

SINGULAR PERTURBATION AND STATIONARY SOLUTIONS OF PARABOLIC EQUATIONS IN GAUSS-SOBOLEV SPACES

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ABSTRACT. The paper is concerned with a class of parabolic equations in a Gauss-Sobolev space containing a small parameter $\varepsilon > 0$. They degenerate into elliptic equations as ε tends to zero. It is proven that, under appropriate conditions, the solution to the Cauchy problem for such a parabolic equation converges, as $\varepsilon \downarrow 0$, to a limit that satisfies the reduced elliptic equation. This singular perturbation problem is shown to be closely related to the stationary solution of the parabolic problem as $t \rightarrow \infty$. An application of this result to the asymptotic evaluation of a certain functional integral is given.

1. Introduction

The subject of parabolic equations in infinite dimensions has been studied by many authors, (see e.g., the papers [6, 10, 13] and the books [3, 7, 8]). In finite dimensions, the singular perturbation problems for partial differential equations have been studied extensively [9]. It is natural to initiate the investigation of such problems in infinite dimensions. In a recent paper [4], we treated the singular perturbation problem for the Kolmogorov equation in a Gauss-Sobolev space when the diffusion coefficient approaches zero. In contrast the current paper is concerned with a class of parabolic equations containing a small positive parameter ε such that, as $\varepsilon \downarrow 0$, the equations become elliptic. In particular we are interested in the limiting behavior of solutions to the associated Cauchy (initial-value) problems as the parameter tends to zero. As it turns out, such problems are closely related to the existence question of the stationary solutions to the parabolic Cauchy problems as the time $t \rightarrow \infty$. In both of the linear and the semilinear cases, the existence of strong solutions to the associated Cauchy problems was proved in a recent paper [2]. Here the main technical problem is to show that such strong solutions will converge in a proper sense to the reduced elliptic equations. Moreover we will show the connection of the singular perturbation problems to that of asymptotic solutions of the parabolic equations as $t \rightarrow \infty$.

To be specific, the paper is organized as follows. In Section 2, we recall some basic results in the Gauss-Sobolev spaces to be needed in the subsequent sections. Section 3 pertains to the strong solutions of parabolic equations in Gauss-Sobolev spaces. Some a priori estimates and inequalities for the linear equations are given

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by Lemmas 3.1 to 3.3, and the existence of strong solutions to the semilinear parabolic equations is ensured by Theorem 3.4. Then a singular perturbation problem and the related stationary solutions for linear parabolic equation are considered in Section 4. The results on the convergence of solutions are presented in Theorem 4.1 and Theorem 4.2, respectively. In Section 5 the corresponding results for the semilinear parabolic equations are given in Theorem 5.4 and Theorem 5.5. As an application, Theorem 5.5 is applied to the asymptotic evaluation of a certain functional integral in Section 6.

2. Preliminaries

Let H be a real separable Hilbert space with inner product (\cdot, \cdot) and norm $|\cdot|$ and let $V \subset H$ be a Hilbert subspace with norm $\|\cdot\|$. Denote the dual space of V by V' and their duality pairing by $\langle \cdot, \cdot \rangle$. Assume that the inclusions $V \subset H \cong H' \subset V'$ are dense and continuous [11].

Suppose that $A : V \rightarrow V'$ is a continuous closed linear operator with domain $\mathcal{D}(A)$ dense in H , and W_t is a H -valued Wiener process with the covariance operator R . Consider the linear stochastic equation in a distributional sense:

$$\begin{aligned} du_t &= Au_t dt + dW_t, \quad t \geq 0, \\ u_0 &= h \in H. \end{aligned} \tag{2.1}$$

Assume that the following conditions (A.1)–(A.3) hold:

(A.1) Let $A : V \rightarrow V'$ be a self-adjoint, coercive operator such that

$$\langle Av, v \rangle \leq -\beta \|v\|^2,$$

for some $\beta > 0$, and $(-A)$ has positive eigenvalues $0 < \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n \leq \dots$, counting the finite multiplicity, with $\gamma_n \uparrow \infty$ as $n \rightarrow \infty$. The corresponding orthonormal set of eigenfunctions $\{e_n\}$ is complete.

(A.2) The resolvent operator $\mathcal{R}_\lambda(A)$ and covariance operator R commute, so that $\mathcal{R}_\lambda(A)R = R\mathcal{R}_\lambda(A)$, where $\mathcal{R}_\lambda(A) = (\lambda I - A)^{-1}$, $\lambda \geq 0$, with I being the identity operator in H .

(A.3) The covariance operator $R : H \rightarrow H$ is a self-adjoint operator with a finite trace such that $\text{Tr } R < \infty$.

Then, by a direct calculation or applying Theorem 4.1 in [5] for invariant measures, we have the following theorem.

Theorem 2.1. *Under conditions (A.1)–(A.3), the stochastic equation (2.1) has a unique invariant measure μ on H , which is a centered Gaussian measure supported in V with covariance operator $\Gamma = -\frac{1}{2}A^{-1}R$.*

Remark 2.2. Let e^{tA} denote the semigroup of operators generated by A . Without condition (A.2), the covariance operator of the invariant measure μ is given by $R = \int_0^\infty e^{tA} R e^{tA} dt$, which cannot be evaluated in a closed form. Though a $L^2(\mu)$ -theory can be developed in the subsequent analysis, one needs to impose some other conditions which are not easily verifiable.

Let $\mathcal{H} = L^2(H, \mu)$ with norm defined by

$$\|\Phi\| = \left\{ \int_H |\Phi(v)|^2 \mu(dv) \right\}^{1/2},$$

and the inner product $[\cdot, \cdot]$ given by

$$[\Theta, \Phi] = \int_H \Theta(v)\Phi(v)\mu(dv), \quad \text{for } \Theta, \Phi \in \mathcal{H}.$$

Let $\mathbf{n} = (n_1, n_2, \dots, n_k, \dots)$, where $n_k \in \mathbb{Z}^+$, the set of nonnegative integers, and let $\mathbf{Z} = \{\mathbf{n} : n = |\mathbf{n}| = \sum_{k=1}^\infty n_k < \infty\}$, so that $n_k = 0$ except for a finite number of n'_k 's. Let $h_m(r)$ be the standard one-dimensional Hermite polynomial of degree m . For $v \in H$, define a Hermite (polynomial) functional of degree n by

$$H_{\mathbf{n}}(v) = \prod_{k=1}^\infty h_{n_k}[\ell_k(v)],$$

where we set $\ell_k(v) = (v, \Gamma^{-1/2}e_k)$ and $\Gamma^{-1/2}$ denotes a pseudo-inverse, by restricting it to the range of $\Gamma^{1/2}$. For a smooth functional Φ on H , let $D\Phi$ and $D^2\Phi$ denote the Fréchet derivatives of the first and second orders, respectively. The differential operator

$$\mathcal{A}\Phi(v) = \frac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v) \rangle \tag{2.2}$$

is well defined for a polynomial functional Φ with $D\Phi(v)$ lies in the domain $\mathcal{D}(A)$ of A . It was shown in [1] that the following holds.

Proposition 2.3. *The set of all Hermite functionals $\{H_{\mathbf{n}} : \mathbf{n} \in \mathbf{Z}\}$ forms a complete orthonormal system in \mathcal{H} . Moreover we have*

$$\mathcal{A}H_{\mathbf{n}}(v) = -\lambda_{\mathbf{n}}H_{\mathbf{n}}(v), \quad \forall \mathbf{n} \in \mathbf{Z},$$

where $\lambda_{\mathbf{n}} = \mathbf{n} \cdot \gamma = \sum_{k=1}^\infty n_k \gamma_k$.

