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APPLICABILITY OF MULTIPLICATIVE RENORMALIZATION METHOD FOR A CERTAIN FUNCTION

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ABSTRACT. We characterize the class of probability measures for which the multiplicative renormalization method can be applied for the function $h(x) = \frac{1}{\sqrt{1-x}}$ to obtain orthogonal polynomials. It turns out that this class consists of only uniform probability measures on intervals and probability measures being supported by one or two points.

1. Multiplicative Renormalization Method

Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders. Then we can apply the Gram-Schmidt orthogonalization process to the sequence $\{x^n\}_{n=0}^{\infty}$ to get an orthogonal sequence $\{P_n(x)\}_{n=0}^{\infty}$ in the real Hilbert space $L^2(\mathbb{R}, \mu)$. Here the leading coefficient of P_n is 1 for each n. It is well known that these polynomials satisfy the recursion formula:

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_{n-1}P_{n-1}(x), \quad n \geq 0, \tag{1.1}$$

where by convention $\omega_{-1} = 1$ and $P_{-1} = 0$. The numbers α_n, ω_n, $n \geq 0$, are known as the Jacobi–Szegö parameters of μ.

It is natural to ask whether there is a method for deriving $\{P_n(x)\}_{n=0}^{\infty}$ from μ. A method, called multiplicative renormalization method, has been introduced in [3, 4] to answer this question. This method starts with an analytic function $h(x)$ at 0. Then we define two functions

$$\theta(t) = \int_{\mathbb{R}} h(tx)\, d\mu(x), \tag{1.2}$$

$$\widetilde{\theta}(t, s) = \int_{\mathbb{R}} h(tx)h(sx)\, d\mu(x), \tag{1.3}$$

Theorem 1.1. [3, 4] *Let $\rho(t)$ be an analytic function at 0 with $\rho(0) = 0$ and $\rho'(0) \neq 0$. Then the function*

$$\Theta_\rho(t, s) := \frac{\widetilde{\theta}\big(\rho(t), \rho(s)\big)}{\theta\big(\rho(t)\big)\theta\big(\rho(s)\big)} \tag{1.4}$$

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defined in some neighborhood of $(0,0)$ *is a function of the product* ts *if and only if the multiplicative renormalization*

$$\psi(t,x) := \frac{h\big(\rho(t)x\big)}{\theta\big(\rho(t)\big)} \tag{1.5}$$

is an orthogonal polynomial generating function for μ*, namely, it has the power series expansion*

$$\psi(t,x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n, \tag{1.6}$$

where a_n*'s are nonzero constants and* $\{P_n(x)\}_{n=0}^{\infty}$ *is the above sequence.*

Thus for a given μ and a fixed function $h(x)$, we need to compute the functions $\theta(t)$ and $\widetilde{\theta}(t,s)$, and then find a function $\rho(t)$ such that the function $\Theta_\rho(t,s)$ defined by Equation (1.4) is a function of ts. In that case, we can expand the function $\psi(t,x)$ defined by Equation (1.5) as a power series in t to find the polynomials $P_n(x)$'s as given by Equation (1.6).

The method in Theorem 1.1 for deriving orthogonal polynomials $P_n(x)$'s from μ is called the *multiplicative renormalization method.* For convenience, we use the following definition from [8].

Definition 1.2. A probability measure μ is called *MRM-applicable* for a function $h(x)$ if there exists an analytic function $\rho(t)$ at 0 such that $\rho(0)=0$, $\rho'(0)\neq 0$, and the function $\Theta_\rho(t,s)$ in Equation in (1.4) is a function of ts.

The probability measures such as Gaussian, Poisson, gamma, uniform, arcsine, semi-circle, beta, and Pascal are MRM-applicable for functions of the form

$$h(x) = e^x, \quad h(x) = (1-x)^c,$$

where c is a constant. The resulting $P_n(x)$'s are the well-known classical orthogonal polynomials. For the derivation, see [3, 4]. Conversely, we have the following

Problem. *Given a fixed function* $h(x)$*, find all MRM-applicable probability measures* μ *for* $h(x)$*.*

Kubo solved this problem for the case $h(x) = e^x$ in [6] and showed that the class of MRM-applicable measures for $h(x) = e^x$ is the same as the Meixner class [1, 10]. On the other hand, we have recently solved this problem for the case $h(x) = (1-x)^{-1}$ in [8]. The class of MRM-applicable measures for $h(x) = (1-x)^{-1}$ includes those distributions obtained in [2, 5, 7, 9], in particular, the arcsine and semi-circle distributions.

The purpose of the present paper is to solve the above problem for the case when $h(x)$ is given by

$$h(x) = \frac{1}{\sqrt{1-x}}.$$

The result (see Theorem 5.4) is somewhat surprising because it says that this class of MRM-applicable probability measures consists of uniform probability measures on intervals, plus the degenerate ones being supported by one or two points.

Recall from [3, 4] that the uniform probability measure μ on the interval $[-1,1]$ is MRM-applicable for $h(x) = \frac{1}{\sqrt{1-x}}$ with the associated functions:

$$\theta(t) = \frac{2}{\sqrt{1-t}+\sqrt{1+t}}, \tag{1.7}$$

$$\widetilde{\theta}(t,s) = \begin{cases} \frac{1}{\sqrt{ts}} \log \frac{\sqrt{|s|}\sqrt{1+|t|}+\sqrt{|t|}\sqrt{1+|s|}}{\sqrt{|s|}\sqrt{1-|t|}+\sqrt{|t|}\sqrt{1-|s|}}, & \text{if } ts > 0, \\ \frac{1}{\sqrt{-ts}} \tan^{-1}\left[\frac{2\sqrt{-ts}\,(s-t)}{\left(s\sqrt{1-t^2}-t\sqrt{1-s^2}\right)\left(\sqrt{(1+s)(1-t)}+\sqrt{(1-s)(1+t)}\right)}\right], & \text{if } ts < 0, \\ \frac{4}{\left(\sqrt{1+t}+\sqrt{1-t}\right)\left(\sqrt{1+s}+\sqrt{1-s}\right)}, & \text{if } ts = 0, \end{cases}$$

$$\rho(t) = \frac{2t}{1+t^2}, \tag{1.8}$$

$$\Theta_\rho(t,s) = \begin{cases} \frac{1}{2\sqrt{ts}} \log \frac{1+\sqrt{ts}}{1-\sqrt{ts}}, & \text{if } ts > 0, \\ \frac{1}{2\sqrt{-ts}} \tan^{-1} \frac{2\sqrt{-ts}}{1+ts}, & \text{if } ts < 0, \\ 1, & \text{if } ts = 0, \end{cases}$$

$$\psi(t,x) = \frac{1}{\sqrt{1-2tx+t^2}}, \tag{1.9}$$

where $|t|, |s| < 1$ for all functions. The orthogonal polynomial generating function $\psi(t,x)$ has the power series expansion in t,

$$\psi(t,x) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{[n/2]} (-1)^n 2^{n-2k} \binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} x^{n-2k} \right] t^n.$$

(see [4] for the derivation.) Therefore, the polynomials $P_n(x)$'s are given by

$$P_n(x) = \frac{n!}{(2n-1)!!} \sum_{k=0}^{[n/2]} (-1)^n 2^{n-2k} \binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} x^{n-2k},$$

which, by Theorem 4.4 [4], is related to the Legendre polynomials $L_n(x)$ by

$$P_n(x) = \frac{n!}{(2n-1)!!} L_n(x).$$

Moreover, the Jacobi–Szegö parameters are given by

$$\alpha_n = 0, \quad \omega_n = \frac{(n+1)^2}{(2n+3)(2n+1)}, \quad n \geq 0. \tag{1.10}$$

2. Affine Transformation

In this section we prepare something to be used in Sections 5. From now on we will always assume that $h(x)$ is the function: $h(x) = \frac{1}{\sqrt{1-x}}$.

