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PAC COMMUTATORS AND THE R–TRANSFORM

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Abstract. We develop an algorithmic method for working out moments of a probability measure on the real line from the preservation–annihilation–creation operator commutator relations for the measure. The method is applied to prove a result of Voiculescu on the $\mathcal{R}$–transform.

1. Introduction

Expanding on a program introduced in [1] and continued in [2], it was proven in [8] that the moments of a probability distribution can be recovered from the commutator between its annihilation and creation operators, and the commutator between its annihilation and preservation operators, provided that the first order moment is given. Moreover, a simple, concrete method for computing the moments was introduced in [8]. In the present paper we apply this method to some classic distributions and to give another proof of a theorem of Voiculescu concerning the $\mathcal{R}$–transform. There are already some known techniques for recovering the moments or even the probability distribution of a random variable X, having finite moments of all orders, from its Szegö–Jacobi parameters. One method uses a continued fraction expansion of the Cauchy–Stieltjes transform of X and is very useful when the random variable has a compact support. Another powerful way is the method of renormalization introduced in [3, 5, 4] and pushed almost to the limits in [7]. However, our method is based on the Lie algebra structure of the algebra generated by the annihilation, preservation, and creation operators.

In section 2 we introduce very quickly the annihilation, preservation, and creation operators for a one–dimensional distribution having finite moments of any order. For brevity we will call these operators the PAC *operators*. We also present the commutator method and its dual developed in [8]. In section 3 we apply this method to two families of distributions. Finally, in section 4, we use the commutator method and its dual to give a proof of an important theorem, of Voiculescu, about the analytic function theory tools for computing the $\mathcal{R}$–transform.

2. Background

Let X be a random variable having finite moments of any order, i.e., $E[|X|^p] < \infty$, for all $p > 0$, where E denotes the expectation. It is well–known that by applying the Gram–Schmidt orthogonalization procedure to the sequence of monomial

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random variables: $1, X, X^2, \ldots$, we can obtain a sequence of orthogonal polynomial random variables: $f_0(X), f_1(X), f_2(X), \ldots$, chosen such that for each $n \geq 0$, f_n has degree n and a leading coefficient equal to 1. We assume that the probability distribution of X has an infinite support so that the sequence f_0, f_1, f_n, $\ldots$ never terminates. There exist two sequences of real numbers: $\{\alpha_k\}_{k\geq 0}$ and $\{\omega_k\}_{k\geq 1}$, called the *Szegö–Jacobi parameters* of X, such that for all $n \geq 0$, we have:

$$Xf_n(X) = f_{n+1}(X) + \alpha_n f_n(X) + \omega_n f_{n-1}(X). \tag{2.1}$$

See [6] and [9]. When $n = 0$, in this recursive relation $f_{-1} = 0$ (the null polynomial) and $\omega_0 := 0$ by agreement. The terms of the sequence $\{\omega_k\}_{k\geq 1}$ are called the *principal* Szegö–Jacobi parameters of X and they must all be positive since, for all $n \geq 1$,

$$E\left[f_n(X)^2\right] = \omega_1\omega_2\cdots\omega_n. \tag{2.2}$$

Moreover, by Favard's theorem, given any sequence of real numbers $\{\alpha_k\}_{k\geq 0}$ and any sequence of positive numbers $\{\omega_k\}_{k\geq 1}$, there exists a random variable X having these sequences as its Szegö–Jacobi parameters.