We now introduce the Gauss-Sobolev spaces. For $\Phi \in \mathcal{H}$, by Proposition 2.2, it can be expressed as

$$\Phi = \sum_{\mathbf{n} \in \mathbf{Z}} \Phi_{\mathbf{n}} H_{\mathbf{n}},$$

where $\Phi_{\mathbf{n}} = [\Phi, H_{\mathbf{n}}]$ and $\|\Phi\|^2 = \sum_{\mathbf{n}} |\Phi_{\mathbf{n}}|^2 < \infty$. Let \mathcal{H}_m denote the Gauss-Sobolev space of order m defined as

$$\mathcal{H}_m = \{\Phi \in \mathcal{H} : \|\Phi\|_m < \infty\}$$

for any integer m , where the norm

$$\|\Phi\|_m = \|(I - \mathcal{A})^{m/2}\Phi\| = \left\{ \sum_{\mathbf{n}} (1 + \lambda_{\mathbf{n}})^m |\Phi_{\mathbf{n}}|^2 \right\}^{1/2}, \tag{2.3}$$

with I being the identity operator in $\mathcal{H} = \mathcal{H}_0$. For $m \geq 1$, the dual space \mathcal{H}'_m of \mathcal{H}_m is given by \mathcal{H}_{-m} , and the duality pairing between them will be denoted by $\langle\langle \cdot, \cdot \rangle\rangle_m$ with $\langle\langle \cdot, \cdot \rangle\rangle_1 = \langle\langle \cdot, \cdot \rangle\rangle$. Clearly, the sequence of norms $\{\|\Phi\|_m\}$ is increasing, that is,

$$\|\Phi\|_m < \|\Phi\|_{m+1},$$

for any integer m , and, by identify \mathcal{H} with its dual \mathcal{H}' , we have

$$\mathcal{H}_m \subset \mathcal{H}_{m-1} \subset \cdots \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \cdots \subset \mathcal{H}_{-m+1} \subset \mathcal{H}_{-m}, \quad \text{for } m \geq 1,$$

and the inclusions are dense and continuous. Of course the spaces \mathcal{H}_m can be defined for any real number m , but they are not needed in this paper.

Owing to the use of the invariant measure μ , it is possible to develop a L^2 -theory of infinite-dimensional parabolic and elliptic equations connected to stochastic PDEs similar to the finite-dimensional ones. In particular the following properties of \mathcal{A} are crucial in the subsequent analysis. So far the differential operator \mathcal{A} given by (2.2) is defined only in the set of Hermite polynomial functionals. In fact it can be extended to a self-adjoint linear operator in \mathcal{H} . To this end, let \mathcal{P}_N be a projection operator in \mathcal{H} onto its subspace spanned by the Hermite polynomial functionals of degree N and define $\mathcal{A}_N = \mathcal{P}_N \mathcal{A}$. Then the following theorem holds (Theorem 3.1, [1]).

Theorem 2.4. *The sequence $\{\mathcal{A}_N\}$ converges strongly to a linear symmetric operator $\mathcal{A} : \mathcal{H}_2 \rightarrow \mathcal{H}$, so that, for $\Phi, \Psi \in \mathcal{H}_2$, the following identity holds:*

$$\int_H (\mathcal{A}\Phi, \Psi) d\mu = \int_H (\mathcal{A}\Psi, \Phi) d\mu = -\frac{1}{2} \int_H (RD\Phi, D\Psi) d\mu. \quad (2.4)$$

Moreover, \mathcal{A} has a self-adjoint extension, still denoted by \mathcal{A} , with domain dense in \mathcal{H} .

For $\Phi \in \mathbf{C}_b^2(H)$ being a bounded \mathbf{C}^2 -continuous functional on H , let P_t denote the transition operator defined by

$$[P_t\Phi](v) = E\{\Phi(u_t)|u_0 = v\} = \Psi_t(v).$$

Then, for $v \in \mathcal{D}(A)$, $\Psi_t(v)$ satisfies the Kolmogorov equation in the classical sense:

$$\begin{aligned} \frac{\partial}{\partial t} \Psi_t(v) &= \mathcal{A}\Psi_t(v), \quad t > 0, \\ \Psi_0(v) &= \Phi(v). \end{aligned} \quad (2.5)$$

In fact the transition operator P_t can be extended to be a bounded linear operator on \mathcal{H} and it is possible to define the equation (2.5) for μ -a.e. $v \in \mathcal{H}$.

Theorem 2.5. *Under conditions (A.1)–(A.3), the transition operator P_t is defined on \mathcal{H} for all $t \geq 0$ and $\{P_t : t \geq 0\}$ forms a strongly continuous semigroup of linear contraction operators on \mathcal{H} with the infinitesimal generator $\tilde{\mathcal{A}} = \mathcal{A}$ in \mathcal{H}_2 .*

3. Basic Estimates and Existence Theorem

Consider the Cauchy problem for the linear parabolic equation:

$$\begin{aligned} \frac{\partial}{\partial t} \Psi_t(v) &= \mathcal{A}_\alpha \Psi_t(v) + Q_t(v), \quad \mu - a.e. v \in H, \quad t > 0, \\ \Psi_0(v) &= \Phi(v), \end{aligned} \quad (3.1)$$

for $Q \in L^2((0, \infty); \mathcal{H})$ and $\Phi \in \mathcal{H}$, where α is a positive parameter and

$$\mathcal{A}_\alpha = (\mathcal{A} - \alpha I). \tag{3.2}$$

It was shown in [1] that the solution of (3.1) has the regularity:

$$\Psi \in \mathbf{C}([0, T]; \mathcal{H}) \cap L^2((0, T); \mathcal{H}_1).$$

Let \mathcal{G}_t denote the Green's operator associated with equation (3.1) given by

$$\mathcal{G}_t = e^{-\alpha t} P_t.$$

In view of Theorem 2.4, the solution of (3.1) can be expressed as

$$\Psi_t(v) = [\mathcal{G}_t \Phi](v) + \int_0^t [\mathcal{G}_{t-s} Q_s](v) ds. \tag{3.3}$$

In what follows, we assume that conditions (A.1)–(A.3) are satisfied. Then we have the following technical lemmas.

Lemma 3.1. *The Green's operator $\mathcal{G}_t : \mathcal{H} \rightarrow \mathcal{H}$ is linear and bounded such that, for $\Phi \in \mathcal{H}$ and $Q \in L^2((0, \infty); \mathcal{H})$, we have*

$$\|\|\mathcal{G}_t \Phi\|\| < e^{-\alpha t} \|\|\Phi\|\|, \tag{3.4}$$

and

$$\|\|\int_0^t \mathcal{G}_{t-s} Q_s ds\|\|^2 \leq \frac{1}{\alpha} \int_0^t e^{-\alpha(t-s)} \|\|Q_s\|\|^2 ds, \quad t \geq 0. \tag{3.5}$$

Proof. For $\Phi \in \mathcal{H}$, in terms of the Hermite functionals, we can write

$$\mathcal{G}_t \Phi = \sum_{\mathbf{n}} e^{-\beta_{\mathbf{n}} t} [\Phi, H_{\mathbf{n}}] H_{\mathbf{n}}, \quad \text{with } \beta_{\mathbf{n}} = \alpha + \lambda_{\mathbf{n}}, \tag{3.6}$$