Let $\theta(t)$, $\widetilde{\theta}(t,s)$, $\rho(t), \Theta_\rho(t,s)$, and $\psi(t,x)$ be the associated functions of μ as defined in Section 1. Consider an affine transformation $\tau_{v,q}(x) = vx+q$, $v \neq 0$ and define a probability measure

$$\mu_{v,q} = \mu \circ \tau_{v,q}.$$

Obviously, $\mu_{v,q}$ has finite moments of all orders. It can be easily checked that the associated θ-, $\widetilde{\theta}$-, and Θ-functions of $\mu_{v,q}$ are given by

$$\theta_{v,q}(t) = \sqrt{\frac{v}{v+qt}}\,\theta\Big(\frac{t}{v+qt}\Big), \tag{2.1}$$

$$\widetilde{\theta}_{v,q}(t,s) = \sqrt{\frac{v}{v+qt}}\sqrt{\frac{v}{v+qs}}\,\widetilde{\theta}\Big(\frac{t}{v+qt}, \frac{s}{v+qs}\Big),$$

$$\Theta_r^{v,q}(t,s) = \frac{\widetilde{\theta}\Big(\frac{r(t)}{v+qr(t)}, \frac{r(s)}{v+qr(s)}\Big)}{\theta\Big(\frac{r(t)}{v+qr(t)}\Big)\,\theta\Big(\frac{r(s)}{v+qr(s)}\Big)}. \tag{2.2}$$

In view of Equation (2.2) we introduce a ρ-function $\rho_{v,q;\sigma}(t)$ for $\mu_{v,q}$ by

$$\frac{\rho_{v,q;\sigma}(t)}{v+q\rho_{v,q;\sigma}(t)} = \rho(\sigma t),$$

or equivalently,

$$\rho_{v,q;\sigma}(t) = \frac{v\rho(\sigma t)}{1-q\rho(\sigma t)}. \tag{2.3}$$

Here we have an extra parameter σ in order to allow the scaling of the variable t, which will be crucial in Section 5. By direct calculations, we can prove the following theorem and corollary.

Theorem 2.1. *With the function $r(t) = \rho_{v,q;\sigma}(t)$ for $\mu_{v,q}$ in Equation (2.2), the following equalities hold:*

$$\theta_{v,q}(\rho_{v,q;\sigma}(t)) = \sqrt{1-q\rho(\sigma t)}\,\theta\big(\rho(\sigma t)\big), \tag{2.4}$$

$$\Theta^{v,q}_{\rho_{v,q;\sigma}}(t,s) = \Theta_\rho(\sigma t, \sigma s). \tag{2.5}$$

Corollary 2.2. *The probability measure $\mu_{v,q}$ is MRM-applicable for $h(x) = \frac{1}{\sqrt{1-x}}$ with a ρ-function $\rho_{v,q;\sigma}(t)$ for some number σ if and only if μ is MRM-applicable for $h(x) = \frac{1}{\sqrt{1-x}}$ with a ρ-function $\rho(t)$. Moreover, their orthogonal polynomial generating functions, orthogonal polynomials, and Jacobi–Szegö parameters are related by*

$$\psi_{v,q;\sigma}(t,x) = \psi(\sigma t, vx+q), \tag{2.6}$$

$$P_n^{v,q;\sigma}(x) = \frac{1}{v^n}P_n(vx+q), \quad n \geq 0,$$

$$\alpha_n^{v,q;\sigma} = \frac{\alpha_n - q}{v}, \quad n \geq 0,$$

$$\omega_n^{v,q;\sigma} = \frac{\omega_n}{v^2}, \quad n \geq 0. \tag{2.7}$$

3. Key Lemmas

In this section, we will prove several lemmas for Section 4. Let $\widetilde{\theta}(t,s)$ denote the function defined by Equation (1.3) for $h(x) = \frac{1}{\sqrt{1-x}}$, namely,

$$\widetilde{\theta}(t,s) = \int_{\mathbb{R}} \frac{1}{\sqrt{1-tx}} \frac{1}{\sqrt{1-sx}} \, d\mu(x). \tag{3.1}$$

Lemma 3.1. *The function $\widetilde{\theta}(t,s)$ satisfies the partial differential equation*

$$\frac{\partial\widetilde{\theta}}{\partial t} - \frac{\partial\widetilde{\theta}}{\partial s} = 2(t-s)\frac{\partial^2\widetilde{\theta}}{\partial t\partial s}. \tag{3.2}$$

Proof. Differentiate Equation (3.1) with respect to s to get

$$2s\frac{\partial\widetilde{\theta}}{\partial s} = \int_{\mathbb{R}} \frac{1}{\sqrt{1-tx}} \frac{sx}{(1-sx)^{3/2}} \, d\mu(x). \tag{3.3}$$

Add up Equations (3.1) and (3.3) to show that

$$\widetilde{\theta} + 2s\frac{\partial\widetilde{\theta}}{\partial s} = \int_{\mathbb{R}} \frac{1}{\sqrt{1-tx}\,(1-sx)^{3/2}} \, d\mu(x). \tag{3.4}$$

Then differentiate Equation (3.4) with respect to t to check that

$$2t\Big(\frac{\partial\widetilde{\theta}}{\partial t} + 2s\frac{\partial^2\widetilde{\theta}}{\partial t\partial s}\Big) = \int_{\mathbb{R}} \frac{tx}{(1-tx)^{3/2}(1-sx)^{3/2}} \, d\mu(x). \tag{3.5}$$

Sum up Equations (3.4) and (3.5) to find that

$$\widetilde{\theta} + 2s\frac{\partial\widetilde{\theta}}{\partial s} + 2t\Big(\frac{\partial\widetilde{\theta}}{\partial t} + 2s\frac{\partial^2\widetilde{\theta}}{\partial t\partial s}\Big) = \int_{\mathbb{R}} \frac{1}{(1-tx)^{3/2}(1-sx)^{3/2}} \, d\mu(x). \tag{3.6}$$

Now use partial fractions to rewrite the integrand as

$$\frac{1}{(1-tx)^{3/2}(1-sx)^{3/2}} = \frac{1}{t-s}\Big(t\frac{1}{\sqrt{1-sx}(1-tx)^{3/2}} - s\frac{1}{\sqrt{1-tx}(1-sx)^{3/2}}\Big).$$

Then use Equation (3.4) to show that

$$\int_{\mathbb{R}} \frac{1}{(1-tx)^{3/2}(1-sx)^{3/2}} \, d\mu(x) = \widetilde{\theta} + \frac{1}{t-s}\Big(2t^2\frac{\partial\widetilde{\theta}}{\partial t} - 2s^2\frac{\partial\widetilde{\theta}}{\partial s}\Big). \tag{3.7}$$

Finally, put Equations (3.6) and (3.7) together to derive Equation (3.2). □

Notation 3.2. For convenience, we define a function J by

$$J(ts) = \Theta_\rho(t,s).$$

Lemma 3.3. *Assume that $\Theta_\rho(t,s)$ in Equation (1.4) is a function of ts. Then the function $f(t) = \theta\big(\rho(t)\big)$ satisfies the equation*

$$\frac{f'(t)}{f(t)} = \frac{2b\rho(t) + (a+bt)\rho'(t)}{1 - 2(a+bt)\rho(t)}, \tag{3.8}$$

where $a = \theta'(0)$ and $b = J'(0)/\rho'(0)$.