Let F be the space of all random variables of the form $f(X)$, where f is a polynomial function, and for each $n \geq 0$, let F_n be the subspace of F consisting of all random variables $f(X)$, such that f is a polynomial of degree at most n. Let $G_0 := F_0$, and for all $n \geq 1$, let $G_n := F_n \ominus F_{n-1}$, i.e., the orthogonal complement of F_{n-1} into F_n. For each $n \geq 0$, $G_n = \mathbb{C}f_n(X)$ (scalar multiples of $f_n(X)$) and G_n is called the *homogenous chaos space of degree* n generated by X. The space $\mathcal{H} := \oplus_{n\geq 0} G_n$ is called the *chaos space* generated by X. It is clear that F is dense in $\mathcal{H}$. For each $n \geq 0$, we denote by P_n the orthogonal projection of $\mathcal{H}$ onto G_n. If we look back to the recursive formula (2.1), then we can easily see that, for all $n \geq 0$:

$$P_{n+1}\left[Xf_n(X)\right] = f_{n+1}(X),$$

$$P_n\left[Xf_n(X)\right] = \alpha_n f_n(X),$$

and

$$P_{n-1}\left[Xf_n(X)\right] = \omega_n f_{n-1}(X).$$

Let us regard now X not as a random variable, but as a multiplication operator from F to F, which maps a polynomial random variable $f(X)$ into $Xf(X)$. We can see that applying the multiplication operator X to a polynomial from G_n we get three polynomials: one in G_{n+1}, one in G_n, and one in G_{n-1}. That means:

$$X|G_n = P_{n+1}X|G_n + P_nX|G_n + P_{n-1}X|G_n.$$

We define $D_n^+ := P_{n+1}X|G_n$, $D_n^0 := P_nX|G_n$, and $D_n^- := P_{n-1}X|G_n$. Since D_n^+ maps G_n into G_{n+1}, it increases the degree of f_n, and thus it is called a *creation operator*. Similarly, since $D_n^0 : G_n \to G_n$, it is called a *preservation operator*, and since $D_n^- : G_n \to G_{n-1}$, it is called an *annihilation operator*. So far the annihilation, preservation, and creation operators have been defined only on each individual homogenous chaos space G_n. We extend their definition as linear operators from F to F, and define the operators: a^-, a^0, and a^+, such that for any $n \geq 0$, $a^-|G_n := D_n^-$, $a^0|G_n := D_n^0$, and $a^+|G_n := D_n^+$. As a multiplication operator X is the sum of these three operators. Thus:

$$X = a^- + a^0 + a^+. \tag{2.3}$$

It is known that X is *polynomially symmetric*, i.e., $E[X^{2k-1}] = 0$, for all positive integers k, if and only if $a^0 = 0$, see [1], or equivalently $\alpha_n = 0$, for all $n \geq 0$. In this case $X = a^- + a^+$.

We will briefly explain now the commutator method introduced in [8], used to recover the moments and if possible the probability distribution of a random variable X, from the commutator between its annihilation and creation operators, commutator between its annihilation and preservation operators, and its first moment, $E[X]$. We would like to make the reader aware of the fact that some times we regard X as a random variable, and other times we view it as a multiplication operator. We hope that this will not create any confusion, since most of the time, when we refer to it as being a multiplication operator, we will write $X^n 1$, where 1 is the constant (vacuum) polynomial equal to 1. When we write $E[X^n]$, for some $n \geq 1$, we regard X as a random variable. The commutator of two operators A and B is defined as:

$$[A, B] := AB - BA. \tag{2.4}$$

Commutator Method

Let X be a random variable, having finite moments of all orders. We assume that $[a^-, a^+]$, $[a^-, a^0]$, and $E[X]$ are given (known). Then, in order to compute the higher moments of X, we will follow the following three steps.

