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ABSOLUTE CONTINUITY OF LAWS FOR SEMILINEAR STOCHASTIC EQUATIONS WITH ADDITIVE NOISE

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ABSTRACT. We present the Girsanov theorem for a non linear Itô equation in an infinite dimensional Hilbert space with a non linearity of polynomial growth and an infinite dimensional additive noise. We assume a condition weaker than Novikov one, as done by Mikulevicius and Rozovskii in the study of more general stochastic PDE's. The equivalence of the laws of the linear equation and of the non linear equation implies results on weak solutions and on invariant measures for the given non linear equation. Two examples are presented: a stochastic Kuramoto–Sivashinsky equation and a stochastic hyperviscosity-regularized Navier–Stokes equation.

1. Introduction

The study of non linear equations requires some skill to deal successfully with the non linearity. As far as stochastic differential equations are concerned, a possible technique is given by the Girsanov transform. Indeed, a non linear stochastic Itô equation

$$du(t) + [\ Lu(t) + F(u(t))\ ]\, dt = Gdw(t), \quad t \in]0,T];\ u(0) = x \tag{1.1}$$

can be considered as a perturbation of the linear equation

$$dz(t) + Lz(t)\ dt = Gdw(t), \quad t \in]0,T];\ z(0) = x. \tag{1.2}$$

Most of the results available in the literature assume Novikov condition

$$\mathbb{E}\Big[\exp\big(\tfrac{1}{2}\int_0^T |G^{-1}F(z(t))|^2 dt\big)\Big] < \infty$$

in order to apply Girsanov theorem (see, e.g., [2], [10]). However, in [12] Mikulevicius and Rozovskii studied Girsanov transform for general stochastic PDE's, assuming the much weaker condition

$$\mathbb{P}\{\int_0^T |G^{-1}F(z(t))|^2 dt < \infty\} = 1.$$

Notice that this assumption is enough to define all the terms appearing in the density

$$\rho_T(z) := \exp\Big(- \int_0^T \langle G^{-1}F(z(s)), dw(s)\rangle - \tfrac{1}{2}\int_0^T |G^{-1}F(z(s))|^2 ds\Big)$$

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which comes in on the change of measure by Girsanov transform; indeed the stochastic integral is a locally square integrable martingale.

Since the setting of [12] is very general, we shall present it in the particular case of the stochastic equation (1.1), set in a Hilbert space, where w is an infinite dimensional Wiener process and G is independent of u. Equation (1.1) can be seen as the abstract formulation of a stochastic PDE. As examples, we shall consider a stochastic Kuramoto–Sivashinsky equation and a stochastic hyperviscosity-regularized Navier–Stokes equation (the modification with respect to the Navier–Stokes equation consists in substituting the Laplacian $-\Delta$ with $(-\Delta)^\alpha$).

We recall the importance of Girsanov theorem. First, from the existence of a weak solution for the linear equation (1.2) we get existence of a weak solution for the non linear equation (1.1). Moreover the law $\mathcal{L}^F$ of the process solving (1.1) is absolutely continuous with respect to the law $\mathcal{L}^0$ of the process solving (1.2) (we write $\mathcal{L}^F \prec \mathcal{L}^0$). It may be possible to prove the converse too, so to get the equivalence of $\mathcal{L}^F$ and $\mathcal{L}^0$ ($\mathcal{L}^F \sim \mathcal{L}^0$). We recall that if $\mathcal{L}^F \prec \mathcal{L}^0$, uniqueness for equation (1.2) implies uniqueness in law for equation (1.1). Moreover, if $\mathcal{L}^F \sim \mathcal{L}^0$, each property holding a.s. for the process z must also hold for the process u and vice versa. And given $\mathcal{L}^F \sim \mathcal{L}^0$ for the equations on any finite time interval $[0,T]$, we can deduce some information also on the asymptotic behaviour (for $T \to \infty$).

As to the structure of the paper, in Section 2 we formalize the analysis of Girsanov transform, in the setting of stochastic equations in a Hilbert space with an infinite dimensional additive noise and a non linearity F of polynomial growth; this is Theorem 2.1 and its consequences about the equivalence $\mathcal{L}^F \sim \mathcal{L}^0$ are presented in Theorem 2.3 and Corollary 2.4 . Then, in the other two sections these results are applied to a stochastic Kuramoto–Sivashinsky equation and to a stochastic hyperviscosity-regularized Navier–Stokes equation, respectively.

2. Girsanov Theorem

First, we define the operators and the Wiener process w appearing in equations (1.1)-(1.2).

Let H and E be two separable Hilbert spaces, with E continuously and densely embedded in H. We denote by $|\cdot|_H$ the norm in H and by ${}_H\langle\cdot,\cdot\rangle_H$ the scalar product in H; similarly in E.

We assume that L and G are linear operators in H and G is invertible; moreover, the operator $G^{-1}F : E \to H$ is measurable. When dealing with a Polish space, i.e. a complete separable metric space, the σ-algebra associated is the Borel σ-algebra.

Any probability space $(\Omega, \mathbb{F}, \mathbb{P})$ is assumed to be complete and the filtration $\{\mathbb{F}_t\}_{t\geq 0}$ right continuous. We denote by $\mathbb{E}$ the expectation with respect to the measure $\mathbb{P}$.

Given a stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$, we say that

$$w = \{w(t)\}_{t\geq 0}$$

is an H-cylindrical Wiener process with respect to $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ if, for any $h, h' \in H$, we have that ${}_H\langle w(\cdot), h\rangle_H$ is a continuous $\{\mathbb{F}_t\}$-martingale with $w(0) = 0$ $\mathbb{P}$-a.s. and $\mathbb{E}[{}_H\langle w(t), h'\rangle_H\, {}_H\langle w(s), h\rangle_H] = {}_H\langle h', h\rangle_H\ (t \wedge s)$ for any $s, t \geq 0$.

The operator G in front of w in equations (1.1)-(1.2) makes it a "coloured" Wiener process. Since G is invertible, Gw is an infinite dimensional Wiener process. Notice that $w(t)$ is not H-valued; but for each $h \in H$, ${}_H\langle w(t), h\rangle_H$ is well defined $\mathbb{P}$-a.s.

We begin with the result on the existence of the probability density appearing in the Girsanov transform and on the change of drift. This is based on [12] for the crucial part i), which shows that the probability density ρ_T is well defined assuming (2.1) instead of Novikov condition (see also a similar argument in [13], dealing with the easier case of one-dimensional processes). We give all the details of the proof, since we do not work in the general setting of [12]. Indeed, our presentation refers to the particular case of equation (1.1) where the noise is additive and defined by a Wiener process, and the solution of the linear equation (1.2) is a continuous process.

Theorem 2.1 (Girsanov theorem)**.** *Assume we are given an H-cylindrical Wiener process w defined on $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ and an $\{\mathbb{F}_t\}$-adapted process $z : \Omega \to C([0,T]; E)$ such that*

$$\mathbb{P}\{\textstyle\int_0^T |G^{-1}F(z(s))|_H^2 ds < \infty\} = 1. \tag{2.1}$$

Then
(i) the stochastic process

$$\rho_t(z) = \exp\Big(- \textstyle\int_0^t {}_H\langle G^{-1}F(z(s)), dw(s)\rangle_H - \frac{1}{2}\int_0^t |G^{-1}F(z(s))|_H^2 ds\Big), \ \ t \in [0,T],$$

is a positive $\{\mathbb{F}_t\}$-martingale; in particular, $\mathbb{E}[\rho_T(z)] = 1$.
(ii) the stochastic process

$$w^*(t) = w(t) + \int_0^t G^{-1}F(z(s)) \ ds \ , \quad t \in [0,T], \tag{2.2}$$

is an H-cylindrical Wiener process with respect to $\mathbb{P}^*$, where the probability measure $\mathbb{P}^*$ is defined on $(\Omega, \mathbb{F}_T)$ by

$$d\mathbb{P}^* = \rho_T(z) \ d\mathbb{P}.$$

Proof. (i) The stochastic integral in the exponent of $\rho_t(z)$ is well defined $\mathbb{P}$-a.s.; indeed, it is a locally square integrable martingale (see, e.g., [12] in the infinite dimensional setting and [10] in the finite dimensional setting). Therefore $\rho_t(z)$ is a positive and continuous process. This implies that

$$\int_0^T \rho_s(z)^2 |G^{-1}F(z(s))|_H^2 ds \leq (\sup_{s\in[0,T]} \rho_s(z)^2) \int_0^T |G^{-1}F(z(s))|_H^2 ds < \infty$$

$\mathbb{P}$-a.s.. By Itô calculus, we have

$$\rho_t(z) = 1 + \int_0^t \rho_s(z) \, {}_H\langle G^{-1}F(z(s)), dw(s)\rangle_H \ , \qquad t \in [0,T].$$

Then $\rho_\cdot(z)$ is a local $\{\mathbb{F}_t\}$-martingale. To show that it is indeed a martingale, we need to show that $\mathbb{E}[\rho_t(z)] = 1$ for all $t \in [0,T]$.