Proof. Let F denote the function $F(t,s)=\widetilde{\theta}\big(\rho(t),\rho(s)\big)$. Then by Lemma 3.1,

$$\rho'(s)\frac{\partial F}{\partial t}-\rho'(t)\frac{\partial F}{\partial s}=2\big(\rho(t)-\rho(s)\big)\frac{\partial^2 F}{\partial t\partial s}. \tag{3.9}$$

On the other hand, by Equation (1.4) and the assumption in this lemma,

$$F(t,s)=J(ts)f(t)f(s), \tag{3.10}$$

which yields the following partial derivatives,

$$\begin{aligned}\frac{\partial F}{\partial t}&=sJ'(ts)f(t)f(s)+J(ts)f'(t)f(s),\\ \frac{\partial F}{\partial s}&=tJ'(ts)f(t)f(s)+J(ts)f(t)f'(s),\end{aligned} \tag{3.11}$$

$$\begin{aligned}\frac{\partial^2 F}{\partial t\partial s}&=\Big\{J'(ts)f(t)f(s)+tsJ''(ts)f(t)f(s)+sJ'(ts)f(t)f'(s)\Big\}\\ &\quad+tJ'(ts)f'(t)f(s)+J(ts)f'(t)f'(s).\end{aligned} \tag{3.12}$$

Evaluate these derivatives at $s=0$ and note that $J(0)=f(0)=1$ to get

$$\frac{\partial F}{\partial t}\Big|_{s=0}=f'(t), \tag{3.13}$$

$$\frac{\partial F}{\partial s}\Big|_{s=0}=tJ'(0)f(t)+f(t)f'(0),$$

$$\frac{\partial^2 F}{\partial t\partial s}\Big|_{s=0}=J'(0)f(t)+tJ'(0)f'(t)+f'(t)f'(0). \tag{3.14}$$

Put the values of these derivatives into Equation (3.9) with $s=0$ and then simplify the resulting equation to obtain Equation (3.8). $\square$

Lemma 3.4. *Under the same assumption as that in Lemma 3.3, the function* $f(t)=\theta\big(\rho(t)\big)$ *also satisfies the equation*

$$\frac{f'(t)}{f(t)}=\frac{-3b+4(ab+c_3t)\rho(t)+(c_2+2abt+c_3t^2)\rho'(t)}{3a+c_1+3bt-2(c_2+2abt+c_3t^2)\rho(t)}, \tag{3.15}$$

where $c_1=\rho''(0)/\rho'(0)^2$, $c_2=f''(0)/\rho'(0)^2$, *and* $c_3=J''(0)/\rho'(0)^2$.

Proof. Differentiate Equation (3.9) with respect to s to get

$$\rho''(s)\frac{\partial F}{\partial t}+3\rho'(s)\frac{\partial^2 F}{\partial t\partial s}-\rho'(t)\frac{\partial^2 F}{\partial s^2}=2\big(\rho(t)-\rho(s)\big)\frac{\partial^3 F}{\partial t\partial s^2}. \tag{3.16}$$

Differentiate Equation (3.11) with respect to s and then put $s=0$ to find that

$$\frac{\partial^2 F}{\partial s^2}\Big|_{s=0}=t^2J''(0)f(t)+2tJ'(0)f(t)f'(0)+f(t)f''(0). \tag{3.17}$$

Similarly, differentiate Equation (3.12) with respect to s and then put $s=0$ to get

$$\begin{aligned}\frac{\partial^3 F}{\partial t\partial s^2}\Big|_{s=0}&=2tJ''(0)f(t)+t^2J''(0)f'(t)+2J'(0)f'(0)f(t)\\ &\quad+2tJ'(0)f'(0)f'(t)+f''(0)f'(t).\end{aligned} \tag{3.18}$$

Evaluate Equation (3.16) at $s = 0$ and use the values from Equations (3.13), (3.14), (3.17), and (3.18). Then we can easily simplify the resulting equality to derive Equation (3.15). □

Lemma 3.5. *Under the same assumption as that in Lemma 3.3, the function* $f(t) = \theta\big(\rho(t)\big)$ *also satisfies the equation*

$$\frac{f'(t)}{f(t)} = \frac{A\big(\rho(t), \rho'(t), t\big)}{B\big(\rho(t), t\big)}, \tag{3.19}$$

where A *and* B *are functions defined by*

$$\begin{aligned} A(X, Y, t) &= -4bc_1 - 10ab - 10c_3 t + 6\Big\{bc_2 + 2ac_3 t + c_6 t^2\Big\}X \\ &\quad + \Big\{c_5 + 3bc_2 t + 3ac_3 t^2 + c_6 t^3\Big\}Y \qquad (3.20) \\ B(X, t) &= c_4 + 4ac_1 + 5c_2 + (4bc_1 + 10ab)t + 5c_3 t^2 \\ &\quad - 2\Big\{c_5 + 3bc_2 t + 3ac_3 t^2 + c_6 t^3\Big\}X, \qquad (3.21) \end{aligned}$$

with c_1, c_2, c_3 *being the constants in Lemma 3.4 and* c_4, c_5, c_6 *given by*

$$c_4 = \rho'''(0)/\rho'(0)^3, \quad c_5 = f'''(0)/\rho'(0)^3, \quad c_6 = J'''(0)/\rho'(0)^3.$$

Proof. Just differentiate Equation (3.16) with respect to s and then evaluate at $s = 0$. However, the calculations are quite tedious and rather lengthy. Thus we omit the details. □

Note that from the above lemmas there are constants $a, b, c_1, c_2, c_3, c_4, c_5, c_6$. But there are restrictions on them. For example, they must satisfy the conditions in the next two lemmas. Additional conditions will be specified in Theorem 4.1.

Lemma 3.6. *Suppose* μ *is MRM-applicable for the function* $h(x) = \frac{1}{\sqrt{1-x}}$ *and has a support of at least three points. Then* $b \neq 0$ *and* $c_3 > 0$.