It follows that

$$\begin{aligned} \|\|\mathcal{G}_t \Phi\|\|^2 &= \sum_{\mathbf{n}} e^{-2\beta_{\mathbf{n}} t} [\Phi, H_{\mathbf{n}}]^2 \\ &\leq e^{-2\alpha t} \sum_{\mathbf{n}} [\Phi, H_{\mathbf{n}}]^2 = e^{-2\alpha t} \|\|\Phi\|\|^2, \end{aligned}$$

which verifies (3.4). Similar to (3.6), for $Q \in L^2((0, T); \mathcal{H})$, we have

$$\mathcal{G}_{t-s} Q_s = \sum_{\mathbf{n}} e^{-\beta_{\mathbf{n}}(t-s)} [Q_s, H_{\mathbf{n}}] H_{\mathbf{n}}, \quad \text{with } \beta_{\mathbf{n}} = \alpha + \lambda_{\mathbf{n}},$$

so that

$$\|\|\int_0^t \mathcal{G}_{t-s} Q_s ds\|\|^2 = \sum_{\mathbf{n}} \left\{ \int_0^t e^{-\beta_{\mathbf{n}}(t-s)} Q_s^{\mathbf{n}} ds \right\}^2, \tag{3.7}$$

where we let $Q_s^{\mathbf{n}} = [Q_s, H_{\mathbf{n}}]$. Since

$$\begin{aligned} \left\{ \int_0^t e^{-\beta_{\mathbf{n}}(t-s)} Q_s^{\mathbf{n}} ds \right\}^2 &\leq \frac{1}{\beta_{\mathbf{n}}} \int_0^t e^{-\beta_{\mathbf{n}}(t-s)} |Q_s^{\mathbf{n}}|^2 ds, \\ &\leq \frac{1}{\alpha} \int_0^t e^{-\alpha(t-s)} |Q_s^{\mathbf{n}}|^2 ds, \end{aligned}$$

the inequality (3.5) can be obtained from (3.7) as follows

$$\begin{aligned} \left\| \int_0^t \mathcal{G}_{t-s} Q_s ds \right\|^2 &\leq \frac{1}{\alpha} \sum_{\mathbf{n}} \int_0^t e^{-\alpha(t-s)} |Q_s^{\mathbf{n}}|^2 ds \\ &= \frac{1}{\alpha} \int_0^t e^{-\alpha(t-s)} \|Q_s\|^2 ds, \end{aligned}$$

where the interchange of the summation and integration can be justified due to the monotone convergence. \square

Lemma 3.2. For $Q \in L^2((0, T); \mathcal{H}_{m-1})$, the following inequality holds:

$$\left\| \int_0^t \mathcal{G}_{t-s} Q_s ds \right\|_m^2 \leq \frac{1}{\alpha_1} \int_0^t e^{-\alpha(t-s)} \|Q_s\|_{m-1}^2 ds, \tag{3.8}$$

where $\alpha_1 = \min\{\alpha, 1\}$ and $\|\cdot\|_0 = \|\cdot\|$.

Proof. By definition (2.3) and an expansion in terms of the Hermite functionals, similar to (3.7), we can obtain

$$\left\| \int_0^t \mathcal{G}_{t-s} Q_s ds \right\|_m^2 = \sum_{\mathbf{n}} (1 + \alpha_{\mathbf{n}})^m \left\{ \int_0^t e^{-\beta_{\mathbf{n}}(t-s)} Q_s^{\mathbf{n}} ds \right\}^2, \tag{3.9}$$

where we let $Q_s^{\mathbf{n}} = [Q_s, H_{\mathbf{n}}]$. Since

$$\left\{ \int_0^t e^{-\beta_{\mathbf{n}}(t-s)} Q_s^{\mathbf{n}} ds \right\}^2 \leq \frac{1}{\beta_{\mathbf{n}}} \int_0^t e^{-\beta_{\mathbf{n}}(t-s)} |Q_s^{\mathbf{n}}|^2 ds,$$

the equation (3.9) yields

$$\begin{aligned} \left\| \int_0^t \mathcal{G}_{t-s} Q_s ds \right\|_m^2 &\leq \sum_{\mathbf{n}} \frac{(1 + \lambda_{\mathbf{n}})^m}{\beta_{\mathbf{n}}} \int_0^t e^{-\beta_{\mathbf{n}}(t-s)} |Q_s^{\mathbf{n}}|^2 ds \\ &\leq \frac{1}{\alpha_1} \int_0^t e^{-\beta_{\mathbf{n}}(t-s)} \|Q_s\|_{m-1}^2 ds, \end{aligned} \tag{3.10}$$

after changing the order of the summation and integration. The inequality (3.8) now follows from (3.9). \square

Lemma 3.3. The linear operator $\mathcal{A}_{\alpha} : \mathcal{H}_2 \rightarrow \mathcal{H}$, as defined by (3.2), has a bounded inverse $\mathcal{A}_{\alpha}^{-1}$ such that

$$\|\mathcal{A}_{\alpha}^{-1} \Theta\|^2 \leq \frac{1}{\alpha^2} \|\Theta\|^2, \quad \text{for } \Theta \in \mathcal{H}, \tag{3.11}$$

and

$$\|\mathcal{A}_{\alpha}^{-1} \Phi\|^2 \leq \frac{1}{\alpha \alpha_1} \|\Phi\|_{-1}^2, \quad \text{for } \alpha_1 = \min\{\alpha, 1\}, \Phi \in \mathcal{H}_{-1}. \tag{3.12}$$

Further, for any $\Theta \in \mathcal{H}$, the following equation holds

$$\lim_{t \rightarrow \infty} \int_0^t \mathcal{G}_s \Theta ds = -\mathcal{A}_{\alpha}^{-1} \Theta. \tag{3.13}$$

Proof. The linear operator is clearly invertible since, by equation (2.4),

$$\langle\langle -\mathcal{A}_\alpha \Phi, \Phi \rangle\rangle \geq \alpha_1 \|\Phi\|_1^2$$

for $\Phi \in \mathcal{H}_2$.

The inequalities (3.11) and (3.12) can be verified similarly by expanding Φ in terms of the Hermite functionals. So, for brevity, we will only show the latter one. We can get

$$\begin{aligned} \|\mathcal{A}_\alpha^{-1} \Phi\|^2 &= \sum_{\mathbf{n}} (\alpha + \lambda_{\mathbf{n}})^{-2} \Phi_{\mathbf{n}}^2 \\ &\leq \frac{1}{\alpha \alpha_1} \sum_{\mathbf{n}} (1 + \lambda_{\mathbf{n}})^{-1} \Phi_{\mathbf{n}}^2, \end{aligned}$$

or

$$\|\mathcal{A}_\alpha^{-1} \Phi\|^2 \leq \frac{1}{\alpha \alpha_1} \|\Phi\|_{-1}^2.$$

Since \mathcal{A}_α generates a contraction semigroup \mathcal{G}_t in \mathcal{H} , the equation (3.13) is known to be true in the semigroup theory [12]. \square

Now consider the Cauchy problem for the nonlinear evolution equation in \mathcal{H} in a distributional sense:

$$\begin{aligned} \frac{d}{dt} \Psi_t &= \mathcal{A}_\alpha \Psi_t + \mathcal{B}(\Psi_t) + Q_t, \quad t > 0, \\ \Psi_0 &= \Theta, \end{aligned} \tag{3.14}$$

where $\mathcal{B} : \mathcal{H}_1 \rightarrow \mathcal{H}$ is bounded and continuous. Then it follows from Theorem 4.2 [2] that the following theorem holds.

