda_i}^{(i)}(t)/\mathbf{Var}X_{p,m_i-r,\lambda_i}^{(i)}(t). \tag{4.8}$$

Recall that $\forall$ fixed real $t \geq 0$, r.v.'s $\xi_0(t), \xi_1(t), ..., \xi_k(t)$ represent the amounts of shares of the corresponding securities which are held in the portfolio at time $t-$ (see Section 1).

The next assertion is not applicable to Hougaard processes which are constructed starting from *stable* members of PVF (The latter ones include subclass $\{Tw_0(\mu, \lambda), \mu \in \Omega_0, \lambda \in \Lambda\}$ as well as the subclasses characterized by (2.13)–(2.14).) Hence, the following result does not overlap with Theorem 2.5, but it is of the same spirit.

Theorem 4.2. *Given a constant risk-free rate $r \geq 0$ and the values of parameters $p \in [1, \infty)$, m_i's $> r$, λ_i's $\in \Lambda$, consider k equities whose chaotic movements are assumed to be independent and satisfy SDE's (4.1). Suppose that W is defined by (1.6). Let us introduce the cumulative price process*

$$S_p^{(W)}(t) := \xi_0(t) \cdot S^{(b)}(t) + \sum_{i=1}^{k} \xi_i(t) \cdot S_{p,m_i,\lambda_i,r}^{(i)}(t). \tag{4.9}$$

(i) Suppose that $\kappa \in \mathbf{R}^1_+$ is a constant. Then the scalar multiple of (constant) Merton's allocation of weights $w_i := \widehat{w}_i^{(\kappa)}$ (which are given by (4.8)) in the continuously traded self-financing portfolio (4.9) implies the validity of the following SDE for the resulting process $\{\widehat{S}_p^{(W)}(t),\ t \geq 0\}$:

$$d\widehat{S}_p^{(W)}(t)/\widehat{S}_p^{(W)}(t-) = r \cdot dt + dX_{p,m-r,\lambda}(t). \tag{4.10}$$

Here, the pair $\{m, \lambda\}$ denotes the unique solution to the next system of two equations:

$$m - r = \sum_{i=1}^{k} \widehat{w}_i^{(\kappa)} \cdot (m_i - r); \tag{4.11}$$

$$\lambda \cdot (m - r)^{2-p} = \sum_{i=1}^{k} \lambda_i \cdot (m_i - r)^{2-p}. \tag{4.12}$$

(ii) Given an arbitrary admissible allocation of weights $W = (w_0, w_1, ..., w_k)$, consider the cumulative price process $\widehat{S}_p^{(W)}(t)$. Suppose that this process satisfies SDE (4.10) with its own Hougaard process characterized by the same value of parameter p. Assume that parameters $m > r$ and $\lambda \in \Lambda$ of the Hougaard process that emerges in (4.10) are known. Then the values $(w_1, ..., w_k)$ must coincide with the specific scalar multiple of Merton's weights $\widehat{w}_i^{(\kappa)}$ given by (4.8), such that

$$\kappa = \kappa_p\ (= \kappa_p(m, r, \lambda))\ := (m - r)^{p-1}/\lambda. \tag{4.13}$$

Proof. It involves Lemma 2.2, formulas (2.18)–(2.21), and Vinogradov [23, Th. 2.1, Prop. 2.1 and Rmrk 2.1.i].

(i) Combine (2.10)–(2.11), Vinogradov [23, Prop. 2.1], the allocation of weights (4.8), Definition 3.1 and Proposition 3.2 to conclude that

$$
\begin{aligned}
d\widehat{S}_p^{(W)}(t) &= \big(1 - \sum_{i=1}^{k} \widehat{w}_i^{(\kappa)} \big) \cdot \widehat{S}_p^{(W)}(t-) \cdot r \cdot dt \\
&\quad + \sum_{i=1}^{k} \widehat{w}_i^{(\kappa)} \cdot \widehat{S}_p^{(W)}(t-) \cdot \big(r \cdot dt + dX_{p,m_i-r,\lambda_i}^{(i)}(t) \big) \\
&= \widehat{S}_p^{(W)}(t-) \cdot \Big(r \cdot dt + d \Big(\sum_{i=1}^{k} \widehat{w}_i^{(\kappa)} \cdot X_{p,m_i-r,\lambda_i}^{(i)}(t) \Big) \Big) \\
&\stackrel{\mathrm{d}}{=} \widehat{S}_p^{(W)}(t-) \cdot \big(r \cdot dt + dX_{p,m-r,\lambda}(t) \big).
\end{aligned}
\tag{4.14}
$$

Here, m and λ admit representations (4.11) and (4.12), respectively. This implies that for the Merton-type allocation of weights, the combined movement of the equities is governed by a geometric Lévy process, which is related to the same PVF.

(ii) By Vinogradov [23, Prop. 2.1.ii], the existence of the decomposition of Hougaard process $X_{p,m-r,\lambda}(t)$ into a linear combination of k independent Hougaard processes with the same p necessitates the selection of weights (4.8) with κ given by (4.13). Compare to the rightmost equation in (4.14). This implies the invariance of ratio (4.8), which is attained through the Merton-type selection of weights. The rest is trivial. $\qquad\square$

Let us stress that the validity of SDE (4.10) implies the additivity property for a class of geometric Lévy processes $\{S_{p,\mu_i,\lambda_i,0}^{(i)}(t),\ 1 \leq i \leq k\}$ given by (4.2). This property holds under the allocation of constant weights $\kappa \cdot \lambda_i \cdot \mu_i^{1-p}$.

The following corollary to Lemma 4.1 and Theorem 4.2 is an analogue of Remark 2.6.

Corollary 4.3. *Let all the conditions of Theorem 4.2.i be fulfilled. Then the expected excess growth rate $g_p^{(\kappa)}(m_1, ..., m_k, r, \lambda_1, ..., \lambda_k) - r$ of the portfolio $\widehat{S}_p^{(W)}(t)$ equals*

$$
\kappa \cdot SR_{port}^2 = \kappa \cdot (g_p^{(\kappa)}(m_1, ..., m_k, r, \lambda_1, ..., \lambda_k) - r) = \kappa \cdot \phi_p(X_{p,m-r,\lambda}(1))
$$

$$
= \kappa \cdot \lambda \cdot (m - r)^{2-p} = \kappa \cdot \sum_{i=1}^{k} \lambda_i \cdot (m_i - r)^{2-p}
\tag{4.15}
$$

$$
= \kappa \cdot \sum_{i=1}^{k} \phi_p(X_{p,m_i-r,\lambda_i}^{(i)}(1)) = \kappa \cdot \sum_{i=1}^{k} \big(g_p^{(\kappa)}(m_i, r, \lambda_i) - r \big) = \kappa \cdot \sum_{i=1}^{k} SR_i^2.
$$

Proof. By (4.7), $\log \mathbf{E} \exp\{\mathcal{L}_{p,m_i-r,\lambda_i,r}(1)\} = m_i$. The rest is derived by applying (2.11) and following the arguments of the proof of Lemma 4.1 and Theorem 4.2. $\qquad\square$

We interpret Corollary 4.3 as the *Pythagorean theorem* for *Sharpe measure*, since the independence of SR_i's implies their orthogonality. Also, compare (4.15) with (2.30).

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