For each $N = 1, 2, \dots$, define the truncation function χ^N as follows:

$$\chi_t^N(z) = \begin{cases} 1 & \text{if } \int_0^t |G^{-1}F(z(s))|_H^2 ds \le N \\ 0 & \text{otherwise} \end{cases}$$

Notice that $\chi_\cdot^N(z)$ is a progressively measurable process and

$$\lim_{N\to\infty} \mathbb{P}\{\int_0^T |G^{-1}F(z(s))|_H^2 ds \le N\} = 1$$

by (2.1).

By the definition of χ_t^N we have

$$\int_0^T |G^{-1}\chi_s^N(z)F(z(s))|_H^2 ds \le N \qquad \mathbb{P}-a.s.;$$

therefore Novikov condition

$$\mathbb{E}\left[\exp\left(\tfrac{1}{2}\int_0^T |G^{-1}\chi_s^N(z)F(z(s))|_H^2 ds\right)\right] < \infty$$

is satisfied. This implies (see, e.g., Theorem 6.1 in [10] or Proposition 3.2 in [12]) that, for any $N = 1, 2, \dots$

$$\rho_t^N(z) = \exp\left(-\int_0^t \chi_s^N(z)\; {}_H\langle G^{-1}F(z(s)), dw(s)\rangle_H - \tfrac{1}{2}\int_0^t \chi_s^N(z)|G^{-1}F(z(s))|_H^2 ds\right), \qquad t \in [0,T]$$

is an $\{\mathbb{F}_t\}$-martingale and in particular $\mathbb{E}[\rho_T^N(z)] = 1$.

Let us prove that $\mathbb{E}[\rho_T(z)] = 1$. As in [12] (see the proof of Theorem 3.1), we write

$$\begin{aligned} 1 = \mathbb{E}[\rho_T^N(z)] &= \mathbb{E}[\chi_T^N(z)\rho_T^N(z)] + \mathbb{E}[(1-\chi_T^N(z))\rho_T^N(z)] \\ &= \mathbb{E}[\chi_T^N(z)\rho_T(z)] + P\{\chi_T^N(z) = 0\}. \end{aligned} \tag{2.3}$$

By monotone convergence, $\lim_{N\to\infty} \mathbb{E}[\chi_T^N(z)\rho_T(z)] = \mathbb{E}[\rho_T(z)]$. On the other hand, $\lim_{N\to\infty} P\{\chi_T^N(z) = 0\} = \lim_{N\to\infty} P\{\int_0^T |G^{-1}F(z(s))|_H^2 ds > N\} = 0$. Passing to the limit as $N \to \infty$ in (2.3), we conclude that $\mathbb{E}[\rho_T(z)] = 1$. In the same way we prove that $\mathbb{E}[\rho_t(z)] = 1$ for $t < T$.

(ii) This is Theorem 10.14 in [2]. □

Now, we apply Girsanov transform to study equation (1.1). We need to recall what is a weak solution.

Definition 2.2. We say that there exists a *weak solution* to equation (1.1) on the time interval $[0,T]$ if there exists a stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\ge 0}, \mathbb{P})$, an H-cylindrical Wiener process w and an $\{\mathbb{F}_t\}$-adapted process u defined in it such that

$$u(t) + \int_0^t [\; Lu(s) + F(u(s)) \;]\, ds = x + Gw(t) \qquad \mathbb{P}-a.s.$$

holds as an equality in some Hilbert space[1] for every $t \in [0,T]$.

We denote this solution by the triplet $\big((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathbb{P}),\ w,\ u\big)$. On the other hand, a *strong solution* $\big((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathbb{P}),\ w,\ u\big)$ will be a process u solving (1.1) on a (a priori) given stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathbb{P})$ with a given H-cylindrical Wiener process w.

Let $\big((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathbb{P}), w, z\big)$ be a weak solution of equation (1.2), fulfilling the assumptions of Theorem 2.1. This means that for any fixed $t \in [0,T]$

$$z(t) + \int_0^t Lz(s)\ ds = x + Gw(t) \qquad \mathbb{P} - a.s.$$

By (2.2) we get also that

$$z(t) + \int_0^t Lz(s)\ ds + \int_0^t F(z(s))\ ds = x + Gw^*(t) \qquad \mathbb{P}^* - a.s.$$

where w^* is the H-cylindrical Wiener process (with respect to $\mathbb{P}^*$) defined in (2.2). This equality holds $\mathbb{P}^*$-a.s., since we know that it holds $\mathbb{P}$-a.s. Therefore, equation (1.1) has a weak solution; this is $\big((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P}^*),\ w^*,\ z\big)$. The law of this solution of equation (1.1) is

$$\mathcal{L}^F(\Lambda) = \mathbb{P}^*\{z \in \Lambda\} \tag{2.4}$$

and $\mathcal{L}^F \prec \mathcal{L}^0$ as measures on the Borel subsets of $C([0,T];E)$ (the law of (1.2) is by definition $\mathcal{L}^0(\Lambda) = \mathbb{P}\{z \in \Lambda\}$). In fact, $\mathbb{P}^* \prec \mathbb{P}$ with Radon-Nykodim derivative $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \rho_T(z)$.

Also the converse is true. We summarize the result in the following theorem. Here we denote by $\sigma_T(z)$ the σ-algebra generated by $\{z(t)\}_{0\leq t\leq T}$.

Theorem 2.3. *Assume there exists a weak solution* $\big((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathbb{P}), w, z\big)$ *of equation* (1.2). *If*

$$\mathbb{P}\{z \in C([0,T];E)\} = 1 \tag{2.5}$$

and equation (2.1) is satisfied, then there exists a weak solution to equation (1.1)*; this solution is*

$$\big((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathbb{P}^*),\ w^*,\ z\big)$$

where $w^*(t)$ *and* $d\mathbb{P}^*$ *are given by*

$$w^*(t) = w(t) + \int_0^t G^{-1}F(z(s))\ ds$$

$$d\mathbb{P}^* = e^{-\int_0^T {}_H\langle G^{-1}F(z(s)), dw(s)\rangle_H - \frac{1}{2}\int_0^T |G^{-1}F(z(s))|_H^2 ds}\ d\mathbb{P}.$$

In particular, the laws are defined on the Borel subsets Λ *of* $C([0,T];E)$ *as*

$$\mathcal{L}^0(\Lambda) = \mathbb{P}\{z \in \Lambda\}\ , \qquad \mathcal{L}^F(\Lambda) = \mathbb{P}^*\{z \in \Lambda\}.$$

[1] More precisely, given an Hilbert space $\tilde{H} \supseteq H$ such that the embedding is continuous and dense, we consider its dual $\tilde{H}'$ ($\tilde{H}' \subseteq H' \simeq H \subseteq \tilde{H}$) . The equality holds in the Hilbert space $\tilde{H}$ if, for every $h \in \tilde{H}'$

$$\langle u(t), h\rangle + \int_0^t \langle Lu(s) + F(u(s)), h\rangle\ ds = \langle x, h\rangle + \langle Gw(t), h\rangle \qquad \mathbb{P} - a.s.$$

for every $t \in [0,T]$. Here $\langle\cdot,\cdot\rangle$ is the $\tilde{H}$-$\tilde{H}'$ duality pairing.