Step 1. Let $\langle \cdot, \cdot \rangle$ denote the inner product defined as:

$$\langle f(X), g(X) \rangle := E[f(X)\overline{g(X)}],$$

for all polynomials f and g. For any fixed positive integer n, we have:

$$\begin{aligned} E[X^n] &= \langle XX^{n-1}1, 1\rangle \\ &= \langle (a^- + a^0 + a^+)X^{n-1}1, 1\rangle \\ &= \langle a^- X^{n-1}1, 1\rangle + \langle a^0 X^{n-1}1, 1\rangle + \langle a^+ X^{n-1}1, 1\rangle. \end{aligned}$$

Since $(a^0)^* = a^0$, $(a^+)^* = a^-$, $a^0 1 = E[X]1$, and $a^- 1 = 0$, we have:

$$\begin{aligned} \langle a^+ X^{n-1}1, 1\rangle &= \langle X^{n-1}1, a^- 1\rangle \\ &= \langle X^{n-1}1, 0\rangle \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \langle a^0 X^{n-1}1, 1\rangle &= \langle X^{n-1}1, a^0 1\rangle \\ &= \langle X^{n-1}1, E[X]1\rangle \\ &= E[X]\langle X^{n-1}1, 1\rangle \\ &= E[X]E[X^{n-1}]. \end{aligned}$$

Thus

$$E[X^n] = E[X]E[X^{n-1}] + \langle a^- X^{n-1}1, 1\rangle.$$

Step 2. Swap (permute) a^- and X^{n-1}, using the simple formula:

$$AB = BA + [A, B]$$

and the product rule for commutators:

$$[A, B^k] = \sum_{j=0}^{k-1} B^{k-1-j}[A, B]B^j,$$

for all operators A and B, and any $k \geq 2$. Use also the fact that

$$\begin{aligned}
[a^-, X] &= [a^-, a^- + a^0 + a^+] \\
&= [a^-, a^-] + [a^-, a^0] + [a^-, a^+] \\
&= [a^-, a^0] + [a^-, a^+].
\end{aligned}$$

Thus we get:

$$\begin{aligned}
E[X^n] &= E[X]E[X^{n-1}] + \langle a^- X^{n-1}1, 1\rangle \\
&= E[X]E[X^{n-1}] + \langle \left(X^{n-1}a^- + [a^-, X^{n-1}]\right) 1, 1\rangle \\
&= E[X]E[X^{n-1}] + \langle X^{n-1}a^-1, 1\rangle + \langle [a^-, X^{n-1}]1, 1\rangle \\
&= E[X]E[X^{n-1}] + \langle [a^-, X^{n-1}]1, 1\rangle \\
&= E[X]E[X^{n-1}] + \sum_{j=0}^{n-2} \langle X^{n-2-j}[a^-, X]X^j1, 1\rangle \\
&= E[X]E[X^{n-1}] + \sum_{j=0}^{n-2} \langle X^{n-2-j}\left([a^-, a^0] + [a^-, a^+]\right) X^j1, 1\rangle.
\end{aligned}$$

Step 3. If necessary, go back to **Step 2** and repeat the procedure, until a recursive formula expressing the n–th moment in terms of lower order moments is obtained.

The idea in this method (algorithm) is very simple: move each annihilator a^- stepwise to the right, using the commutator relations with a^+ and a^0, until it acts on the vacuum polynomial 1 and kills it. There is also a dual of this method, using the creation operator a^+, instead of the annihilation operator a^-. We will briefly explain it now.

Dual Commutator Method

Step 1. For any fixed positive integer n, we have:

$$\begin{aligned}
E[X^n] &= \langle X^{n-1}X1, 1\rangle \\
&= \langle X^{n-1}(a^- + a^0 + a^+)1, 1\rangle \\
&= \langle X^{n-1}a^-1, 1\rangle + \langle X^{n-1}a^01, 1\rangle + \langle X^{n-1}a^+1, 1\rangle \\
&= 0 + \langle X^{n-1}E[X]1, 1\rangle + \langle X^{n-1}a^+1, 1\rangle \\
&= E[X]E[X^{n-1}] + \langle X^{n-1}a^+1, 1\rangle.
\end{aligned}$$

Step 2. Swap X^{n-1} and a^+.

Step 3. Repeat **Step 2** if necessary.