Theorem 3.4. *Suppose that \mathcal{A}_α is given as before, and \mathcal{B} satisfies the following conditions:*

- (1) $\|\mathcal{B}(\Phi)\|^2 \leq C_1 \{1 + |\Phi|^2 + |R^{1/2} D\Phi|^2\}$,
 - (2) $\|\mathcal{B}(\Phi) - \mathcal{B}(\Phi')\|^2 \leq C_2 \{|\Phi - \Phi'|^2 + |R^{1/2} D(\Phi - \Phi')|^2\}$,
- for some constants $C_1, C_2 > 0$ and for any $\Phi, \Phi' \in \mathcal{H}_1$.*

Then, for $\Theta \in \mathcal{H}$ and $Q \in L^2((0, \infty); \mathcal{H})$, the Cauchy problem (3.14) has a unique strong solution $\Psi \in \mathbf{C}([0, T]; \mathcal{H}) \cap L^2((0, T); \mathcal{H}_1)$, for any $T > 0$, so that the following equation holds for every $t \in [0, T]$, $\Phi \in \mathcal{H}_1$:

$$[\Psi_t, \Phi] = [\Theta, \Phi] + \int_0^t \langle\langle \mathcal{A}_\alpha \Psi_s, \Phi \rangle\rangle ds + \int_0^t [\mathcal{B}(\Psi_s), \Phi] ds + \int_0^t [Q_s, \Phi] ds.$$

Lemma 3.5. *Let F_t be a bounded continuous \mathcal{H} -valued function on $[0, \infty)$. Suppose there is $F_0 \in \mathcal{H}$ such that*

$$\lim_{t \rightarrow \infty} \|F(t) - F_0\| = 0.$$

Then, for any $\alpha > 0$, we have

$$\lim_{t \rightarrow \infty} \int_0^t e^{-\alpha(t-s)} \|F(s) - F_0\|^2 ds = 0. \tag{3.15}$$

Proof. Let $f_t = \|F(t) - F_0\|^2$. Then f_t is a bounded continuous real-valued function on $[0, \infty)$ with $f_t \geq 0$ and $f_t \rightarrow 0$ as $t \rightarrow \infty$. Thus the equation

$$\lim_{t \rightarrow \infty} \int_0^t e^{-\alpha(t-s)} f_s ds = 0,$$

can be verified easily. □

4. Linear Parabolic Equations

Consider the singularly perturbed form for the linear parabolic equation (3.1), which will be regarded as an evolution equation in \mathcal{H} (in a distributional sense):

$$\begin{aligned} \varepsilon \frac{d}{dt} \Psi_t^\varepsilon &= \mathcal{A}_\alpha \Psi_t^\varepsilon + Q_t^\varepsilon, \quad t > 0, \\ \Psi_0^\varepsilon &= \Theta, \end{aligned} \tag{4.1}$$

where $\varepsilon \in (0, 1]$ is a small parameter and Q_t^ε is a given \mathcal{H} -valued function depending on ε . Therefore the solution, denoted by Ψ_t^ε , also depends on ε . Suppose that Q_t^ε is bounded and continuous in $t \in [0, \infty)$ for each $\varepsilon \in (0, 1]$, and there exists $Q \in \mathcal{H}$ such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [\delta, T]} \|Q_t^\varepsilon - Q\| = 0, \tag{4.2}$$

for any finite interval $[\delta, T] \subset (0, \infty)$.

For a fixed $t > 0$ we are interested in the asymptotic behavior of the solution as $\varepsilon \rightarrow 0$. Suppose that, as $\varepsilon \rightarrow 0$, the solution Ψ_t^ε converges in \mathcal{H}_2 to a limit Ψ for $t > 0$. Then formally, in view of condition (4.2), the equation (4.1) reduces to the elliptic equation for Ψ :

$$\mathcal{A}_\alpha \Psi = -Q, \tag{4.3}$$

This result can be made precise by the following theorem.

Theorem 4.1. *Let $\Theta \in \mathcal{H}$ and let Q_t^ε be a \mathcal{H} -valued function on $[0, \infty)$ such that it is bounded and continuous for all $\varepsilon \in (0, 1]$. Suppose that conditions (A.1)–(A.3) and condition (4.2) hold. Then, for any $t > 0$, there exists $\Psi \in \mathcal{H}_2$ such that*

$$\lim_{\varepsilon \rightarrow 0} \|\Psi_t^\varepsilon - \Psi\| = 0, \tag{4.4}$$

and the limit is uniform in any finite interval $[a, b] \subset [\delta, \infty)$. Moreover the limit Ψ satisfies the elliptic equation (4.3).

Proof. In view of equation (3.3), the solution of equation (4.1) can be written as

$$\Psi_t^\varepsilon = [\mathcal{G}_t^\varepsilon \Theta] + \frac{1}{\varepsilon} \int_0^t [\mathcal{G}_{t-s}^\varepsilon Q_s^\varepsilon] ds, \tag{4.5}$$

where we set $\mathcal{G}_t^\varepsilon = \mathcal{G}_{t/\varepsilon}$. By Lemma (3.1), we have, for $\Theta \in \mathcal{H}$,

$$\|\mathcal{G}_t^\varepsilon \Theta\| \leq e^{-\alpha t/\varepsilon} \|\Theta\|$$

so that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{G}_t^\varepsilon \Theta = 0. \tag{4.6}$$

Let

$$U_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t \mathcal{G}_{t-s}^\varepsilon Q_s^\varepsilon ds, \tag{4.7}$$

and

$$V_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t \mathcal{G}_{t-s}^\varepsilon Q ds = \int_0^{t/\varepsilon} \mathcal{G}_s Q ds. \tag{4.8}$$

By (3.13) in Lemma 3.3, we get

$$\lim_{\varepsilon \rightarrow 0} V_t^\varepsilon = \lim_{t \rightarrow \infty} \int_0^t \mathcal{G}_s Q ds = -\mathcal{A}_\alpha^{-1} Q, \quad Q \in \mathcal{H}. \tag{4.9}$$

By invoking Lemma (3.2), we can deduce from (4.7) and (4.8) that

$$\begin{aligned} & \|U_t^\varepsilon - V_t^\varepsilon\| \leq \frac{1}{\varepsilon} \int_0^t e^{-\alpha(t-s)/\varepsilon} \|Q_s^\varepsilon - Q\| ds \\ &= \frac{1}{\varepsilon} \int_0^{\varepsilon T} e^{-\alpha(t-s)/\varepsilon} \|Q_s^\varepsilon - Q\| ds + \frac{1}{\varepsilon} \int_{\varepsilon T}^t e^{-\alpha(t-s)/\varepsilon} \|Q_s^\varepsilon - Q\| ds \\ &= \int_0^T e^{-\alpha\tau} \|Q_{t-\varepsilon\tau}^\varepsilon - Q\| d\tau + \int_T^{t/\varepsilon} e^{-\alpha(t-s)} \|Q_{\varepsilon s}^\varepsilon - Q\| ds \\ &\leq \int_0^T e^{-\alpha\tau} \|Q_{t-\varepsilon\tau}^\varepsilon - Q\| d\tau + \frac{M}{\alpha} e^{-\alpha T}, \end{aligned} \tag{4.10}$$

where $M = \sup_{t \geq 0, 0 < \varepsilon \leq 1} (\|Q_t^\varepsilon\| + \|Q\|)$.