Proof. Use Notation 3.2 and Equations (1.4) and (1.6) to find that

$$\begin{aligned} J(ts) = \Theta_\rho(t, s) &= \frac{\widetilde{\theta}\big(\rho(t), \rho(s)\big)}{\theta\big(\rho(t)\big)\theta\big(\rho(s)\big)} \\ &= \int_{\mathbb{R}} \frac{h\big(\rho(t)x\big)h\big(\rho(t)x\big)}{\theta\big(\rho(t)\big)\theta\big(\rho(s)\big)}\, d\mu(x) \\ &= \int_{\mathbb{R}} \psi(t, x)\psi(s, x)\, d\mu(x) \\ &= \sum_{n=0}^{\infty} a_n^2 \|P_n\|^2 (ts)^n, \end{aligned}$$

where $\|P_n\|^2 = \int_{\mathbb{R}} P_n(x)^2\, d\mu(x)$. Therefore,

$$J'(0) = a_1^2\|P_1\|^2, \quad J''(0) = 2a_2^2\|P_2\|^2, \tag{3.22}$$

which are positive numbers since μ is assumed to be supported by at least three points. But we have $J'(0) = b\rho'(0)$ and $J''(0) = c_3\rho'(0)^2$. It follows that $b \neq 0$ and $c_3 > 0$. □

Lemma 3.7. *Suppose μ is MRM-applicable for the function $h(x) = \frac{1}{\sqrt{1-x}}$ and has a support of at least two points. Then the following inequality holds:*

$$c_2 > 3a^2 + ac_1.$$

Proof. First differentiate the function $\theta(t)$ and evaluate at $t = 0$ to get

$$\theta'(0) = a = \frac{1}{2}\int_{\mathbb{R}} x\, d\mu(x), \tag{3.23}$$

$$\theta''(0) = \frac{3}{4}\int_{\mathbb{R}} x^2\, d\mu(x). \tag{3.24}$$

Then differentiate the function $f(t) = \theta\big(\rho(t)\big)$ and evaluate at $t = 0$ to see that

$$f''(0) = \theta''(0)\rho'(0)^2 + \theta'(0)\rho''(0),$$

which yields that

$$c_2 = \frac{f''(0)}{\rho'(0)^2} = \theta''(0) + ac_1. \tag{3.25}$$

From Equations (3.23), (3.24), and (3.25), we have

$$\begin{aligned} c_2 - 3a^2 - ac_1 &= \theta''(0) - 3\theta'(0)^2 \\ &= \frac{3}{4}\Big\{\int_{\mathbb{R}} x^2\, d\mu(x) - \Big(\int_{\mathbb{R}} x\, d\mu(x)\Big)^2\Big\} \\ &= \frac{3}{4}\operatorname{var}(\mu). \end{aligned}$$

Note that $\operatorname{var}(\mu) > 0$ since μ is assumed to be supported by at least two points. Hence we must have $c_2 - 3a^2 - ac_1 > 0$ and the lemma is proved. □

4. Derivation of the ρ-function

Recall that we are assuming that μ is MRM-applicable for $h(x) = \frac{1}{\sqrt{1-x}}$ with a ρ-function $\rho(t)$. In this section we will derive the function $\rho(t)$ which is a candidate of the ρ-function satisfying the equations in the key Lemmas 3.3, 3.4, and 3.5.

Recall that $\frac{f'(t)}{f(t)}$ must satisfy Equations (3.8), (3.15), and (3.19). Replace $\rho(t)$ and $\rho'(t)$ by X and Y, respectively, in these equations and consider the following algebraic equations:

$$\frac{2bX + (a+bt)Y}{1 - 2(a+bt)X} = \frac{-3b + 4(ab + c_3t)X + (c_2 + 2abt + c_3t^2)Y}{3a + c_1 + 3bt - 2(c_2 + 2abt + c_3t^2)X}, \tag{4.1}$$

$$\frac{2bX + (a+bt)Y}{1 - 2(a+bt)X} = \frac{A(X,Y,t)}{B(X,t)}, \tag{4.2}$$

where $A(X,Y,t)$ and $B(X,t)$ are functions defined by Equations (3.20) and (3.21), respectively. Note that if $f(t) = \theta\big(\rho(t)\big)$ satisfies Equations (3.8), (3.15), and (3.19), then $X = \rho(t)$ and $Y = \rho'(t)$ must satisfy Equations (4.1) and (4.2).

Observe that if we multiply out the numerators and denominators in Equations (4.1) and (4.2), then we have two linear equations in X, Y, and X^2. Solve for X^2 from one of the equation and put it into the other equation to obtain a linear equation in X and Y. Upon putting $X = \rho(t)$ and $Y = \rho'(t)$, we have a first order linear differential equation, which can be solved to get $\rho(t)$. But this differential equation is rather complicated. Moreover, there are hidden conditions on the constants $a, b, c_1, c_2, c_3, c_4, c_5, c_6$, which seem to be very hard, if not impossible, to derive from this differential equation.

To overcome the above difficulties, we use another approach, namely, we will solve the algebraic equations (4.1) and (4.2) to get the solution $X = \rho(t)$. At the same time, we will derive the conditions on these eight constants. Then in the next section we will derive $f(t) = \theta\big(\rho(t)\big)$ and show that $\rho(t)$ and $f(t)$ satisfy Equations (3.8), (3.15), and (3.19).

Theorem 4.1. *Let μ be MRM-applicable for the function $h(x) = \frac{1}{\sqrt{1-x}}$. Assume that μ is supported by at least three points. Then the associated constants satisfy the following conditions:*

$$b \neq 0, \quad 4a + c_1 = 0, \quad a^2 + c_2 > 0, \quad c_3 = \frac{18}{5} b^2,$$
$$c_4 = 6(3a^2 - c_2), \quad c_5 = -3ac_2, \quad c_6 = \frac{162}{7} b^2.$$

Moreover, Equations (4.1) *and* (4.2) *have a unique solution $X = \rho(t), Y = \rho'(t)$ with $\rho(t)$ given by*

$$\rho(t) = \frac{3bt}{a^2 + c_2 + 6abt + 9b^2t^2}, \tag{4.3}$$

Proof. First note that by Lemmas 3.6 and 3.7 we must have the conditions $b \neq 0$ and $c_2 > 3a^2 + ac_1$. Solve Equation (4.1) for Y to get its solution denoted by $Y_1(X, t)$. Similarly, let $Y_2(X, t)$ be the solution of Equation (4.2) for Y. Note that we must have the equality $Y_1(0, 0) = Y_2(0, 0)$, which implies that

$$c_5 = \frac{1}{3}(-30a^3 - 10a^2c_1 - 4ac_1^2 + 25ac_2 + 4c_1c_2 + 3ac_4). \tag{4.4}$$

Under this condition, Equations (4.1) and (4.2) have a unique solution X, denoted by $X = \widetilde{\rho}(t)$, such that

$$Y_1(\widetilde{\rho}(t), t) = Y_2(\widetilde{\rho}(t), t).$$

Since $\widetilde{\rho}(t)$ is supposed to satisfy Equation (3.8), the equality $\widetilde{\rho}\,'(t) = Y_1\big(\widetilde{\rho}(t), t\big)$ must hold. In particular, $\widetilde{\rho}\,'(0) = Y_1(0, 0)$, which implies that

$$c_4 = \frac{2}{9b^2}(15a^2b^2 + 3ab^2c_1 + 6b^2c_1^2 - 9b^2c_2 + 15a^2c_3 + 5ac_1c_3 - 5c_2c_3). \tag{4.5}$$