Moreover, $\mathcal{L}^F \sim \mathcal{L}^0$ and the Radon-Nykodim derivatives are

$$\frac{d\mathcal{L}^F}{d\mathcal{L}^0}(z) = \mathbb{E}[e^{-\int_0^T {}_H\langle G^{-1}F(z(s)),dw(s)\rangle_H - \frac{1}{2}\int_0^T |G^{-1}F(z(s))|_H^2 ds} | \sigma_T(z)] \qquad \mathbb{P}-a.s. \tag{2.6}$$

$$\frac{d\mathcal{L}^0}{d\mathcal{L}^F}(z) = \mathbb{E}^*[e^{+\int_0^T {}_H\langle G^{-1}F(z(s)),dw^*(s)\rangle_H - \frac{1}{2}\int_0^T |G^{-1}F(z(s))|_H^2 ds} | \sigma_T(z)] \qquad \mathbb{P}^*-a.s. \tag{2.7}$$

Finally, $\mathcal{L}^F$ is unique if and only if $\mathcal{L}^0$ is unique.

Proof. The first part on $\mathbb{P}^* \prec \mathbb{P}$ comes from Theorem 2.1. This implies that z satisfies conditions (2.5) and (2.1) also with respect to $\mathbb{P}^*$. Applying the first part of Theorem 2.1 but considering the probability measure $\mathbb{P}^*$, we get that $d\mathbb{P} = \rho_T^*(z)d\mathbb{P}^*$, with

$$\rho_T^*(z) = \exp\Big(+ \int_0^T {}_H\langle G^{-1}F(z(s)), dw^*(s)\rangle_H - \tfrac{1}{2}\int_0^T |G^{-1}F(z(s))|_H^2 ds\Big).$$

The sign plus in the first integral of the exponent comes in, because we start from equation (1.1) and see (1.2) as a perturbation of (1.1) by the term $-F(z)dt$. Of course we have $\rho_T(z)\rho_T^*(z) = 1$.

As far as the laws are concerned, $\mathcal{L}^F$ is defined as in (2.4). Since $\mathbb{P} \sim \mathbb{P}^*$, then $\mathcal{L}^0 \sim \mathcal{L}^F$. In particular

$$\int_\Omega \phi(z)d\mathbb{P}^* = \int_\Omega \phi(z)\rho_T(z)d\mathbb{P}$$

for every Borel bounded function $\phi : C([0,T];E) \to \mathbb{R}$. Denoting by $\frac{d\mathcal{L}^F}{d\mathcal{L}^0}(v)$ the Radon-Nykodim derivative of $\mathcal{L}^F$ with respect to $\mathcal{L}^0$ evaluated at $v \in C([0,T];E)$, we obtain (2.6); in the same way we prove (2.7).

Uniqueness is trivial, since $\mathcal{L}^F \sim \mathcal{L}^0$. $\square$

In the next sections, we shall present examples for which Theorem 2.3 holds. The linear equation will be easily analyzed; it will have a unique *strong* solution satisfying (2.5). Starting from this solution z, defined on any stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ with any H-cylindrical Wiener process w, we shall define the law $\mathcal{L}^F$ by means of (2.4).

Assumption (2.1) is satisfied if there exist two positive constants p and c such that

$$|G^{-1}F(v)|_H \leq c\,(1 + |v|_E^p) \qquad \forall v \in E.$$

Actually the interesting case is for $p > 1$, whereas the case $p \leq 1$ of at most linear growth of F is usually studied in the literature. In our examples, this estimate will hold for $p = 2$. Therefore, Girsanov theorem can be formulated also in a more convenient way for stochastic equations.

Corollary 2.4. *Assume that*

$$|G^{-1}F(v)|_H \leq c\,(1 + |v|_E^p) \qquad \forall v \in E \tag{2.8}$$

for some constants $p > 0$ and $c > 0$. If there exists a weak solution

$$\big((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathbb{P}),\ w,\ z\big)$$

of equation (1.2) *such that*

$$\mathbb{P}\{z \in C([0,T];E)\} = 1, \tag{2.9}$$

then condition (2.1) *is fulfilled and therefore Theorem 2.3 holds true.*

3. A Stochastic 1D Kuramoto–Sivashinsky Equation

We refer to [7] for the abstract setting, in which we studied a stochastic 1D Kuramoto–Sivashinsky equation written as

$$\begin{cases} du(t) + [\nu A^2 u(t) - Au(t) + B(u(t),u(t))]\, dt = A^\gamma dw(t) \\ u(0) = x \end{cases} \tag{3.1}$$

and the linear equation associated with it is

$$\begin{cases} dz(t) + [\nu A^2 z(t) - Az(t) + \alpha z(t)]\, dt = A^\gamma dw(t) \\ z(0) = x \end{cases} \tag{3.2}$$

The unknown u can be interpreted as a one-dimensional velocity field in a compressible fluid (see [17]). Actually, this stochastic equation is presented in the physical literature in relation to a model for erosion by ion sputtering (see [7] and references therein).

With respect to the setting of Section 2, we have that the linear operator is

$$Lu = \nu A^2 u - Au + \alpha u$$

with $\nu > 0$ and $\alpha > 0$, and the non linear operator is

$$F(u) = B(u,u) - \alpha u.$$

The operator G in front of the Wiener process is taken to be of the form A^γ ($\gamma \in \mathbb{R}$). w is an H-cylindrical Wiener process defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$; $\{\mathbb{F}_t\}_{t\geq 0}$ is the canonical filtration associated to the Wiener process: $\mathbb{F}_t = \sigma_t(w)$. We shall denote by $\mathcal{L}_{KS}$ the law of a process solving (3.1) and by $\mathcal{L}_O$ that of (3.2).

We present the abstract setting. The functional spaces are (given $L > 0$, so the spatial domain is $[-\frac{L}{2}, \frac{L}{2}]$):

$$H = \{u = u(\xi) \in L^2(-\tfrac{L}{2}, \tfrac{L}{2}) : \textstyle\int_{-L/2}^{L/2} u \, d\xi = 0\},$$
$$E = D(A^\theta) \text{ for some } \theta > 0,$$

where

$$Au = -u''$$
$$D(A) = H \cap \{u = u(\xi) \in H^2(-\tfrac{L}{2}, \tfrac{L}{2}) : u(-\tfrac{L}{2}) = u(\tfrac{L}{2}), u'(-\tfrac{L}{2}) = u'(\tfrac{L}{2})\}.$$

The operator A is a strictly positive unbounded self-adjoint operator in H, whose eigenvectors $\{e_j\}_{j=1}^\infty$ form a complete orthonormal basis of the space H. The powers A^θ are defined for any $\theta \in \mathbb{R}$: if $Ae_j = \lambda_j e_j$ then $A^\theta v = \sum_j \lambda_j^\theta \langle v, e_j\rangle e_j$, $D(A^\theta) = \{v = \sum_j v_j e_j : \sum_j \lambda_j^{2\theta} v_j^2 < \infty\}$. Moreover, $\lambda_j \sim j^2$ as $j \to \infty$.

If $\alpha > \frac{1}{4\nu}$, the operator $-(\nu A^2 - A + \alpha)$ generates in H (and in any $D(A^\beta)$) an analytic semigroup of negative type of class C_0. Therefore, from now on we assume $\nu > 0$ and $\alpha > \frac{1}{4\nu}$.

The operator B is the bilinear operator defined by

$$B(u,v) = uv'.$$

For instance, B maps $D(A^{1/2}) \times D(A^{1/2})$ into H; other domains of definition of B are given in [7].

The H-cylindrical Wiener process can be represented as $w(t) = \sum_j \beta_j(t) e_j$, where $\{\beta_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. one dimensional Wiener processes defined on $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$.