For any $\eta > 0$ and $t \in [a, b]$, choose $T > b$ such that

$$\frac{M}{\alpha} e^{-\alpha T} < \eta/2.$$

For this fixed T , we let ε be so small that $(a - \varepsilon T) \leq \delta$. Then we have

$$\sup_{a \leq t \leq b} \int_0^T e^{-\alpha\tau} \|Q_{t-\varepsilon\tau}^\varepsilon - Q\| d\tau \leq \frac{1}{\alpha} \sup_{\delta \leq t \leq T} \|Q_t^\varepsilon - Q\|,$$

which, in view of condition (4.2), can be made less than $\eta/2$ for a sufficiently small ε . Hence it follows from (4.10) and the above estimates that, for any $\eta > 0$, there exists $\varepsilon_0 > 0$ such that

$$\sup_{a \leq t \leq b} \|U_t^\varepsilon - V_t^\varepsilon\| < \eta,$$

for $0 < \varepsilon < \varepsilon_0$. That is, by noting (4.9),

$$\lim_{\varepsilon \rightarrow 0} U_t^\varepsilon = \lim_{\varepsilon \rightarrow 0} V_t^\varepsilon = -\mathcal{A}_\alpha^{-1} Q, \tag{4.11}$$

for any $t \in [a, b]$. By taking (4.6), (4.7) and (4.11) into account, we deduce from (4.5) that, for $t > 0$, the limit (4.2) holds true and Ψ_t^ε converges to $\Psi = -\mathcal{A}_\alpha^{-1} Q \in \mathcal{H}_2$ as $\varepsilon \rightarrow 0$ so that it satisfies equation (4.3) as asserted. \square

Now let us consider the stationary solution of the following Cauchy problem as $t \rightarrow \infty$:

$$\frac{d}{dt} \Phi_t = \mathcal{A}_\alpha \Phi_t + Q_t, \quad \Phi_0 = \Theta. \tag{4.12}$$

As it turns out, this can be reformulated as a singular perturbation problem so that the above theorem is applicable. In fact we have the following theorem.

Theorem 4.2. *Let $Q_t \in \mathcal{H}$ be bounded and continuous on $[0, \infty)$ such that the following condition holds in \mathcal{H}*

$$\lim_{t \rightarrow \infty} Q_t = Q. \tag{4.13}$$

Then under conditions (A.1)–(A.3), for any $\Theta \in \mathcal{H}$, there exists the limit

$$\lim_{t \rightarrow \infty} \Phi_t = \Psi, \tag{4.14}$$

and Ψ satisfies the elliptic equation (4.3).

Proof. Let us re-scale the time t by letting $\tau = \varepsilon t$. We set $\Phi_t = \Phi_{\tau/\varepsilon} \doteq \Psi_\tau^\varepsilon$ and $Q_t = Q_{\tau/\varepsilon} \doteq Q_\tau^\varepsilon$. Then, after the change of time, the equation (4.12) yields equation (4.1). Notice that, for a fixed τ , the limit $\varepsilon \rightarrow 0$ implies that $t \rightarrow \infty$. Therefore it is enough to show that, for a fixed $\tau > 0$,

$$\lim_{\varepsilon \rightarrow 0} \Psi_\tau^\varepsilon = \Psi.$$

By definition and condition (4.13), we have

$$\lim_{\varepsilon \rightarrow 0} \sup_{a \leq \tau \leq b} \| \|Q_\tau^\varepsilon - Q\| \| = \lim_{\varepsilon \rightarrow 0} \sup_{a \leq \tau \leq b} \| \|Q_{\tau/\varepsilon} - Q\| \| = \lim_{t \rightarrow \infty} \| \|Q_t - Q\| \| = 0.$$

This shows that the condition (4.2) for Theorem 4.1 is fulfilled. So we can apply this theorem to conclude that the limit (4.14) exists and it satisfies equation (4.3). \square

Remark 4.3. Two special cases for the function Q_t^ε are of special interest. For $Q_t^\varepsilon = Q_{t/\varepsilon}$, the condition (4.2) yields

$$Q = \lim_{\varepsilon \rightarrow 0} Q_t^\varepsilon = \lim_{t \rightarrow \infty} Q_t = Q_\infty,$$

while, for $Q_t^\varepsilon = Q_{\varepsilon t}$,

$$Q = \lim_{\varepsilon \rightarrow 0} Q_t^\varepsilon = \lim_{t \rightarrow 0} Q_t = Q_0.$$

5. Semilinear Parabolic Equations

Now consider the Cauchy problem for the nonlinear parabolic equation:

$$\begin{aligned} \varepsilon \frac{\partial}{\partial t} \Psi_t^\varepsilon(v) &= \mathcal{A}_\alpha \Psi_t^\varepsilon(v) + \mathcal{F}(v, \Psi_t^\varepsilon, D\Psi_t^\varepsilon) + Q_t^\varepsilon(v), & 0 < t < T, \\ \Psi_0(v) &= \Phi(v), \end{aligned} \tag{5.1}$$

where, under suitable conditions, the nonlinear term $\mathcal{F} : V \times \mathbb{R} \times H \rightarrow \mathcal{H}$ can be defined μ -a.e.. Similar to the linear case (4.1), we regard (5.1) as the following evolution equation in \mathcal{H} :

$$\begin{aligned} \varepsilon \frac{d}{dt} \Psi_t^\varepsilon &= \mathcal{A}_\alpha \Psi_t^\varepsilon + \mathcal{B}(\Psi_t^\varepsilon) + Q_t^\varepsilon, & t > 0, \\ \Psi_0^\varepsilon &= \Theta, \end{aligned} \tag{5.2}$$

where we let $\mathcal{B}(\phi) = \mathcal{F}(\cdot, \phi, D\phi)$. As before we are interested in proving the convergence of the solution Ψ_t^ε to a limit as $\varepsilon \rightarrow 0$ and the related stationary

solution. Unlike the linear case, under certain conditions on \mathcal{B} and Q_t^ε , we can only show that the solution Ψ_t^ε converges to a limit Ψ in a weaker sense and it is a mild solution of the stationary equation:

$$\mathcal{A}_\alpha \Psi + \mathcal{B}(\Psi) + Q = 0. \tag{5.3}$$

To proceed we rewrite the equation (5.3) in the integral form:

$$\Psi_t = \mathcal{G}_t \Theta + \int_0^t \mathcal{G}_{t-s} \mathcal{B}(\Psi_s^\varepsilon) ds + \int_0^t \mathcal{G}_{t-s} Q_s ds. \tag{5.4}$$

In particular we impose the following conditions:

- (B.1) Let $\mathcal{B}(\cdot) : \mathcal{H}_1 \rightarrow \mathcal{H}$ a continuous mapping with $\mathcal{B}(0) = 0$. Suppose there exist positive constants b_1, b_2 , such that $b_2 < \sqrt{b_1} < \alpha$ and

$$\|\mathcal{B}(\phi) - \mathcal{B}(\psi)\|^2 \leq b_1 \|\phi - \psi\|^2 + b_2 \|R^{\frac{1}{2}} D(\phi - \psi)\|,$$

for any $\phi, \psi \in \mathcal{H}_1$.

- (B.2) The map \mathcal{B} can be extended to be a continuous operator from \mathcal{H} into \mathcal{H}_{-1} such that, for some constant $\kappa > 0$, the following holds

$$\|\mathcal{B}(\phi) - \mathcal{B}(\psi)\|_{-1} \leq \kappa \|\phi - \psi\|,$$

for any $\phi, \psi \in \mathcal{H}$.

- (B.3) Let $Q_t^\varepsilon = Q_{t/\varepsilon}$ such that Q_t is a bounded continuous \mathcal{H} -valued function on $[0, \infty)$ with the limit

$$\lim_{t \rightarrow \infty} Q_t = Q.$$

Similar to the linear case, by a simple change of the time-scale, we define $\Phi_t = \Psi_{\varepsilon t}^\varepsilon$. Then it follows from equation (5.2) and condition (B.3), that Φ_t satisfies the equation:

$$\begin{aligned} \frac{d}{dt} \Phi_t &= \mathcal{A}_\alpha \Phi_t + \mathcal{B}(\Phi_t) + Q_t, \quad t > 0, \\ \Phi_0 &= \Theta, \end{aligned} \tag{5.5}$$

To show $\Psi_t^\varepsilon \rightarrow \Psi$ as $\varepsilon \rightarrow 0$, it suffices to prove that $\lim_{t \rightarrow \infty} \Phi_t = \Psi$. To this end we shall first present some lemmas. In what follows, the conditions (A.1)–(A.3) are always assumed to be true without further mentioning.