In the dual commutator method the creation operator a^+ is moved stepwise to the left, until it arrives to the left most possible position. In that moment, for any polynomial f, we have:

$$\begin{aligned}\langle a^+ f(X)1, 1\rangle &= \langle f(X)1, a^- 1\rangle \\ &= 0.\end{aligned}$$

3. Some Calculations

In this section we apply the commutator method to two concrete examples.

Example 3.1. Let us consider now a random variable X, having finite moments of all orders, whose Szegö–Jacobi parameters are $\alpha_n = 0$, for all $n \geq 0$, and the principal Szegö–Jacobi parameters are: c, $c+d$, $2c+d$, $2c+2d$, $3c+2d$, $3c+3d$, $\ldots$. That means, for all $n \geq 1$, $\omega_{2n-1} = nc + (n-1)d$ and $\omega_{2n} = nc + nd$, where c and d are fixed real numbers, such that $c > 0$ and $c + d > 0$.

Since $\alpha_n = 0$, for all $n \geq 0$, we know that X is symmetric and thus the space spanned by the monomial random variables of even degree: 1, X^2, X^4, $\ldots$, is orthogonal to the space spanned by the monomial random variables of odd degrees: X, X^3, X^5, $\ldots$. In fact the closures of these two spaces are $\mathcal{H}_e := G_0 \oplus G_2 \oplus G_4 \oplus \cdots$ and $\mathcal{H}_o := G_1 \oplus G_3 \oplus G_5 \oplus \cdots$. Let $\mathcal{E} : \mathcal{H} \to \mathcal{H}_e$ and $\mathcal{O} : \mathcal{H} \to \mathcal{H}_o$ denote the orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_e$ and $\mathcal{H}_o$, respectively. Since, for all $n \geq 0$, $[a^-, a^+] f_n(X) = (\omega_{n+1} - \omega_n) f_n(X)$, where $\{f_n\}_{n \geq 0}$ are the orthogonal polynomials generated by X, we can see that the commutator of the annihilation and creation operators is:

$$[a^-, a^+] = c\mathcal{E} + d\mathcal{O}. \tag{3.1}$$

Because $a^0 = 0$, all the odd moments vanish. Applying now our commutator method, for all $n \geq 1$, we have:

$$\begin{aligned}E[X^{2n}] &= \sum_{j=0}^{2n-2} \langle X^{2n-2-j}[a^-, a^+]X^j 1, 1\rangle \\ &= c\sum_{j=0}^{2n-2} \langle X^{2n-2-j}\mathcal{E}X^j 1, 1\rangle + d\sum_{j=0}^{2n-2} \langle X^{2n-2-j}\mathcal{O}X^j 1, 1\rangle.\end{aligned}$$

Since $\mathcal{E}X^{2k} = X^{2k}$, $\mathcal{E}X^{2k+1} = 0$, $\mathcal{O}X^{2k+1} = X^{2k+1}$, and $\mathcal{O}X^{2k} = 0$, for all $k \geq 0$, we get:

$$\begin{aligned}E[X^{2n}] &= c\sum_{k=0}^{n-1} \langle X^{2n-2-2k}X^{2k}1, 1\rangle + d\sum_{k=0}^{n-2} \langle X^{2n-2-2k-1}X^{2k+1}1, 1\rangle \\ &= cnE[X^{2n-2}] + d(n-1)E[X^{2n-2}] \\ &= [(c+d)n - d]E[X^{2n-2}].\end{aligned}$$

Iterating this recursive relation, we obtain:

$$\begin{aligned}E[X^{2n}] &= [(c+d)n - d]E[X^{2n-2}] \\ &= [(c+d)n - d][(c+d)(n-1) - d]E[X^{2n-4}] \\ \cdots & \ldots \\ &= [(c+d)n - d][(c+d)(n-1) - d]\cdots[(c+d)1 - d]E[X^0].\end{aligned}$$