On the other hand, differentiate the solution $\widetilde{\rho}(t)$ and then subtract $Y_1\big(\widetilde{\rho}(t), t\big)$ from the resulting derivative $\widetilde{\rho}\,'(t)$ (very complicated and lengthy) to derive the numerator of $\widetilde{\rho}\,'(t) - Y_1\big(\widetilde{\rho}(t), t\big)$ as given by

$$3b^2t^2\big\{K_0 + K_1t + K_2t^2 + K_3t^3 + K_4t^4\big\},$$

where K_i, $i = 1, 2, 3, 4$, are the following numbers,

$$\begin{aligned}
K_0 &= 10c_3(3a^2 + ac_1 - c_2)\big\{561a^2b^2c_3 + 123ab^2c_1c_3 - 24b^2c_1^2c_3 - 315b^2c_2c_3 \\
&\quad + 750a^2c_3^2 + 250ac_1c_3^2 - 250c_2c_3^2 - 567a^2bc_6 - 189abc_1c_6 + 189bc_2c_6\big\}, \\
K_1 &= 12bc_3(4a + c_1)\big\{303a^2b^2c_3 + 69ab^2c_1c_3 - 12b^2c_1^2c_3 - 165b^2c_2c_3 \\
&\quad + 1050a^2c_3^2 + 350ac_1c_3^2 - 350c_2c_3^2 - 621a^2bc_6 - 207abc_1c_6 + 207bc_2c_6\big\}, \\
K_2 &= 3\big\{1449a^2b^4c_3^2 + 387ab^4c_1c_3^2 - 36b^4c_1^2c_3^2 - 675b^4c_2c_3^2 + 16800a^2b^2c_3^3 \\
&\quad + 7200ab^2c_1c_3^3 + 600b^2c_1^2c_3^3 - 2400b^2c_2c_3^3 + 1500a^2c_3^4 + 500ac_1c_3^4 \\
&\quad - 500c_2c_3^4 - 8658a^2b^3c_3c_6 - 3654ab^3c_1c_3c_6 - 288b^3c_1^2c_3c_6 \\
&\quad + 1350b^3c_2c_3c_6 - 3960a^2bc_3^2c_6 - 1320abc_1c_3^2c_6 + 1320bc_2c_3^2c_6 \\
&\quad + 1701a^2b^2c_6^2 + 567ab^2c_1c_6^2 - 567b^2c_2c_6^2\big\}, \\
K_3 &= 25920ab^3c_3^3 + 6480b^3c_1c_3^3 - 11664ab^4c_3c_6 - 2916b^4c_1c_3c_6 \\
&\quad - 7776ab^2c_3^2c_6 - 1944b^2c_1c_3^2c_6 + 3888ab^3c_6^2 + 972b^3c_1c_6^2, \\
K_4 &= 27b\big\{225b^3c_3^3 - 50bc_3^4 - 90b^4c_3c_6 - 90b^2c_3^2c_6 + 30c_3^3c_6 + 54b^3c_6^2 - 15bc_3c_6^2\big\}.
\end{aligned}$$

Now, since we must have $\widetilde{\rho}\,'(t) = Y_1\big(\widetilde{\rho}(t), t\big)$ for all small t, all of the above coefficients $K_i, 0 \le i \le 4$, must be zero. From $K_0 = 0$, we get

$$c_6 = \frac{c_3\big\{561a^2b^2 + 123ab^2c_1 - 24b^2c_1^2 - 315b^2c_2 + 750a^2c_3 + 250ac_1c_3 - 250c_2c_3\big\}}{189b(3a^2 + ac_1 - c_2)}. \tag{4.6}$$

Put this value of c_6 into K_1 and simplify to obtain

$$\begin{aligned}
K_1 = -\frac{80}{7}bc_3^2(4a + c_1)\big\{&327a^2b^2 + 69ab^2c_1 - 15b^2c_1^2 \\
&- 189b^2c_2 - 240a^2c_3 - 80ac_1c_3 + 80c_2c_3\big\}.
\end{aligned} \tag{4.7}$$

Since $c_3 > 0$ by Lemma 3.6, the condition $K_1 = 0$ leads to the following two cases: (i) $4a + c_1 \ne 0$ and (ii) $4a + c_1 = 0$.

Case (i) $4a + c_1 \ne 0$.

Then $K_1 = 0$ in Equation (4.7) implies that the value of c_3 is given by

$$c_3 = \frac{3b^2(109a^2 + 23ac_1 - 5c_1^2 - 63c_2)}{80(3a^2 + ac_1 - c_2)} \tag{4.8}$$

Put this value of c_3 into K_2, K_3, and K_4 to get

$$\begin{aligned}
K_2 &= \frac{729b^8(109a^2 + 23ac_1 - 5c_1^2 - 63c_2)^2(17a^2 + 3ac_1 - c_1^2 - 11c_2)}{2560(3a^2 + ac_1 - c_2)^2}, \\
K_3 &= \frac{243b^9(4a + c_1)(109a^2 + 23ac_1 - 5c_1^2 - 63c_2)^2\big\{483a^2 + 89ac_1 - 27c_1^2 - 305c_2\big\}}{32000(3a^2 + ac_1 - c_2)^3}, \\
K_4 &= \frac{243b^{10}\big(109a^2 + 23ac_1 - 5c_1^2 - 63c_2)^2L}{512000(3a^2 + ac_1 - c_2)^4},
\end{aligned}$$

where L is given by

$$\begin{aligned} L = {} & 26787a^4 + 16690a^3c_1 + 1829a^2c_1^2 - 386ac_1^3 - 45c_1^4 \\ & - 20194a^2c_2 - 7622ac_1c_2 - 334c_1^2c_2 + 2475c_2^2. \end{aligned}$$

Observe from the numerators of K_2 and K_3 that if

$$17a^2 + 3ac_1 - c_1^2 - 11c_2 = 483a^2 + 89ac_1 - 27c_1^2 - 305c_2 = 0,$$

then $4a + c_1 = 0$. But we are assuming that $4a + c_1 \neq 0$ in this case. Hence in order for $K_2 = K_3 = K_4 = 0$ to hold, we must have

$$109a^2 + 23ac_1 - 5c_1^2 - 63c_2 = 0,$$

which implies that $c_3 = 0$ in view of Equation (4.8). But this contradicts the fact that $c_3 > 0$ by Lemma 3.6. Thus this case is impossible.

Case (ii) $4a + c_1 = 0$.

In this case, we have $c_1 = -4a$. Hence the assumption $c_2 > 3a^2 + ac_1$ becomes $a^2 + c_2 > 0$. Moreover, the values of K_2, K_3, and K_4 are simplified to

$$\begin{aligned} K_2 &= -\frac{3200}{21}(a^2 + c_2)(18b^2 - 5c_3)c_3^3, \\ K_3 &= 0, \\ K_4 &= \frac{1600}{441}(63b^2 - 20c_3)(18b^2 - 5c_3)c_3^3. \end{aligned}$$