Lemma 5.1. *Suppose the conditions (B.1) to (B.3) hold. Then, for $\Theta \in \mathcal{H}$, the Cauchy problem (5.5) has a strong solution $\Phi \in \mathbf{C}([0, T]; \mathcal{H}) \cap L^2((0, T); \mathcal{H}_1)$. Moreover the following energy equation holds*

$$\begin{aligned} \frac{d}{dt} \|\Phi_t\|^2 &= -\|R^{1/2} D\Phi_t\|^2 - 2\alpha \|\Phi_t\|^2 \\ &\quad + 2[\mathcal{B}(\Phi_t), \Phi_t] + 2[Q_t, \Phi_t], \\ \|\Phi_0\|^2 &= \|\Theta\|^2. \end{aligned} \tag{5.6}$$

Proof. Under the conditions (B.1)–(B.3), we can apply Theorem 3.4 to assert the existence of a unique strong solution with the indicated regularity property. So we have $\Phi_t \in \mathcal{H}_1$ and $\frac{d}{dt}\Phi_t \in \mathcal{H}_{-1}$. The equation (5.6) can be derived easily from (5.5). We have

$$\left\langle \left\langle \frac{d}{dt}\Phi_t, \Phi_t \right\rangle \right\rangle = \left\langle \left\langle \mathcal{A}_\alpha \Phi_t, \Phi_t \right\rangle \right\rangle + [\mathcal{B}(\Phi_t), \Phi_t] + [\Phi_t, Q_t],$$

which yields equation (5.6) with the aid of the integration by parts formula given in Theorem 2.3. \square

Lemma 5.2. *As in Lemma 5.1, under conditions (B.1)–(B.3), the the following inequality holds*

$$\|\Phi_t\|^2 \leq \|\Theta\|^2 e^{-\gamma t} + \frac{1}{\gamma} \int_0^t e^{-\gamma(t-s)} \|Q_s\|^2 ds, \quad (5.7)$$

where $\gamma = (\alpha - \sqrt{b_1})$.

Proof. For any $\beta, \gamma > 0$, we can deduce from equation (5.6) that

$$\begin{aligned} \frac{d}{dt} \|\Phi_t\|^2 &\leq -\|R^{1/2} D\Phi_t\|^2 - 2\alpha \|\Phi_t\|^2 + (\beta \|\Phi_t\|^2 + \frac{1}{\beta} \|\mathcal{B}(\Phi_t)\|^2) \\ &\quad + (\gamma \|\Phi_t\|^2 + \frac{1}{\gamma} \|Q_t\|^2), \end{aligned}$$

which, by invoking condition (B.1), yields the following

$$\begin{aligned} \frac{d}{dt} \|\Phi_t\|^2 &\leq -(1 - \frac{b_2}{\beta}) \|R^{1/2} D\Phi_t\|^2 - (2\alpha - \beta - \frac{b_1}{\beta} - \gamma) \|\Phi_t\|^2 \\ &\quad + \frac{1}{\gamma} \|Q_t\|^2. \end{aligned} \quad (5.8)$$

Now we set $\beta = \sqrt{b_1}$ and $\gamma = \alpha - \sqrt{b_1}$ in (5.8) to get

$$\begin{aligned} \frac{d}{dt} \|\Phi_t\|^2 &\leq -(1 - \frac{b_2}{\sqrt{b_1}}) \|R^{1/2} D\Phi_t\|^2 - (\alpha - \sqrt{b_1}) \|\Phi_t\|^2 + \frac{1}{\gamma} \|Q_t\|^2 \\ &\leq -\gamma \|\Phi_t\|^2 + \frac{1}{\gamma} \|Q_t\|^2, \end{aligned}$$

which implies the desired result (5.7). \square

To show the convergence of Φ_t as $t \rightarrow \infty$, we shall adopt a backward asymptotic approach which was used effectively in the stochastic case (Lemma 11.2.2, [8]). To this end, by extending the initial time backward in (5.5), consider the Cauchy problem:

$$\begin{aligned} \frac{d}{dt} \Phi_t(s) &= \mathcal{A}_\alpha \Phi_t(s) + \mathcal{B}(\Phi_t(s)) + \widehat{Q}_t, \quad t > -s, \\ \Phi_{-s}(s) &= \Theta, \quad \text{for } s \geq 0, \end{aligned} \quad (5.9)$$

where the extended function \widehat{Q}_t is defined by $\widehat{Q}_t = Q_t$ for $t \geq 0$, and $\widehat{Q}_t = Q_{-t}$ for $t < 0$. Then we have $\Phi_t = \Phi_0(t)$. Hence $\Phi_0(t) \rightarrow \Psi$ implies $\Phi_t \rightarrow \Psi$ as $t \rightarrow \infty$.

Theorem 5.3. *Suppose the conditions (B.1)–(B.3) are satisfied. For $\Theta \in \mathcal{H}$, the solution Φ_t of the Cauchy problem (5.9) converges to a limit $\Psi \in \mathcal{H}$ independent of Θ as $t \rightarrow \infty$.*

Proof. Consider the Cauchy problem (5.9). In view of (5.7) in Lemma 5.2, we shift the initial time 0 to $-s$ to get

$$\begin{aligned} \|\Phi_t(s)\|^2 &\leq \|\Theta\|^2 e^{-\gamma(t+s)} + \frac{1}{\alpha} \int_{-s}^t e^{-\gamma(t-\tau)} \|\widehat{Q}_\tau\|^2 d\tau \\ &\leq \|\Theta\|^2 + \frac{M}{\gamma} \int_{-s}^t e^{-\gamma(t-\tau)} d\tau \leq \|\Theta\|^2 + \frac{M}{\alpha\gamma}, \end{aligned} \tag{5.10}$$

where $M = \sup_{t \geq 0} \|Q_t\|^2$. It follows from (5.10) that, for any $t \geq 0$,

$$\sup_{s > 0} \|\Phi_t(s)\|^2 < \infty. \tag{5.11}$$

Now, for $r > s$, let $\Phi_t(r)$ be the solution of (5.9) with s replaced by r and define

$$\Phi_t(s, r) = \Phi_t(s) - \Phi_t(r).$$

Then it satisfies the equation

$$\begin{aligned} \frac{d}{dt} \Phi_t(s, r) &= \mathcal{A}_\alpha \Phi_t(s, r) + [\mathcal{B}(\Phi_t(s)) - \mathcal{B}(\Phi_t(r))] \quad t > -s, \\ \Phi_{-s}(s, r) &= \Theta - \Phi_s(r), \quad \text{for } s \geq 0. \end{aligned} \tag{5.12}$$

As in Lemma 5.1, we can obtain from (5.12) the following equation

$$\begin{aligned} \frac{d}{dt} \|\Phi_t(s, r)\|^2 &= -\|R^{1/2} D\Phi_t(s, r)\|^2 - 2\alpha \|\Phi_t(s, r)\|^2 \\ &\quad + 2[\mathcal{B}(\Phi_t(s)) - \mathcal{B}(\Phi_t(r)), \Phi_t(s) - \Phi_t(r)], \end{aligned} \tag{5.13}$$

$$\|\Phi_s(s, r)\|^2 = \|\Theta - \Phi_s(r)\|^2.$$

In view of condition (B.1), similar to (5.8), equation (5.13) yields the following inequality

$$\begin{aligned} \frac{d}{dt} \|\Phi_t(s, r)\|^2 &\leq -\left(1 - \frac{b_2}{\sqrt{b_1}}\right) \|R^{1/2} D\Phi_t(s, r)\|^2 - (\alpha - \sqrt{b_1}) \|\Phi_t(s, r)\|^2 \\ &\leq -\gamma \|\Phi_t\|^2, \end{aligned}$$

which implies

$$\begin{aligned} \|\Phi_t(s, r)\|^2 &\leq \|\Theta - \Phi_s(r)\|^2 e^{-\gamma(t+s)} \\ &\leq 2(\|\Theta\|^2 + \|\Phi_s(r)\|^2) e^{-\gamma(t+s)}. \end{aligned}$$