First, let us consider the linear equation (3.2). We denote by $z(t;x)$ the solution evaluated at time t, by $R(t,x,\cdot)$ the transitions functions, i.e. $R(t,x,\Gamma) = P\{z(t;x) \in \Gamma\}$, and by R_t the Markovian semigroup , i.e. $(R_t\phi)(x) = \mathbb{E}[\phi(z(t;x))]$.

We recall some definitions.

(a) z is *irreducible* in $D(A^\theta)$ at time t if $R(t,x,\Gamma) > 0$ for every $x \in D(A^\theta)$ and Γ non empty open subset of $D(A^\theta)$.

(b) z is *Feller* in $D(A^\theta)$ at time $t > 0$ if $R_t\phi \in C_b(D(A^\theta))$ for every $\phi \in C_b(D(A^\theta))$ and *strongly Feller* in $D(A^\theta)$ at time $t > 0$ if $R_t\phi \in C_b(D(A^\theta))$ for every $\phi \in B_b(D(A^\theta))$.

(c) A probability measure m is *invariant* for equation (3.2) if $\int R_t\phi \, dm = \int \phi \, dm$ for every $t \geq 0, \phi \in C_b(D(A^\theta))$.

We collect the results on the linear equation (3.2) in the following proposition; the first part is needed for using Corollary 2.4, the other results will be used in the final part of this section for further analysis of equation (3.1).

Proposition 3.1. *If $\theta + \gamma < \frac{3}{4}$, then for any $x \in D(A^\theta)$ equation* (3.2) *has a unique strong solution z such that*

$$\mathbb{P}\{z \in C([0,T]; D(A^\theta))\} = 1 \tag{3.3}$$

for any $T < \infty$; this is a Markov process, strongly Feller and irreducible in $D(A^\theta)$ for any $t > 0$. The Gaussian measure $\mu_l = \mathcal{N}(0, \frac{1}{2}A^{2\gamma}[\nu A^2 - A + \alpha]^{-1})$ is the unique invariant measure, all transition functions $R(t,x,\cdot)$ are equivalent to μ_l and

$$\lim_{t\to+\infty} R_t\phi(x) = \int \phi \, d\mu_l, \tag{3.4}$$

$$\lim_{t\to+\infty} R(t,x,\Gamma) = \mu_l(\Gamma) \tag{3.5}$$

for any $x \in D(A^\theta), \phi \in C_b(D(A^\theta))$ and Borel set $\Gamma \subset D(A^\theta)$.

Proof. From (3.10) in [7], we know that, given $x \in D(A^\theta)$, if $\theta + \gamma < \frac{3}{4}$ equation (3.2) has a unique strong solution z given by

$$z(t) = e^{-(\nu A^2 - A + \alpha)t} x + \int_0^t e^{-(\nu A^2 - A+\alpha)(t-s)} A^\gamma dw(s).$$

The paths are, $\mathbb{P}$-a.s., in $C([0,T]; D(A^\theta))$. This is a Markov process; many of its properties are easy to check, since the semigroup $\{e^{-(\nu A^2 - A+\alpha)t}\}_{t\geq 0}$ and the covariance of the noise are diagonal operators and commute.

We recall the basic steps for checking the regularity of z (the result follows rigorously, e.g., from [2], Chapter 5, and is proved in [7]):

$$|A^\theta e^{-(\nu A^2-A+\alpha)t}x|_H \le |A^\theta x|_H \qquad \forall t\ge 0$$

$$\begin{aligned}\mathbb{E}\Big|\int_0^t A^\theta e^{-(\nu A^2-A+\alpha)(t-s)}A^\gamma dw(s)\Big|_H^2 &\\ &= \mathbb{E}\Big|\sum_{j=1}^\infty \lambda_j^{\theta+\gamma}\int_0^t e^{-(\nu\lambda_j^2-\lambda_j+\alpha)(t-s)}\ d\beta_j(s)\ e_j\Big|_H^2\\ &= \sum_{j=1}^\infty \lambda_j^{2(\theta+\gamma)}\int_0^t e^{-2(\nu\lambda_j^2-\lambda_j+\alpha)(t-s)}\ ds\\ &\le \sum_{j=1}^\infty \frac{\lambda_j^{2(\theta+\gamma)}}{2(\nu\lambda_j^2-\lambda_j+\alpha)} \qquad \forall t>0.\end{aligned}$$

The latter series is convergent if $\theta+\gamma<\frac34$, since $\lambda_j\sim j^2$ as $j\to\infty$.

The result on the invariant measure is obtained as in [2], Chapter 11. Actually, the result is trivial if we work first on each component z_j and then we recover the infinite dimensional result for z ($z(t)=\sum_{j=1}^\infty z_j(t)e_j$). Indeed, each component z_j satisfies

$$dz_j(t)+[\nu\lambda_j^2-\lambda_j+\alpha]z_j(t)\ dt=\lambda_j^\gamma d\beta_j(t), \qquad z_j(0)=x_j;$$

its law is $\mathcal{N}\big(e^{-(\nu\lambda_j^2-\lambda_j+\alpha)t}x_j, \frac12\frac{\lambda_j^{2\gamma}}{\nu\lambda_j^2-\lambda_j+\alpha}(1-e^{-2(\nu\lambda_j^2-\lambda_j+\alpha)t})\big)$ and for $t\to+\infty$ the density of this Gaussian measure converges to the density of the Gaussian measure $\mathcal{N}\big(0,\frac12\frac{\lambda_j^{2\gamma}}{\nu\lambda_j^2-\lambda_j+\alpha}\big)$, which is the unique stationary measure for z_j. Therefore, equation (3.2) has a unique invariant measure; this is the Gaussian measure with mean 0 and covariance operator $Q_\infty=\frac12 A^{2\gamma}[\nu A^2-A+\alpha]^{-1}$.

Since μ_l is Gaussian, it is easy to check that $\int|A^\theta x|_H^2 d\mu_l(x)<\infty$ and that $\mu_l(\Gamma)>0$ for any open and non empty set $\Gamma\subset D(A^\theta)$.

We expect that irreducibility and strong Feller property hold, because the noise acts on all directions e_j of the Hilbert space and the operator $e^{-(\nu A^2-A+\alpha)t}$ makes $z(t)$ depending very regularly on the initial data x.

As far as the strong Feller property is concerned, from [2] (Chapter 9) we know that the condition $Ran(Q_t^{1/2})\supset Ran(e^{-(\nu A^2-A+\alpha)t})$ is equivalent to the strong Feller property, where Q_t is the covariance operator of the Gaussian random variable $z(t;x)$. Since $Q_t=\frac12 A^{2\gamma}[I-e^{-2(\nu A^2-A+\alpha)t}][\nu A^2-A+\alpha]^{-1}$ and for $t>0$ the range of the operator $e^{-(\nu A^2-A+\alpha)t}$ is contained in any space $D(A^\beta)$ for $\beta>0$, we see that this condition is trivially satisfied.

According to Theorem 11.13 in [2], (3.4) holds and any transition function $R(t,x,\cdot)$ is absolutely continuous with respect to μ_l. Irreducibility comes straightforward. Let us point out that in the proof of this theorem, it is also shown that the law of $z(t;x)$ is equivalent to the law of $z(s;y)$ for any $t,s>0$ and $x,y\in D(A^\theta)$; actually, this follows directly by Feldman-Hajek theorem, which is easy to verify in this case of diagonal operators. □

To set our problem as in Section 2, we have to fix some space $E=D(A^\theta)$. The interesting spaces are $D(A^\theta)$ for $\theta\ge 0$: $D(A^0)=H$ is the basic space of finite energy and, for $\theta > 0$, $D(A^\theta)$ is a subspace of H. In practise, given $\theta \ge 0$ we choose γ as big as possible ($\gamma < \frac{3}{4} - \theta$) so to make to weakest assumption on the covariance of the noise. Or, given $\gamma < \frac{3}{4}$ (the limitation is due to $\theta \ge 0$), we choose θ as big as possible ($\theta < \frac{3}{4} - \gamma$). Decreasing γ, the operator A^γ is "more regular" (in the sense that, for instance, A^γ is a bounded operator for $\gamma \le 0$) and this stronger assumption provides a more regular solution z with paths in $C([0,T]; D(A^\theta))$; indeed, decreasing γ we can increase θ.