Thus in order for the equalities $K_2 = K_4 = 0$ to hold, we must have the value

$$c_3 = \frac{18b^2}{5}.$$

Then put the values of c_1 and c_3 into Equations (4.4), (4.5), and (4.6) to obtain the values of c_4, c_5, and c_6 as stated in the theorem. Put the values of $c_1, c_3, c_4, c_5,$ and c_6 into Equations (4.1) and (4.2) to derive the unique solution

$$X = \widetilde{\rho}(t) = \frac{3bt}{a^2 + c_2 + 6abt + 9b^2t^2}.$$

Moreover, we can verify that Y is indeed equal to $\widetilde{\rho}\,'(t)$. □

5. A Class of MRM-applicable Probability Measures

Suppose μ is an MRM-applicable probability measure for $h(x) = \frac{1}{\sqrt{1-x}}$ and has a support of at least three points. In view of Theorem 4.1, there are three parameters a, b, c_2, which satisfy the condition: $b \neq 0$, $a^2 + c_2 > 0$. For convenience, we make the following change of parameters:

$$\alpha = \frac{2(a^2 + c_2)}{3b}, \quad \beta = 2a, \quad \gamma = 6b.$$

The condition for the new parameters is $\alpha\gamma > 0$. Then we can rewrite the function $\rho(t)$ in Equation (4.3) as

$$\rho(t) = \frac{2t}{\alpha + 2\beta t + \gamma t^2}, \tag{5.1}$$

Lemma 5.1. *Let μ be an MRM-applicable for $h(x) = \frac{1}{\sqrt{1-x}}$. Assume that μ is supported by at least three points. Then the associated functions $f(t) = \theta\big(\rho(t)\big)$ and $\theta(t)$ are given by*

$$f(t) = \sqrt{1 + \frac{2\beta}{\alpha}t + \frac{\gamma}{\alpha}t^2}, \tag{5.2}$$

$$\theta(t) = \frac{2}{\sqrt{1-(\beta+\sqrt{\alpha\gamma})t} + \sqrt{1-(\beta-\sqrt{\alpha\gamma})t}}, \tag{5.3}$$

for t near 0.

Proof. Use the new parameters α, β, γ, and the function $\rho(t)$ in Equation (5.1) to rewrite Equation (3.8) as

$$\frac{f'(t)}{f(t)} = \frac{\beta + \gamma t}{\alpha + 2\beta t + \gamma t^2}$$

with the initial condition $f(0) = 1$. Integrate both sides to get the solution $f(t)$ as given in Equation (5.2).

In order to derive Equation (5.3), we first find the inverse function $\rho^{-1}(s)$ of $\rho(t)$ from Equation (5.1):

$$\rho^{-1}(s) = \frac{\alpha s}{1 - \beta s + \sqrt{(1-\beta s)^2 - \alpha\gamma s^2}}.$$

Observe from Equation (5.2) that we have the equality

$$\theta\big(\rho(t)\big) = f(t) = \sqrt{\frac{2t}{\alpha\rho(t)}},$$

which yields that

$$\theta(s) = \sqrt{\frac{2}{\alpha s}\rho^{-1}(s)} = \sqrt{\frac{2}{1 - \beta s + \sqrt{(1-\beta s)^2 - \alpha\gamma s^2}}}.$$

Then note an algebraic identity

$$\sqrt{\frac{2}{a + \sqrt{a^2 - b^2}}} = \frac{2}{\sqrt{a+b} + \sqrt{a-b}}$$

for $a > 0$ and $|b| \le a$. By using this identity we see that $\theta(s)$ is exactly Equation (5.3) with $t = s$. Thus the lemma is proved. □

Theorem 5.2. *A probability measure μ being supported by at least three points is MRM-applicable for the function $h(x) = \frac{1}{\sqrt{1-x}}$ if and only if it is a uniform probability measure on an interval.*

Proof. We first prove the sufficiency. As pointed out in Section 1, the uniform probability measure on $[-1, 1]$, denoted by $\breve{\mu}$, is MRM-applicable for $h(x) = \frac{1}{\sqrt{1-x}}$. Suppose μ is the uniform probability measure on an interval $[c, d], c < d$. It is easy to verify that

$$\mu = \breve{\mu} \circ \tau_{\frac{2}{d-c}, -\frac{d+c}{d-c}},$$

where $\tau_{v,q}$ is the affine transformation defined in Section 2, i.e., $\tau_{v,q}(x) = vx + q$. Hence by Corollary 2.2, μ is also MRM-applicable for $h(x) = \frac{1}{\sqrt{1-x}}$.

Next we prove the necessity. Suppose μ is supported by at least three points and is MRM-applicable for the function $h(x) = \frac{1}{\sqrt{1-x}}$. Its associated functions $\rho(t)$ and $\theta(t)$ are given by Equations (5.1) and (5.3), respectively.

As above, let $\breve{\mu}$ denote the uniform probability measure on $[-1, 1]$. Its associated functions are given by Equations (1.7) and (1.8), i.e.,

$$\breve{\theta}(t) = \frac{2}{\sqrt{1-t} + \sqrt{1+t}},$$
$$\breve{\rho}(t) = \frac{2t}{1+t^2}.$$

Consider the probability measure defined by $\breve{\mu}_{v,q} = \breve{\mu} \circ \tau_{v,q}$. Then we use Equations (2.1) and (2.3) to find the associated ρ- and θ-functions of $\breve{\mu}_{v,q}$:

$$\begin{aligned}\breve{\theta}_{v,q}(t) &= \sqrt{\frac{v}{v+qt}}\,\breve{\theta}\Big(\frac{t}{v+qt}\Big) \\ &= \frac{2}{\sqrt{1+\frac{q-1}{v}t} + \sqrt{1+\frac{q+1}{v}t}},\end{aligned} \tag{5.4}$$

$$\begin{aligned}\breve{\rho}_{v,q;\sigma}(t) &= \frac{v\breve{\rho}(\sigma t)}{1 - q\breve{\rho}(\sigma t)} \\ &= \frac{2t}{\frac{1}{\sigma v} - 2\frac{q}{v}t + \frac{\sigma}{v}t^2}.\end{aligned} \tag{5.5}$$

Now compare the two ρ-functions in Equations (5.1) and (5.5) and set

$$\frac{1}{\sigma v} = \alpha, \quad \frac{q}{v} = -\beta, \quad \frac{\sigma}{v} = \gamma.$$

These equations have two solutions given by

$$v = \frac{1}{\sqrt{\alpha\gamma}}, \quad q = -\frac{\beta}{\sqrt{\alpha\gamma}}, \quad \sigma = \sqrt{\frac{\gamma}{\alpha}}, \tag{5.6}$$
$$v = -\frac{1}{\sqrt{\alpha\gamma}}, \quad q = \frac{\beta}{\sqrt{\alpha\gamma}}, \quad \sigma = -\sqrt{\frac{\gamma}{\alpha}}.$$

We only need to consider the first solution since the second one leads to the same conclusion. With the choice of the values for v, q, and σ in Equation (5.6), the function in Equation (5.4) can be rewritten as

$$\begin{aligned}\breve{\theta}_{v,q}(t) &= \frac{2}{\sqrt{1+\frac{q-1}{v}t} + \sqrt{1+\frac{q+1}{v}t}} \\ &= \frac{2}{\sqrt{1-(\beta+\sqrt{\alpha\gamma})t} + \sqrt{1-(\beta-\sqrt{\alpha\gamma})t}},\end{aligned}$$

which is exactly the right-hand side of Equation (5.3). Therefore,

$$\theta(t) = \breve{\theta}_{\frac{1}{\sqrt{\alpha\gamma}}, -\frac{\beta}{\sqrt{\alpha\gamma}}}(t). \tag{5.7}$$

Finally, we look into the orthogonal polynomial generating function $\psi(t,x)$ of μ. Use Equation (5.2) to find that

$$\psi(t,x) = \frac{h\big(\rho(t)x\big)}{f(t)} = \frac{1}{\sqrt{1-2\big(\frac{1}{\alpha}x-\frac{\beta}{\alpha}\big)t+\frac{\gamma}{\alpha}t^2}}. \tag{5.8}$$