Therefore, in view of (5.11), we have, for any $t \geq 0, r > s$,

$$\lim_{s \rightarrow \infty} \|\Phi_t(s, r)\|^2 = \lim_{r, s \rightarrow \infty} \|\Phi_t(s) - \Phi_t(r)\|^2 = 0. \tag{5.14}$$

In particular we set $t = 0$ in (5.11) and (5.14) to conclude that the family $\{\Phi_s = \Phi_0(s) : t > 0\}$ yields a Cauchy sequence in \mathcal{H} so that

$$\lim_{t \rightarrow \infty} \Phi_t = \Psi. \quad (5.15)$$

To show the limit Ψ is independent of the initial state Θ , suppose that $\Phi_0 = \Theta' \neq \Theta$ and denote the corresponding solution of the Cauchy problem (5.5) by Φ'_t . Then, similar to the estimate in Lemma 5.2, we can show that the difference $(\Phi_t - \Phi'_t)$ has the following bound:

$$\|(\Phi_t - \Phi'_t)\|^2 \leq C\|\Theta - \Theta'\|^2 e^{-\gamma t}$$

for some $C > 0$, which implies that $\Psi = \Psi'$ and the limit is independent of the initial state as claimed. \square

Returning to the singular perturbation problem (5.1) or (5.2), the following theorem asserts that, as $\varepsilon \downarrow 0$, the solution Ψ_t^ε converges to the mild solution of the stationary equation.

Theorem 5.4. *Suppose that conditions (B.1)–(B.3) hold. Then, for any $t > 0$ and $\Theta \in \mathcal{H}$, the solution Ψ_t^ε of the Cauchy problem (5.1) converges to the limit:*

$$\lim_{\varepsilon \rightarrow 0} \Psi_t^\varepsilon = \Psi, \quad (5.16)$$

and Ψ is the mild solution of (5.3) which satisfies the following equation:

$$\Psi = -\mathcal{A}_\alpha^{-1} [\mathcal{B}(\Psi) + Q], \quad (5.17)$$

where $\mathcal{A}_\alpha^{-1} \mathcal{B}(\cdot) \doteq \mathcal{A}_\alpha^{-1} \circ \mathcal{B}(\cdot)$ is a bounded linear operator on \mathcal{H} . Moreover the solution of equation (5.17) is unique if, in condition (B.2), $\kappa < \sqrt{\alpha\alpha_1}$ with $\alpha_1 = \min\{\alpha, 1\}$.

Proof. It follows from Theorem 5.3 that, by re-scaling the time, we can conclude that the limit given by (5.16) exists. To verify the equation (5.17), again we work with the re-scaled equation (5.5) and show that the limit (5.15) satisfies (5.17). To this end we rewrite (5.5) as the integral equation (5.6). We claim that equation (5.17) follows from equation (5.6) by taking the limit term-wise as $t \rightarrow \infty$. Clearly we have $\Phi_t \rightarrow \Psi$ and $\mathcal{G}_t \Theta \rightarrow 0$ by Lemma 3.1. Similar to the proof of equation (4.11), we can show that

$$\lim_{t \rightarrow \infty} \int_0^t \mathcal{G}_{t-s} Q_s ds = -\mathcal{A}_\alpha^{-1} Q.$$

So it remains to prove that $\mathcal{A}_\alpha^{-1} \mathcal{B}$ is bounded and

$$\lim_{t \rightarrow \infty} \int_0^t \mathcal{G}_{t-s} \mathcal{B}(\Phi_s) ds = -\mathcal{A}_\alpha^{-1} \mathcal{B}(\Psi), \quad (5.18)$$

The boundedness follows from Lemma 3.3 and condition (B.2) as follows

$$\begin{aligned} \|\mathcal{A}_\alpha^{-1} \mathcal{B}(\Psi)\| &\leq \frac{1}{\sqrt{\alpha\alpha_1}} \|\mathcal{B}(\Phi)\|_{-1} \\ &\leq \frac{\kappa}{\sqrt{\alpha\alpha_1}} \|\Phi\|. \end{aligned}$$

To show (5.18), we rewrite it as

$$\int_0^t \mathcal{G}_{t-s} \mathcal{B}(\Psi_s) ds = \int_0^t \mathcal{G}_{t-s} \mathcal{B}(\Psi) + \int_0^t \mathcal{G}_{t-s} [\mathcal{B}(\Phi_s) - \mathcal{B}(\Psi)] ds. \tag{5.19}$$

Similar to (4.9), by means of Lemma 3.3, we get

$$\lim_{t \rightarrow \infty} \int_0^t \mathcal{G}_{t-s} \mathcal{B}(\Psi) ds = -\mathcal{A}_\alpha^{-1} \mathcal{B}(\Psi). \tag{5.20}$$

Now, by making use Lemma 3.2 and condition (B.2), we can obtain

$$\begin{aligned} & \int_0^t \|\mathcal{G}_{t-s} [\mathcal{B}(\Phi_s) - \mathcal{B}(\Psi)]\|^2 ds \\ & \leq \frac{1}{\alpha} \int_0^t e^{-\alpha(t-s)} \|\mathcal{B}(\Phi_s) - \mathcal{B}(\Psi)\|_{-1}^2 ds \\ & \leq \frac{\kappa}{\alpha} \int_0^t e^{-\alpha(t-s)} \|\Phi_s - \Psi\|^2 ds, \end{aligned} \tag{5.21}$$

which, by noting Lemma 3.5, converges to zero as $t \rightarrow \infty$. In view of equations (5.19)–(5.21), the result (5.18) is verified. Therefore the equations (5.16) and (5.17) hold true.

To show the uniqueness of solution to equation (5.17), let Φ be another solution. Then we have

$$(\Phi - \Psi) = \mathcal{A}_\alpha^{-1} [\mathcal{B}(\Phi) - \mathcal{B}(\Psi)],$$

from which, with the aid of Lemma 3.3 and condition (B.2), we can deduce that

$$\begin{aligned} \|\Phi - \Psi\| &= \|\mathcal{A}_\alpha^{-1} [\mathcal{B}(\Phi) - \mathcal{B}(\Psi)]\| \\ &\leq \frac{1}{\sqrt{\alpha\alpha_1}} \|\mathcal{B}(\Phi) - \mathcal{B}(\Psi)\|_{-1} \\ &\leq \frac{\kappa}{\sqrt{\alpha\alpha_1}} \|\Phi - \Psi\|. \end{aligned}$$

Since $\kappa/\sqrt{\alpha\alpha_1} < 1$ by assumption, the above inequality implies $\|\Phi - \Psi\| = 0$ and hence the uniqueness. \square

As seen from the above theorem, due to the possible dependence of the nonlinear function $\mathcal{B}(\Phi)$ on the gradient $D\Phi$, we have shown that the limit function Ψ is only a mild solution of the stationary equation. However, if it is independent of $D\Phi$, the limit Ψ may become a classical solution. To show this possibility, instead of conditions (B.1) and (B.2), we assume that

(B.4) Let $\mathcal{B}(\cdot) : \mathcal{H} \rightarrow \mathcal{H}$ be a continuous mapping with $\mathcal{B}(0) = 0$. Suppose there exists a positive constant b_1 such that $b_1 < \alpha^2$ and

$$\|\mathcal{B}(\phi) - \mathcal{B}(\psi)\|^2 \leq b_1 \|\phi - \psi\|^2,$$

for any $\phi, \psi \in \mathcal{H}$.

Then it follows from Theorem 5.4 that the following result holds.