Now, we deal with estimate (2.8). We have the following result.

Lemma 3.2. *Let the parameters γ and θ be chosen as follows:*

$$\begin{array}{ll} \text{for } \frac{1}{4} < \gamma < \frac{3}{4}: & \frac{3}{8} - \frac{\gamma}{2} \le \theta < \frac{3}{4} - \gamma \\ \text{for } 0 \le \gamma \le \frac{1}{4}: & \frac{5}{8} - \gamma \le \theta < \frac{3}{4} - \gamma \\ \text{for } \gamma < 0: & \frac{1}{2} - \gamma \le \theta < \frac{3}{4} - \gamma. \end{array} \tag{3.6}$$

Then there exists a constant c, depending on γ, θ and α, such that

$$|A^{-\gamma}[B(v,v) - \alpha v]|_H \le c\left(1 + |A^\theta v|_H^2\right) \qquad \forall v \in D(A^\theta).$$

Proof. Notice that (3.6) imply the bounds $\gamma < \frac{3}{4}$, $\theta > 0$ and $\theta + \gamma < \frac{3}{4}$. The non linear term is estimated as follows:

$$|A^{-\gamma}B(v,\tilde{v})|_H \le C_1 |A^{\frac{3}{8}-\frac{\gamma}{2}} v|_H |A^{\frac{3}{8}-\frac{\gamma}{2}} \tilde{v}|_H \qquad \text{if } \frac{1}{4} < \gamma < \frac{3}{4} \tag{3.7}$$

$$|A^{-\gamma}B(v,\tilde{v})|_H \le C_2 |A^{\frac{5}{8}-\gamma} v|_H |A^{\frac{5}{8}-\gamma} \tilde{v}|_H \qquad \text{if } 0 \le \gamma \le \frac{1}{4} \tag{3.8}$$

$$|A^{-\gamma}B(v,\tilde{v})|_H \le C_3 |A^{\frac{1}{2}-\gamma} v|_H |A^{\frac{1}{2}-\gamma} \tilde{v}|_H \qquad \text{if } \gamma < 0 \tag{3.9}$$

The two first inequalities come from the proof of Lemma 2.2 in [9]. The latter is proved in Proposition 2.1 in [7]. By the way, recalling that $B(v_1, v_1) - B(v_2, v_2) = B(v_1 - v_2, v_1) + B(v_2, v_1 - v_2)$ by bilinearity, the above inequalities show that the operator $A^{-\gamma}B(v,v)$ is continuous (hence, measurable) in the spaces where it is defined.

Notice that if (3.6) are satisfied, then $\theta > -\gamma$. Therefore, choosing θ as in (3.6) we get

$$\begin{aligned} |A^{-\gamma}[B(v,v) - \alpha v]|_H &\le |A^{-\gamma}B(v,v)|_H + \alpha |A^{-\gamma} v|_H \\ &\le C_4 |A^\theta v|_H^2 + \alpha C_5 |A^\theta v|_H \\ &\le C_6 \left(1 + |A^\theta v|_H^2\right). \end{aligned}$$

□

Remark 3.3. The case $\theta = 0$ is not included. Indeed, we have

$$|A^{-\gamma}B(v,v)|_H \le c|v|_H^2$$

for $\gamma > \frac{3}{4}$, because

$$\begin{aligned}|\langle B(v,v),x\rangle| = |\int_{-L/2}^{L/2}\frac{1}{2}(v^2)'x\,d\xi| &= \frac{1}{2}|\int_{-L/2}^{L/2} v^2x'\,d\xi| \\ &\le \frac{1}{2}|v^2|_{L^1}|x'|_{L^\infty} \\ &\le c|v|^2_{L^2}|x'|_{D(A^m)} \text{ for } m > \frac{1}{4} \\ &= c|v|^2_{L^2}|x|_{D(A^{\frac{1}{2}+m})} \text{ for } m > \frac{1}{4}.\end{aligned}$$

But the condition $\gamma > \frac{3}{4}$ is incompatible with $\theta + \gamma < \frac{3}{4}, \theta = 0$.

Now, we consider the stochastic Kuramoto–Sivashinsky equation (3.1).

Theorem 3.4. *For every $\gamma < \frac{3}{4}$ and choosing θ as in* (3.6)*, we have the following result. Given $x \in D(A^\theta)$ and any finite time interval $[0,T]$, there exist a unique weak solution of equation* (3.1) *on $[0,T]$. Its law $\mathcal{L}_{KS}$ is equivalent to the law $\mathcal{L}_O$ of* (3.2)*, as measures on the Borel subsets of $C([0,T];D(A^\theta))$. Given the strong solution $\big((\Omega,\mathbb{F},\{\mathbb{F}_t\}_{t\in[0,T]},\mathbb{P}),\ w,\ z\big)$ of equation* (3.2) *as in Proposition 3.1, we have*

$$\frac{d\mathcal{L}_{KS}}{d\mathcal{L}_O}(z) = \\ \mathbb{E}\left[e^{-\int_0^T{}_H\langle A^{-\gamma}[B(z(s),z(s))-\alpha z(s)],dw(s)\rangle_H-\frac{1}{2}\int_0^T|A^{-\gamma}[B(z(s),z(s))-\alpha z(s)]|^2_H ds}\Big|\sigma_T(z)\right]$$

$\mathbb{P}$-*a.s.*

Proof. If θ and γ are chosen as in (3.6), the estimate (2.8) holds with $p = 2$ and $E = D(A^\theta)$. In addition to the result of Proposition 3.1, this grants that the assumptions of Corollary 2.4 are satisfied. Therefore, for $x \in D(A^\theta)$ equation (3.1) has a unique weak solution u living in $C([0,T];D(A^\theta))$ and $\mathcal{L}_{KS} \sim \mathcal{L}_O$; the Radon-Nykodim derivatives are given by Theorem 2.3. Uniqueness of $\mathcal{L}_{KS}$ comes from the uniqueness result for z given in Proposition 3.1. □

We conclude with some remarks. First, the solution of equation (3.1) is indeed a strong solution; in fact Theorem 4.3 in [7] provides existence and uniqueness of a strong solution u for any $u(0) \in H = D(A^0)$ and $\gamma < \frac{3}{4}$.

Moreover, as far as the regularity of solutions is concerned, the result of the above Theorem improves that of Proposition 6.5 in [7], since now we can consider any space $D(A^\theta)$ with $\theta > 0$. However, we are not able to prove the absolute continuity result in $H = D(A^0)$, as explained in Remark 3.3, even if we know from [7] that for any $u(0) \in H$ equation (3.1) has a unique strong solution u such that $u \in C([0,T];H)$ ($\mathbb{P}$-a.s.).

Finally, we present some consequences of the equivalence of laws. Let us denote by $P(t,x,\cdot)$ the transitions functions for u.

Proposition 3.5. *For every $\gamma < \frac{3}{4}$ and choosing θ as in* (3.6)*, we have that*
(i) $P(t,x,\cdot) \sim \mu_l$ for any $t > 0, x \in D(A^\theta)$, where $\mu_l = \mathcal{N}(0,\frac{1}{2}A^{2\gamma}[\nu A^2 - A + \alpha]^{-1})$ is the unique invariant measure for (3.2)*. The process u is irreducible in $D(A^\theta)$*

at any time $t > 0$.
(ii) For α *large enough there exists only one invariant measure* μ_{KS} *for* (3.1) *which is equivalent to* μ_l.
(iii) μ_{KS} *is ergodic, i.e.*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \phi(u(t;x))dt = \int \phi \, d\mu_{KS}$$

$\mathbb{P}$*-a.s. for every* $x \in D(A^\theta), \phi \in L^1(\mu_{KS})$*, and strongly mixing, i.e.*

$$\lim_{t\to+\infty} P(t, x, \Gamma) = \mu_{KS}(\Gamma)$$

for every $x \in D(A^\theta)$ *and Borel set* $\Gamma \subset D(A^\theta)$.