On the other hand, use Equations (1.9) and (2.6) to get

$$\breve{\psi}_{\frac{1}{\sqrt{\alpha\gamma}},-\frac{\beta}{\sqrt{\alpha\gamma}};\sqrt{\frac{\gamma}{\alpha}}}(t,x) = \breve{\psi}\Big(\sqrt{\frac{\gamma}{\alpha}}\,t,\,\frac{1}{\sqrt{\alpha\gamma}}x-\frac{\beta}{\sqrt{\alpha\gamma}}\Big) = \frac{1}{\sqrt{1-2\big(\frac{1}{\alpha}x-\frac{\beta}{\alpha}\big)t+\frac{\gamma}{\alpha}t^2}}, \tag{5.9}$$

Thus by Equations (5.8) and (5.9),

$$\psi(t,x) = \breve{\psi}_{\frac{1}{\sqrt{\alpha\gamma}},-\frac{\beta}{\sqrt{\alpha\gamma}};\sqrt{\frac{\gamma}{\alpha}}}(t,x),$$

which implies that the ω-sequences of μ and $\breve{\mu}_{v,q}$ are related by

$$\omega_n = \breve{\omega}_n^{v,q;\sigma}.$$

Then apply Equations (1.10) and (2.7) to get

$$\omega_n = \frac{1}{v^2}\breve{\omega}_n = \frac{1}{v^2}\frac{(n+1)^2}{(2n+3)(2n+1)}, \quad n \geq 0.$$

Hence the λ-sequence of μ is given by $\lambda_0 = 1$ and

$$\lambda_n = \omega_0\omega_1\cdots\omega_{n-1} = \frac{1}{v^{2n}}\frac{2^{2n}(n!)^4}{(2n+1)\big((2n)!\big)^2}, \quad n \geq 1.$$

Using the Stirling formula, we can easily check that

$$\lim_{n\to\infty}\lambda_n^{-1/2n} = 2v > 0,$$

which shows that $\sum_{n=1}^{\infty}\lambda_n^{-1/2n} = \infty$. Thus by Theorem 1.11 in [11], μ is uniquely determined by its moments. This implies that μ is uniquely determined by the function $\theta(t)$. Therefore, in view of Equation (5.7), we must have

$$\mu = \breve{\mu}_{\frac{1}{\sqrt{\alpha\gamma}},-\frac{\beta}{\sqrt{\alpha\gamma}}} = \breve{\mu}\circ\tau_{\frac{1}{\sqrt{\alpha\gamma}},-\frac{\beta}{\sqrt{\alpha\gamma}}}.$$

This proves that μ is a uniform probability measure on an interval. In fact,

$$\big(\tau_{\frac{1}{\sqrt{\alpha\gamma}},-\frac{\beta}{\sqrt{\alpha\gamma}}}\big)^{-1}(-1) = \beta-\sqrt{\alpha\gamma},$$
$$\big(\tau_{\frac{1}{\sqrt{\alpha\gamma}},-\frac{\beta}{\sqrt{\alpha\gamma}}}\big)^{-1}(1) = \beta+\sqrt{\alpha\gamma}.$$

Hence μ is the uniform probability measure on the interval $\big[\beta-\sqrt{\alpha\gamma},\,\beta+\sqrt{\alpha\gamma}\big]$. □

In view of Theorem 5.2 we now need to consider the case when μ is supported by one or two points. If μ is supported by one point, then it is a Dirac delta measure which is obviously MRM-applicable for the function $h(x) = \frac{1}{\sqrt{1-x}}$.

Suppose μ is supported by two points, then $\|P_2\| = 0$. Hence $J''(0) = 0$ by Equation (3.22). It follows that $c_3 = J''(0)/\rho'(0)^2 = 0$. Then Equation (4.1) with $X = \rho(t)$ and $Y = \rho'(t)$ can be simplify to

$$\rho'(t) = \frac{b\big\{-3 + 2(2a - c_1)\rho(t) + 4(c_2 - 2a^2)\rho(t)^2\big\}}{ac_1 + 3a^2 - c_2 + b(4a + c_1)t + 3b^2t^2},$$

which is a separable differential equation and can be solved to find that

$$\rho(t) = \frac{3}{2}\frac{bt\big\{2(c_2 - 3a^2 - ac_1) - b(4a + c_1)t\big\}}{(c_2 - 3a^2 - ac_1)^2 - 2b(a + c_1)(c_2 - 3a^2 - ac_1)t - b^2(5a^2 - 2ac_1 - c_1^2 - 3c_2)t^2}, \tag{5.10}$$

where the condition $c_2 > 3a^2 + ac_1$ must be satisfied by Lemma 3.7.

Lemma 5.3. *For fixed $\lambda \in (0, 1)$, the Bernoulli distribution $\mu = (1 - \lambda)\delta_o + \lambda\delta_1$ is MRM-applicable for $h(x) = \frac{1}{\sqrt{1-x}}$ with a ρ-function given by*

$$\rho(t) = \frac{4t\big\{\lambda(1 - \lambda) + (1 - 2\lambda)t\big\}}{(1 - \lambda)^2(\lambda + 2t)^2}. \tag{5.11}$$

Proof. We need to find the values of the constants in Equation (5.10) in order to find a function $\rho(t)$ for the MRM-application of μ. For this purpose, we first derive the following values from the series expansion of $\rho(t)$ in Equation (5.10),

$$\begin{aligned}
\rho'(0) &= \frac{3b}{c_2 - 3a^2 - ac_1},\\
\rho''(0) &= c_1\rho'(0)^2,\\
\rho'''(0) &= \frac{2}{3}(5a^2 + ac_1 + 2c_1^2 - 3c_2)\rho'(0)^3.
\end{aligned}$$

Next, we have the θ-function of μ,

$$\theta(t) = \int_{\mathbb{R}} \frac{1}{\sqrt{1 - tx}}\, d\mu(x) = 1 - \lambda + \frac{\lambda}{\sqrt{1 - t}}.$$

Hence the function $f(t) = \theta\big(\rho(t)\big)$ is given by

$$f(t) = 1 - \lambda + \frac{\lambda}{\sqrt{1 - \rho(t)}}, \tag{5.12}$$

which yields the following values:

$$\begin{aligned}
f'(0) &= \frac{\lambda}{2}\rho'(0),\\
f''(0) &= \frac{\lambda}{2}\rho''(0) + \frac{3\lambda}{4}\rho'(0)^2,\\
f'''(0) &= \frac{\lambda}{2}\rho'''(0) + \frac{9\lambda}{4}\rho'(0)\rho''(0) + \frac{15\lambda}{8}\rho'(0)^3.
\end{aligned}$$

Now use the above values of the derivatives and the definitions of the constants from Lemmas 3.3, 3.4, and 3.5 to find that

$$\begin{aligned}
a &= \frac{f'(0)}{\rho'(0)} = \frac{\lambda}{2},\\
c_2 &= \frac{f''(0)}{\rho'(0)^2} = \frac{\lambda c_1}{2} + \frac{3\lambda}{4},\\
c_4 &= \frac{\rho'''(0)}{\rho'(0)^3} = \frac{2}{3}(5a^2 + ac_1 + 2c_1^2 - 3c_2)\\
&= \frac{1}{6}(5\lambda^2 - 9\lambda - 4\lambda c_1 + 8c_1^2),\\
c_5 &= \frac{f'''(0)}{\rho'(0)^3} = \frac{\lambda}{2}c_4 + \frac{9\lambda}{4}c_1 + \frac{15}{8}\lambda\\
&= \frac{\lambda}{24}(10\lambda^2 - 18\lambda - 8\lambda c_1 + 16c_1^2 + 54c_1 + 45). \qquad (5.13)
\end{aligned}$$