Thus we obtain that:

$$\frac{1}{(c+d)^n}E[X^{2n}] = \left(n - \frac{d}{c+d}\right)\left(n-1-\frac{d}{c+d}\right)\cdots\left(1-\frac{d}{c+d}\right). \quad (3.2)$$

We recognize that the right–hand side of (3.2) is exactly the n–th moment of a gamma distribution. That means the distribution of the random variable $Y := [1/(c+d)]X^2$ is given by the density function:

$$f(x) = \frac{1}{\Gamma\left(\frac{c}{c+d}\right)} x^{\frac{c}{c+d}-1}e^{-x}1_{(0,\infty)}(x), \quad (3.3)$$

where Γ denotes the Euler's gamma function. Thus X^2 is a re–scaled gamma random variable. Since X is a symmetric random variable, we can compute first its distribution function F_X in the following way:

$$\begin{aligned} F_X(a) &:= P(X \leq a) \\ &= 1 - P(X > a) \\ &= 1 - \frac{1}{2}P(X^2 > a^2) \\ &= \frac{1}{2} + \frac{1}{2}P([1/(c+d)]X^2 \leq a^2/(c+d)) \\ &= \frac{1}{2} + \frac{1}{2}F_Y(a^2/(c+d)), \end{aligned}$$

for all $a > 0$. Differentiating both sides of the last equality with respect to a, we obtain that the density of X is:

$$\begin{aligned} g(a) &= F_X'(a) \\ &= \frac{a}{c+d}F_Y'(a^2/(c+d)) \\ &= \frac{a}{c+d}f(a^2/(c+d)), \end{aligned}$$

for all $a > 0$. Since $g(-a) = g(a)$, we conclude that X is the random variable given by the density function:

$$g(x) = \frac{1}{(c+d)^{\frac{c}{c+d}}\Gamma\left(\frac{c}{c+d}\right)}|x|^{\frac{c-d}{c+d}}e^{-\frac{x^2}{c+d}}. \quad (3.4)$$

Example 3.2. Let us find now the random variable X whose Szegö–Jacobi parameters are:

$$\alpha_n = \begin{cases} \alpha & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \geq 1 \end{cases}$$

and

$$\omega_n = \begin{cases} b & \text{if} \quad n = 1 \\ c & \text{if} \quad n \geq 2, \end{cases}$$

where α, b, and c are fixed real numbers, such that b and c are strictly positive.

Before computing the moments of X, we will find a simple upper bound for $E[|(X-\alpha)^n|]$, for each $n \geq 0$.

Claim 1. For each $n \geq 0$, we have:

$$E[|(X-\alpha)^n|] \quad \leq \quad (3T)^n, \tag{3.5}$$

where $T := \max\{1,\, |\alpha|,\, b,\, c\}$.

Indeed, let $n \geq 0$ be fixed. Let $\{f_n\}_{n\geq 0}$ denote the sequence of orthogonal polynomials, with a leading coefficient equal to 1, generated by X. We have:

$$\begin{aligned} E[(X-\alpha)^{2n}] &= \langle (X-\alpha I)\cdots(X-\alpha I)1, 1\rangle \\ &= \langle \{a^- + a^+ + (a^0 - \alpha I)\}\cdots\{a^- + a^+ + (a^0-\alpha I)\}1, 1\rangle \\ &= \sum_{(a_1,\cdots,a_{2n})\in\{a^-,a^+,a^0-\alpha I\}^{2n}} \langle a_1\cdots a_{2n}1, 1\rangle. \end{aligned}$$