Proof. Since $\mathcal{L}_{KS} \sim \mathcal{L}_0$, it follows that $P(t, x, \cdot) \sim R(t, x, \cdot)$ and from Proposition 3.1 we get that also u is irreducible. Moreover Proposition 3.1 provides that $R(t, x, \cdot) \sim \mu_l$ and therefore $P(t, x, \cdot) \sim \mu_l$; hence, there is equivalence of all transition functions. ¿From Doob's theorem (see, e.g., [3]) follows uniqueness of invariant measures, ergodicity and strongly mixing property. The existence of an invariant measure has been proved in [7] for α large enough. □

Remark 3.6. **(i)** Notice that the support of μ_{KS} is the same as that of μ_l.
(ii) The results of this section hold true if the operator in front of the Wiener process in equation (3.1) is of the form $LA^{\frac{1}{2}}$, where L is an isometry in H and $LA^{\frac{1}{2}}w(t) = \sum_{j=1}^{\infty} \lambda_j^{\frac{1}{2}} \beta_j(t)(-1)^j e_{j+(-1)^{j+1}}$; notice that the covariance of noise is $L^*AL = A$. The case $G = LA^{\frac{1}{2}}$ is interesting from the physical point of view as explained in [7]. On the other hand, the Girsanov transform for a stochastic Kuramoto–Sivashinsky equation with a *finite* dimensional Wiener process has already been studied in [4], even if in a different setting.

4. A Stochastic Hyperviscosity-regularized Navier–Stokes Equation

Since the quadratic term in the Kuramoto–Sivashinsky equation is similar to that in the Navier–Stokes equation, the only difference being that the Navier–Stokes equation is set in spaces of divergence free vectors, it is appealing to investigate whether Girsanov transform holds for the stochastic Navier–Stokes equation. Unfortunately, the answer is negative. Anyway, let us analyse this problem modifying the linear part. Our issue is to determine how to modify the Navier–Stokes equation in order to apply Corollary 2.4.

Therefore, instead of the stochastic Navier–Stokes equation

$$du(t) + \big[\nu A u(t) + B\big(u(t), u(t)\big)\,\big] \; dt = A^\gamma \, dw(t)$$

(studied, e.g., in [1], [18], [8]), we introduce a modification in the linear part; given any $\alpha \geq 1$ we consider

$$\begin{cases} du(t) + \big[\nu A^\alpha u(t) + B\big(u(t), u(t)\big)\,\big] \; dt = A^\gamma \, dw(t) \\ u(0) = x \end{cases} \tag{4.1}$$

For $\alpha > 1$, this corresponds to replace the Laplacian $-\Delta$ with $(-\Delta)^\alpha$ in the Navier–Stokes equations and models hyperviscous fluids (see [15] and references therein).

Notice that our analysis reminds that of [11] to investigate for which values of α the modified *deterministic* Navier–Stokes equation

$$\frac{du}{dt}(t) + \nu A^\alpha u(t) + B\big(u(t), u(t)\big) = f(t) \tag{4.2}$$

is well posed for $d = 3$ (we recall that for $d = 2$ there is no need of modification to get existence and uniqueness of a global solution).

The linear equation associated to (4.1) is the modified stochastic Stokes equation

$$\begin{cases} dz(t) + \nu A^\alpha z(t)dt = A^\gamma \; dw(t) \\ z(0) = x \end{cases} \tag{4.3}$$

With respect to the setting of Section 2, we have that the linear operator is

$$Lu = \nu A^\alpha u$$

with $\nu > 0$, $\alpha \geq 1$, and the non linear operator is

$$F(u) = B(u, u).$$

The operator G in front of the Wiener process is taken to be of the form A^γ ($\gamma \in \mathbb{R}$). w is a cylindrical Wiener process in H on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$; $\{\mathbb{F}_t\}_{t\geq 0}$ is the canonical filtration associated to the Wiener process.

The functional setting is defined as usual (see [16]). The symbols A and B will denote different operators from those of Section 3, but we use the same symbols because of the analogy between these quantities in equations (3.1) and (4.1).

For $d = 2, 3$, let $\mathcal{D}$ be the d-dimensional torus $\mathbb{R}^d/(2\pi\mathbb{Z})^d$, i.e. we consider our problem on the spatial domain $[0, 2\pi]^d$ with periodic boundary conditions.

Set

$$H = \{u = \vec{u}(\vec{\xi}) \in [L^2(\mathcal{D})]^d : \text{ div } u = 0, \gamma_n u \text{ periodic }, \int_{\mathcal{D}} u \; d\vec{\xi} = 0\}$$

$$E = D(A^\theta) \qquad \text{for some } \theta > 0$$

where $\gamma_n u$ is the trace of the normal component of u on $\partial\mathcal{D}$.

Let $[\dot{H}^m_p(\mathcal{D})]^d, m \in \mathbb{N}\backslash\{0\}$, be the space of functions of $[H^m_{loc}(\mathbb{R}^d)]^d$, periodic with period $[0, 2\pi]^d$ and with zero average. Then the Stokes operator is defined as

$$Au = -\Delta u, \qquad u \in D(A) = [\dot{H}^2_p(\mathcal{D})]^d \cap H.$$

A is a strictly positive unbounded self-adjoint operator in H, whose eigenvectors $\{e_j\}_{j=1}^\infty$ form a complete orthonormal basis of the space H. The powers A^α are defined for any $\alpha \in \mathbb{R}$. The operator $-A$ generates in H (and in any $D(A^\beta)$) an analytic semigroup of negative type $\{e^{-tA}\}_{t\geq 0}$ of class C_0. Moreover, $Ae_j = \lambda_j e_j$ with $\lambda_j \sim j^{2/d}$ as $j \to \infty$.

The bilinear operator B, from $D(A^{1/2}) \times D(A^{1/2})$ into $D(A^{-1/2})$, is defined by

$$\langle B(u, v), z\rangle = \int_{\mathcal{D}} z \cdot [(u \cdot \nabla) \, v] \; d\vec{\xi} \qquad \forall \, u, v, z \in D(A^{1/2}).$$

By the incompressibility condition we have

$$\langle B(u,v),v\rangle\ =0, \qquad\qquad \langle B(u,v),z\rangle\ =-\langle B(u,z),v\rangle.$$

Other domains of definition of B are given below in (4.8).

First, let us consider the linear equation. Similarly to the previous section, we have

Proposition 4.1. *If*

$$\alpha-2(\theta+\gamma)>\frac{d}{2}, \tag{4.4}$$

then for any $x\in D(A^\theta)$ equation (4.3) has a unique strong solution z such that

$$\mathbb{P}\{z\in C([0,T];D(A^\theta))\}=1 \tag{4.5}$$

for any $T<\infty$; this is a Markov process, strongly Feller and irreducible in $D(A^\theta)$ for any $t>0$. The transition functions $\tilde R(t,x,\cdot)$ are equivalent to $\tilde\mu_l$ for any $t>0, x\in D(A^\theta)$, where $\tilde\mu_l=\mathcal{N}(0,\frac{1}{2\nu}A^{2\gamma-\alpha})$ is the unique invariant measure, and

$$\lim_{t\to+\infty}\tilde R_t\phi(x)=\int\phi\ d\tilde\mu_l \tag{4.6}$$

$$\lim_{t\to+\infty}\tilde R(t,x,\Gamma)=\tilde\mu_l(\Gamma) \tag{4.7}$$

for any $x\in D(A^\theta),\phi\in C_b(D(A^\theta))$ and Borel set $\Gamma\subset D(A^\theta)$.