Theorem 5.5. *Suppose that conditions (B.3) and (B.4) hold. Then, for any $t > 0$ and $\Theta \in \mathcal{H}$, the solution Ψ_t^ε of the Cauchy problem (5.1) converges to a limit $\Psi \in \mathcal{H}_2$ as $\varepsilon \rightarrow 0$ and Ψ satisfies the following equation:*

$$\mathcal{A}_\alpha \Psi + \mathcal{B}(\Psi) = -Q. \quad (5.22)$$

Furthermore the solution of the above equation is unique.

Proof. Given condition (B.4), following a similar line of proof for Theorem 5.4, we can show that the solution Ψ_t^ε of the problem (5.1) converges to $\Psi \in \mathcal{H}$ as $\varepsilon \rightarrow 0$. Moreover it is a mild solution that satisfies (5.17). Since $\mathcal{B}(\psi) \in \mathcal{H}$, $Q \in \mathcal{H}$ and $\mathcal{A}_\alpha^{-1} : \mathcal{H} \rightarrow \mathcal{H}_2$, the equation (5.17) shows $\Psi \in \mathcal{H}_2$. So we can apply the operator \mathcal{A}_α to (5.17) to obtain the desired equation (5.22).

For the uniqueness question, suppose Φ is another solution of equation (5.22). Then the difference $\Theta = \Psi - \Phi$ satisfies the equation:

$$\mathcal{A}_\alpha \Theta + [\mathcal{B}(\Theta + \Phi) - \mathcal{B}(\Phi)] = 0.$$

It follows that

$$\langle \langle -\mathcal{A}_\alpha \Theta, \Theta \rangle \rangle = [\mathcal{B}(\Theta + \Phi) - \mathcal{B}(\Phi), \Theta]. \quad (5.23)$$

Since $\langle \langle -\mathcal{A}_\alpha \Theta, \Theta \rangle \rangle \geq \alpha \|\Theta\|^2$, we can deduce from (5.23) and condition (B.4) that

$$\begin{aligned} \|\Theta\|^2 &\leq \frac{1}{\alpha} \|\mathcal{B}(\Theta + \Phi) - \mathcal{B}(\Phi)\| \|\Theta\| \\ &\leq \frac{\sqrt{b_1}}{\alpha} \|\Theta\|^2 < \|\Theta\|^2, \end{aligned}$$

which implies $\|\Theta\| = \|\Psi - \Phi\| = 0$ and hence the uniqueness. \square

6. Application

Consider the linear Cauchy problem:

$$\begin{aligned} \frac{\partial}{\partial t} \Phi_t(v) &= (\mathcal{A} - \alpha) \Phi_t(v) + U(v) \Phi_t(v) + Q_t(v), \quad t > 0, \\ \Phi_0(v) &= \Theta(v), \end{aligned} \quad (6.1)$$

where, as before,

$$\mathcal{A}\Psi(v) = \frac{1}{2} \text{Tr}[RD^2\Psi(v)] + \langle Av, D\Psi(v) \rangle,$$

and $U : H \rightarrow \mathbb{R}$ is a bounded continuous function. This equation is a special case of equation (5.5) with $\mathcal{B}(\Phi) = U\Phi$. Assume condition (B.3) and

$$\sup_{v \in \mathcal{H}} |U(v)|^2 = b_1 < \alpha^2. \quad (6.2)$$

Then it is easy to check that condition (B.4) holds. By applying Theorem 5.5, we can conclude that there is $\Psi \in \mathcal{H}_2$ such that

$$\lim_{t \rightarrow \infty} \Phi_t(v) = \Psi(v),$$

and Ψ is the unique strong solution of the stationary equation:

$$\mathcal{A}_\alpha \Psi(v) + U(v) \Psi(v) = -Q(v). \quad (6.3)$$

On the other hand, the solution of the Cauchy problem (6.1) has a probabilistic representation. To this end let $u_t(s, v)$ be the solution of the stochastic equation:

$$\begin{aligned} du_t &= Au_t dt + dW_t, \quad t \geq s \geq 0, \\ u_s &= v, \end{aligned} \tag{6.4}$$

and denote its solution by $u_t(s, v)$. Assume that $\Theta \in \mathbf{C}_b^2(\mathcal{H})$. Then, for $Q \equiv 0$, the corresponding homogeneous equation (6.1) can be represented by the Feynman-Kac formula [7] in terms of the solution of equation (6.4) as follows:

$$\Phi_t(v) = \mathbb{E} \left\{ \Theta(u_t(0, v)) \exp[-\alpha t + \int_0^t U(u_t(s, v)) ds] \right\} \tag{6.5}$$

and equation (6.3) becomes

$$\mathcal{A}_\alpha \Psi(v) + U(v) \Psi(v) = 0. \tag{6.6}$$

Since $\Psi \equiv 0$ is the unique solution of equation (6.6), it follows from (6.5) that

$$\lim_{t \rightarrow \infty} \mathbb{E} \left\{ \Theta(u_t(0, v)) \exp[-\alpha t + \int_0^t U(u_t(s, v)) ds] \right\} = 0. \tag{6.7}$$

For Q not being identically zero, as in the finite-dimensional case, the solution of equation (6.1) can be represented as

$$\begin{aligned} \Phi_t(v) &= \mathbb{E} \left\{ \Theta(u_t(0, v)) \exp[-\alpha t + \int_0^t U(u_t(s, v)) ds] \right\} \\ &+ \mathbb{E} \left\{ \int_0^t Q(u_t(r, s), s) \exp[-\alpha(t-s) + \int_s^t U(u_t(r, v)) dr] ds \right\}. \end{aligned} \tag{6.8}$$

Therefore we can deduce from (6.7) and (6.8) that the following limit exists,

$$\Psi(v) = \lim_{t \rightarrow \infty} \mathbb{E} \left\{ \int_0^t Q(u_t(r, s), s) \exp[-\alpha(t-s) + \int_s^t U(u_t(r, v)) dr] ds \right\}, \tag{6.9}$$

and Ψ is the unique solution of the stationary equation (6.3).

Remark 6.1. The right-hand side of equation (6.9) can be regarded as the asymptotic evaluation of a functional integral by means of the differential equation (6.3). This equation can be solved by the method of iterations. To proceed let it be rewritten as

$$\Psi = \mathcal{K}_\alpha \Psi + \Psi_0,$$

where $\Psi_0 = -\mathcal{A}_\alpha^{-1} Q$ and $\mathcal{K}_\alpha \Psi = -\mathcal{A}_\alpha^{-1} U \Psi$. By setting $\Psi_n = \mathcal{K}_\alpha \Psi_{n-1} + \Psi_0$, for $n = 1, 2, \dots$, we can obtain

$$\Psi_n = \sum_{j=0}^n (\mathcal{K}_\alpha)^j \Psi_0,$$

where $(\mathcal{K}_\alpha)^0 = I$, $(\mathcal{K}_\alpha)^1 = \mathcal{K}_\alpha$ and $(\mathcal{K}_\alpha)^j = \mathcal{K}_\alpha (\mathcal{K}_\alpha)^{j-1}$ for $j \geq 2$. By (3.11) in Lemma 3.3 and condition (6.2), we can show that $\|\mathcal{K}_\alpha \Psi\| \leq \sqrt{b_1}/\alpha \|\Psi\|$. Thus

the linear operator $\mathcal{K}_\alpha : \mathcal{H} \rightarrow \mathcal{H}$ is bounded. Since $\sqrt{b_1}/\alpha < 1$, the sequence Ψ_n converges in \mathcal{H} to the solution of (6.3) given by

$$\Psi = \sum_{j=0}^{\infty} (\mathcal{K}_\alpha)^j \Psi_0.$$

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