On the other hand, note that Equation (4.4) is valid for the general case. Hence we can put the above values of a, c_2, and c_4 into Equation (4.4) to get

$$c_5 = \frac{\lambda}{24}(-20\lambda^2 + 22\lambda c_1 + 57\lambda + 16c_1^2 + 24c_1). \qquad (5.14)$$

Since $\lambda \neq 0$, Equations (5.13) and (5.14) imply that

$$10\lambda^2 - 18\lambda - 8\lambda c_1 + 16c_1^2 + 54c_1 + 45 = -20\lambda^2 + 22\lambda c_1 + 57\lambda + 16c_1^2 + 24c_1,$$

which can be factorized as $(\lambda - 1)(2c_1 - 2\lambda + 3) = 0$. But $\lambda \neq 1$. Hence we have the value of c_1:

$$c_1 = \lambda - \frac{3}{2}.$$

Then we obtain the values of other constants:

$$c_2 = \frac{\lambda^2}{2}, \quad c_4 = \frac{3}{2}(\lambda - 1)(\lambda - 2), \quad c_5 = \frac{3\lambda^2}{4}.$$

Put all values of the constants into Equation (5.10) to get the ρ-function

$$\rho(t) = \frac{4bt\{\lambda(1-\lambda) + (1-2\lambda)bt\}}{(1-\lambda)^2(\lambda + 2bt)^2}.$$

Note that b only gives a scaling of t. Hence for simplicity we may take $b = 1$, which yields the function $\rho(t)$ in Equation (5.11). With this $\rho(t)$, we have

$$\sqrt{1 - \rho(t)} = \frac{\lambda(1 - \lambda - 2t)}{(1-\lambda)(\lambda + 2t)}.$$

Hence by Equation (5.12), we see that

$$\theta\big(\rho(t)\big) = f(t) = \frac{1-\lambda}{1 - \lambda - 2t}. \qquad (5.15)$$

On the other hand, the $\widetilde{\theta}$-function is given by

$$\widetilde{\theta}(t,s) = \int_{\mathbb{R}} \frac{1}{\sqrt{1-tx}} \frac{1}{\sqrt{1-sx}}\, d\mu(x) = 1 - \lambda + \frac{\lambda}{\sqrt{1-t}\sqrt{1-s}}.$$

Therefore, we have

$$\widetilde{\theta}\big(\rho(t),\rho(s)\big) = 1-\lambda+\frac{\lambda}{\sqrt{1-\rho(t)}\sqrt{1-\rho(s)}} = \frac{(1-\lambda)\big\{\lambda(1-\lambda)+4ts\big\}}{\lambda(1-\lambda-2t)(1-\lambda-2s)}. \tag{5.16}$$

Finally, by Equations (5.15) and (5.16),

$$\Theta_\rho(t,s) = \frac{\widetilde{\theta}\big(\rho(t),\rho(s)\big)}{\theta\big(\rho(t)\big)\theta\big(\rho(s)\big)} = \frac{\lambda(1-\lambda)+4ts}{\lambda(1-\lambda)},$$

which is a function of the product ts. Thus by Theorem 1.1, μ is MRM-applicable for the function $h(x)=\frac{1}{\sqrt{1-x}}$. □

Before proceeding to the final theorem, let us make some remarks about the Bernoulli distribution μ in Lemma 5.3. Note that the Hilbert space $L^2(\mu)$ is two dimensional. Hence the associated sequences $\{\alpha_n,\omega_n,P_n\}_{n=0}^\infty$ are reduced to $\{\alpha_n,\omega_n,P_n\}_{n=0}^1$. It is easy to check from the Gram-Schmidt orthogonalization process that

$$\begin{aligned} &\alpha_0=\lambda, && \alpha_1 = 1-\lambda,\\ &\omega_0=\lambda(1-\lambda), && \omega_1=0,\\ &P_0(x)=1, && P_1(x)=x-\lambda. \end{aligned}$$

On the other hand, we can use Equations (5.11), (5.15), and (1.5) with the function $h(x)=\frac{1}{\sqrt{1-x}}$ to find the orthogonal polynomial generating function,

$$\psi(t,x) = \frac{(1-\lambda-2t)(\lambda+2t)}{\sqrt{(1-\lambda)^2(\lambda+2t)^2-4xt\{\lambda(1-\lambda)+(1-2\lambda)t\}}},$$

which has a power series in t with algebraic polynomial coefficients as follows:

$$\begin{aligned}\psi(t,x) = 1 &+ \frac{2}{\lambda(1-\lambda)}(x-\lambda)t + \frac{6}{\lambda^2(1-\lambda)^2}(x^2-x)t^2 \\ &+ \frac{20}{\lambda^3(1-\lambda)^3}\Big(x^3-\frac{3(3-\lambda)}{5}x^2+\frac{4-3\lambda}{5}x\Big)t^3+\cdots.\end{aligned}$$

However, when μ is taken into account, we have

$$\psi(t,x) = 1+\frac{2}{\lambda(1-\lambda)}(x-\lambda)t, \quad \mu\text{–a.e..}$$

Thus we get the above polynomials $P_0(x)$ and $P_1(x)$. Moreover, we have

$$E_\mu\big[\psi(t,\cdot)^2\big] = 1+\frac{4}{\lambda(1-\lambda)}t^2, \quad E_\mu\big[x\psi(t,\cdot)^2\big] = \lambda+4t+\frac{4}{\lambda}t^2,$$

which yields the above numbers $\alpha_0,\alpha_1,\omega_0$, and ω_1 by Theorem 2.6 in [4].

Theorem 5.4. *The class of MRM-applicable probability measures for the function $h(x)=\frac{1}{\sqrt{1-x}}$ consists of uniform probability measures on intervals and probability measures being supported by one or two points.*

Proof. By Theorem 5.2 we only need to show that a probability measure μ with a support of two points is MRM-applicable for $h(x) = \frac{1}{\sqrt{1-x}}$. Such a probability measure μ can be represented by

$$\mu = (1-\lambda)\delta_c + \lambda\delta_d,$$

with $0 < \lambda < 1$ and $c < d$. Let μ_λ be the Bernoulli distribution in Lemma 5.3, i.e., $\mu_\lambda = (1-\lambda)\delta_0 + \lambda\delta_1$. Choose v and q satisfying the equations:

$$vc + q = 0, \quad vd + q = 1,$$

i.e., v and q are given by $v = \frac{1}{d-c}$, $q = -\frac{c}{d-c}$.. Then we have the equality

$$\mu = \mu_\lambda \circ \tau_{v,q},$$

where $\tau_{v,q}(x) = vx + q$. Hence by Corollary 2.2 and Lemma 5.3, we see that μ is MRM-applicable for the function $h(x) = \frac{1}{\sqrt{1-x}}$. □

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