Proof. The solution of equation (4.3) is given by

$$z(t)=e^{-\nu A^\alpha t}x+\int_0^t e^{-\nu A^\alpha(t-s)}A^\gamma dw(s).$$

If (4.4) holds, then there exists a continuous version with values in $D(A^\theta)$. Indeed, the basic estimates are

$$\left|A^\theta e^{-\nu A^\alpha t}x\right|_H\le\left|A^\theta x\right|_H \qquad \forall t\ge 0$$

$$\begin{aligned}\mathbb{E}\Big|\int_0^t A^\theta e^{-\nu A^\alpha(t-s)}A^\gamma dw(s)\Big|_H^2&=\mathbb{E}\Big|\sum_{j=1}^\infty\lambda_j^{\theta+\gamma}\int_0^t e^{-\nu\lambda_j^\alpha(t-s)}\ d\beta_j(s)\ e_j\Big|_H^2\\&=\sum_{j=1}^\infty\lambda_j^{2(\theta+\gamma)}\int_0^t e^{-2\nu\lambda_j^\alpha(t-s)}\ ds\\&\le\sum_{j=1}^\infty\frac{\lambda_j^{2(\theta+\gamma)}}{2\nu\lambda_j^\alpha} \qquad \forall t>0.\end{aligned}$$

The latter series is convergent if (4.4) is fulfilled, since $\lambda_j\sim j^{2/d}$ as $j\to\infty$.

The unique invariant measure is the Gaussian measure with mean 0 and covariance operator $\frac{1}{2\nu}A^{2\gamma-\alpha}$; indeed, each component z_j satisfies

$$dz_j(t)+\nu\lambda_j^\alpha z_j(t)dt=\lambda_j^\gamma\ d\beta_j(t);\qquad z_j(0)=x_j$$

and this equation has only one invariant measure which is the one-dimensional Gaussian measure $\mathcal{N}(0,\frac{1}{2\nu}\lambda_j^{2\gamma-\alpha})$.

(4.6) and the equivalence $\tilde R(t,x,\cdot)\sim\tilde\mu_l$ can be shown as in Proposition 3.1. □

Now, we have to choose the space E. Let us consider $\theta \geq 1$. Why? Because the easiest estimate for $B(v,v)$ is in the spaces $D(A^m)$ with $m \geq \frac{1}{2}$; indeed, for these values the space $D(A^m)$ is a multiplicative algebra and therefore

$$\left|A^m B(v,\tilde{v})\right|_H \leq c_m \left|A^m v\right|_H \left|A^{m+\frac{1}{2}} \tilde{v}\right|_H \tag{4.8}$$

(see, e.g., [16]). This estimate shows that in these spaces the operator $A^m B(v,v)$ is well defined and continuous (for this, we use that B is a bilinear operator). In particular

$$\left|A^{\theta-\frac{1}{2}} B(v,v)\right|_H \leq c'_\theta \left|A^\theta v\right|_H^2 \quad \text{for } \theta \geq 1. \tag{4.9}$$

To check inequality (2.8) in our context, the latter result suggests to set

$$-\gamma = \theta - \frac{1}{2}.$$

In this case, from (4.4) we know that the process z will have paths in the space $C([0,T]; D(A^\theta))$ if

$$\alpha > \frac{d}{2} + 1.$$

Remark 4.2. (i) This condition shows that $\alpha = 1$ is not allowed. That is, our procedure does not work for the Navier–Stokes equation; only taking α sufficiently large we can prove Girsanov theorem and the absolute continuity of the laws. In particular, for $d = 2$ we require $\alpha > 2$ and for $d = 3$ we require $\alpha > \frac{5}{2}$. In the same way we can prove this result of absolute continuity for the stochastic 1-dimensional Burgers equation if $\alpha > \frac{3}{2}$.

(ii) It is interesting to compare which values of α provide that the Navier–Stokes equation is well posed, that is it has a unique global solution. For the *stochastic* problem, when $d = 2$ it is enough to take $\alpha = 1$ (see, e.g., [8], [6]); this holds also for the *deterministic* equation (see [16]). We guess that when $d = 3$ there is well posedness of the stochastic Navier–Stokes equation for $\alpha > \frac{5}{4}$. The value $\frac{5}{4}$ appears in the *deterministic* equation; indeed, in [11] it is proved that equation (4.2) is well posed for $\alpha > \frac{5}{4}$. The corresponding result for equation (4.1) (with $d = 3$) will be proved in a future work.

(iii) By the way, we point out that existence and uniqueness of martingale solutions for a stochastic hyperviscous Navier–Stokes equation with additive or multiplicative noise have been studied in [15] (see Sect. 5); when the noise is additive, the results there hold with $\alpha \geq 2$ and $d = 2, 3$.

At this point, we prefer to fix a value of θ; indeed, there are three quantities involved in the study of equation (4.1): α, γ, θ. To get not too involved relations to determine the "good" values of these parameters, we reduce the number of parameters setting $\theta = 1$. We point out that all the following results can be obtained in the same way for any $\theta > 1$, because of (4.8). However, the technicalities are more involved for $0 \leq \theta < 1$ (see also Remark 5.2 below). Having set $-\gamma = \theta - \frac{1}{2}$, the choice $\theta = 1$ implies $\gamma = -\frac{1}{2}$.

Here is our main result. We denote by $\mathcal{L}_{NS}$ the law of a process solving equation (4.1) and by $\mathcal{L}_S$ that for equation (4.3). We state the result for $d = 2, 3$.

Theorem 4.3. *(i) Consider* $\gamma = -\frac{1}{2}$ *and* $\alpha > \frac{d}{2} + 1$ *in equation* (4.1). *Given* $x \in D(A)$, *on any finite time interval* $[0, T]$ *there exists a unique weak solution of equation* (4.1). *Its law* $\mathcal{L}_{NS}$ *is equivalent to the law* $\mathcal{L}_S$ *of equation* (4.3), *as measures on the Borel subsets of* $C([0, T]; D(A))$.
(ii) Given the strong solution $\big((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathbb{P}),\ w,\ z\big)$ *of equation* (4.3), *we have*

$$\frac{d\mathcal{L}_{NS}}{d\mathcal{L}_S}(z) = \mathbb{E}\left[e^{-\int_0^T {}_H\langle A^{\frac{1}{2}} B(z(s),z(s)),dw(s)\rangle_H - \frac{1}{2}\int_0^T |A^{\frac{1}{2}} B(z(s),z(s))|_H^2 ds} \Big| \sigma_T(z)\right]$$

$\mathbb{P}$-*a.s.*

Proof. For $\theta = 1$, $\gamma = -\frac{1}{2}$ and $\alpha > \frac{d}{2} + 1$, (4.4) shows that the linear equation has a unique strong solution z with paths in $C([0, T]; D(A))$. Moreover, by (4.9) we see that estimate (2.8) holds for $p = 2$ and $E = D(A)$. According to Corollary 2.4 we conclude that equation (4.1) has a weak solution and $\mathcal{L}_{NS} \sim \mathcal{L}_S$ and the Radon-Nykodim derivative $\frac{d\mathcal{L}_{NS}}{d\mathcal{L}_S}$ is given as in Theorem 2.3. □

It is possible to reinforce the result of existence of a weak solution getting existence of a strong solution. Indeed, a result by Yamada–Watanabe states that weak existence and pathwise uniqueness imply the existence of a strong solution (see, e.g., [14], Chapter IX, Theorem 1.7).

Pathwise uniquess will be proved in section 5. Hence we have a unique strong solution u for equation (4.1). We can define the transition function $\tilde{P}(t, x, \Gamma) = \mathbb{P}\{u(t; x) \in \Gamma\} = \mathbb{P}^*\{z(t; x) \in \Gamma\}$ and the Markovian semigroup $(\tilde{P}_t\phi)(x) = \mathbb{E}[\phi(u(t; x))] = \mathbb{E}^*[\phi(z(t; x))]$. We have

Proposition 4.4. *Let* $\gamma = -\frac{1}{2}$ *and* $\alpha > \frac{d}{2} + 1$. *For equation* (4.1) *we have that* $\tilde{P}(t, x, \cdot) \sim \tilde{\mu}_l$ *for any* $t > 0, x \in D(A)$, *where* $\tilde{\mu}_l = \mathcal{N}(0, \frac{1}{2\nu}A^{-1-\alpha})$ *is the unique invariant measure for* (4.3). *In particular, the solution process* u *is Feller and irreducible in* $D(A)$ *at any time* $t > 0$; *hence there exists at most one invariant measure for* (4.1), *which is equivalent to* $\tilde{\mu}_l$.

Proof. ¿From Theorem 4.3 we know that $\mathcal{L}_{NS} \sim \mathcal{L}_S$; for irreducibility and uniqueness of invariant measure the proof goes along the same lines as those of Proposition 3.5. It remains to prove Feller property, that is $\tilde{P}_t : C_b(D(A)) \to C_b(D(A))$ for any t; this follows from the pathwise uniqueness result proved below in Proposition 5.1. Indeed, by (5.2) if $x \to y$ in $D(A)$, then $\mathbb{P}$-a.s. $u(t; x) \to u(t; y)$ in $D(A)$; taking a continuous and bounded function $\phi : D(A) \to \mathbb{R}$, we have that $\mathbb{P}$-a.s. $\phi(u(t; x)) \to \phi(u(t; y))$ as $x \to y$ in $D(A)$. Finally, since ϕ is bounded, by dominated convergence we get that $\mathbb{E}\phi(u(t; x)) \to \mathbb{E}\phi(u(t; y))$ as $x \to y$ in $D(A)$. □

Remark 4.5. In this section we have assumed periodic boundary conditions so to give a meaning to terms as $A^{\frac{1}{2}}B(z, z)$. The reader can consult [5] for instance, to see for which values of β the expression $A^\beta B(z, z)$ is well defined when working in a bounded spatial domain $\mathcal{D} \subset \mathbb{R}^d$, assuming the velocity vanishes on the boundary $\partial\mathcal{D}$. However, no such problem of giving a meaning to $A^{\frac{1}{2}}B(z, z)$ arises in the periodic case.

5. Appendix

Proposition 5.1 (Pathwise uniqueness)**.** *For $\gamma = -\frac{1}{2}$ and $\alpha > \frac{d}{2} + 1$, given $x \in D(A)$ any two $C([0,T];D(A))$-valued strong solutions of* (4.1) *coincide* $\mathbb{P}$*-a.s.*

Proof. Let u_1, u_2 be two strong solutions on the stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$. Set $U = u_1 - u_2$. Then U satisfies, $\mathbb{P}$-a.s.,

$$\frac{dU}{dt}(t) + \nu A^\alpha U(t) + B\big(u_1(t), u_1(t)\big) - B\big(u_2(t), u_2(t)\big) = 0 \tag{5.1}$$

with initial data $U(0) = 0$. We proceed pathwise.

By bilinearity, $B(u_1, u_1) - B(u_2, u_2) = B(u_1, U) + B(U, u_2)$. We multiply both sides of (5.1) by $A^2 U(t)$; then (all the norms are in H)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|AU(t)|^2 + \nu|A^{1+\frac{\alpha}{2}}U(t)|^2 &= -\langle B\big(u_1(t), U(t)\big) + B\big(U(t), u_2(t)\big), A^2U(t)\rangle \\ &= -\langle A^{\frac{1}{2}}\big[B\big(u_1(t), U(t)\big) + B\big(U(t), u_2(t)\big)\big], A^{\frac{3}{2}}U(t)\rangle.\end{aligned}$$

Using (4.8) with $m = \frac{1}{2}$, we have $|A^{\frac{1}{2}}[B(u_1, U) + B(U, u_2)]| \leq c[|Au_1| + |Au_2|]|AU|$; thus

$$\begin{aligned}|\langle A^{\frac{1}{2}}\big[B\big(u_1, U\big) + B\big(U, u_2\big)\big], A^{\frac{3}{2}}U\rangle| &\leq c\big[|Au_1| + |Au_2|\big]\ |AU|\ |A^{\frac{3}{2}}U| \\ &\overset{(*)}{\leq} c\big[|Au_1| + |Au_2|\big]\ |AU|\ |A^{1+\frac{\alpha}{2}}U| \\ &\leq \frac{\nu}{2}|A^{1+\frac{\alpha}{2}}U|^2 + c_\nu\big[|Au_1|^2 + |Au_2|^2\big]|AU|^2.\end{aligned}$$

Hence

$$\frac{d}{dt}|AU(t)|^2 + \nu|A^{1+\frac{\alpha}{2}}U(t)|^2 \leq C_7\big[|Au_1(t)|^2 + |Au_2(t)|^2\big]|AU(t)|^2.$$

In particular

$$\frac{d}{dt}|AU(t)|^2 \leq C_7\big[|Au_1(t)|^2 + |Au_2(t)|^2\big]|AU(t)|^2$$

and by Gronwall lemma

$$\sup_{0\leq t\leq T} |AU(t)|^2 \leq |AU(0)|^2 e^{\int_0^T C_7[|Au_1(s)|^2 + |Au_2(s)|^2]ds}. \tag{5.2}$$

Since the paths u_1, u_2 are in $C([0,T];D(A))$ and $U(0) = 0$, it follows that

$$|AU(t)| = 0 \qquad \forall t \in [0,T],$$

that is $u_1(t) = u_2(t)$ for all $t \in [0,T]$. □

Remark 5.2. The estimates of the proof remain valid for any $\alpha \geq 1$; in fact, inequality $(*)$ holds for $\alpha \geq 1$. Therefore, we could have stated the proposition assuming only $\alpha \geq 1$. This depends strongly on the choice of θ. We point out that for $\theta < 1$ uniqueness in $C([0,T];D(A^\theta))$ can be proved along the same lines, but α must be larger than 1.

For example, in the case $\underline{\theta = 0}$ we estimate the non linearity by

$$|A^{-(\frac{1}{2}+\frac{d}{4}+\varepsilon)}B(v, \tilde{v})|_H \leq c|v|_H|\tilde{v}|_H, \tag{5.3}$$

which holds for any $\varepsilon > 0$. This is proved by means of the embeddings $D(A^{\frac12+\frac d4+\varepsilon}) \subset [H^{1+\frac d2+2\varepsilon}(\mathcal{D})]^d$ and $[H^{1+\frac d2+2\varepsilon}(\mathcal{D})]^d \subset [L^\infty(\mathcal{D})]^d$, that generalize the estimate of Remark 3.3 (proved there for $d=1$). In the proof of pathwise uniqueness (for $\theta=0, \gamma=\frac12+\frac d4+\varepsilon$) we would use

$$\begin{aligned}|\langle B\big(u_1,U\big)+B\big(U,u_2\big),U\rangle| &= |\langle A^{-(\frac12+\frac d4+\varepsilon)}\big[B\big(u_1,U\big)+B\big(U,u_2\big)\big], A^{\frac12+\frac d4+\varepsilon}U\rangle| \\ &\le c\big[|u_1|_H+|u_2|_H\big]\ |U|_H\ |A^{\frac12+\frac d4+\varepsilon}U|_H.\end{aligned}$$

If $|A^{\frac12+\frac d4+\varepsilon}U|_H \le c|A^{\frac\alpha2}U|_H$, that is if $\alpha > 1+\frac d2$, we would get that

$$\frac{d}{dt}|U(t)|_H^2+\nu|A^{\frac\alpha2}U(t)|_H^2 \le C_8\big[|u_1(t)|_H^2+|u_2(t)|_H^2\big]|U(t)|_H^2,$$

so to conclude by Gronwall lemma that $|U(t)|_H=0$ for all $t\in[0,T]$.

Hence, we can prove pathwise uniqueness in $C([0,T];D(A^0))$ if $\alpha>1+\frac d2$. On the other hand, chosen $\theta=0$ and $\gamma=\frac12+\frac d4+\varepsilon$ so to estimate the quadratic term as in (5.3), it follows that inequality (4.4) holds for $\alpha>1+d$.

Summing up, we have checked that to apply our procedure for $\theta=0$ we need a stronger assumption on α: $\alpha>1+d$. This is the reason for choosing $\theta\ge1$ so to make the minimal assumption on α (given $\alpha>1$, the smaller is α the closer is the model to the Navier-Stokes equation).

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