Observe that in the last sum only the terms that contain the same number of annihilation and creation operators could be non–zero, since we start from the vacuum space $\mathbb{R}1$ and we have to return to this space (otherwise $a_1\cdots a_{2n}1 \perp 1$). For these terms, we move from one orthogonal polynomial to another in the following way. If $a_j = a^+$, and we are currently at f_k, then $a_j f_k = f_{k+1}$, and we retain a coefficient $c_j = 1$. If $a_j = a^-$, then $a_j f_k = \omega_k f_{k-1}$ and we retain a coefficient $c_j = \omega_k$. Finally, if $a_j = a^0 - \alpha I$, then $a_j f_k = (\alpha_k - \alpha) f_k$ and we retain a coefficient $c_j = \alpha_k - \alpha$. Observe, that for all j, we have $|c_j| \leq T$. Since at the end we return to the vacuum polynomial 1, we get:

$$\begin{aligned} E[(X-\alpha)^{2n}] &= \sum \langle c_1\cdots c_{2n}1, 1\rangle \\ &= \sum c_1\cdots c_{2n} \\ &\leq \sum |c_1\cdots c_{2n}| \\ &\leq 3^{2n}T^{2n}, \end{aligned}$$

since the cardinality of the set $\{a^-,\, a^0,\, a^0 - \alpha I\}^{2n}$ is 3^{2n}. Using now Jensen's inequality we get:

$$\begin{aligned} E[|X|^n] &\leq \sqrt{E[X^{2n}]} \\ &\leq (3T)^n. \end{aligned}$$

Let us compute now the moments of X. We have:

$$[a^-, a^+]f_n \quad = \quad (\omega_{n+1} - \omega_n) f_n. \tag{3.6}$$

Thus $[a^-,\, a^+]f_0 = bf_0$, $[a^-,\, a^+]f_1 = (c-b)f_1$, and $[a^-,\, a^+]f_n = 0$, for all $n \geq 2$. This means that:

$$[a^-, a^+] \quad = \quad bP_0 + (c-b)P_1, \tag{3.7}$$

where P_k denotes the projection onto the space $G_k = \mathbb{C}f_k$. Moreover, since

$$[a^-, a^0]f_n \quad = \quad (\alpha_n - \alpha_{n-1})\omega_n f_{n-1}, \tag{3.8}$$

for all $n \geq 0$, where $\alpha_{-1} := 0$, we conclude that:

$$[a^-, a^0] \quad = \quad -\alpha a^- P_1. \tag{3.9}$$

From the recursive relation:

$$Xf_0(X) \;=\; f_1(X) + \alpha_0 f_0(X) + \omega_0 f_{-1}(X),$$

since $f_0 = 1$, we conclude that $f_1(X) = X - \alpha$. Moreover $E[f_1(X)^2] = \omega_1 = b$. Thus $\{1\}$ and $\{(1/\sqrt{b})(X-\alpha)\}$ are orthonormal bases of G_0 and G_1, respectively. Hence for all polynomial functions g, we have:

$$\begin{aligned} P_0 g(X) &= \langle g(X), 1\rangle 1 \\ &= E[g(X)]1 \end{aligned}$$

and

$$\begin{aligned} P_1 g(X) &= \langle g(X), (1/\sqrt{b})(X-\alpha)\rangle (1/\sqrt{b})(X-\alpha) \\ &= \frac{1}{b}\langle g(X), X-\alpha\rangle (X-\alpha) \\ &= \frac{1}{b}E[(X-\alpha)g(X)](X-\alpha). \end{aligned}$$

We apply now our commutator method to compute the moments of X. Actually, it is easier to compute the moments of $X - \alpha$ than those of X. For any fixed natural number n, we have:

$$\begin{aligned} E[(X-\alpha)^n] &= \langle (a^+ + a^0 + a^- - \alpha I)(X-\alpha I)^{n-1}1, 1\rangle \\ &= \langle a^+(X-\alpha I)^{n-1}1, 1\rangle \\ &+ \langle (a^0 - \alpha I)(X-\alpha I)^{n-1}1, 1\rangle \\ &+ \langle a^-(X-\alpha I)^{n-1}1, 1\rangle. \end{aligned}$$

Here I denotes the identity operator of $\mathcal{H}$. We have:

$$\begin{aligned} \langle a^+(X-\alpha I)^{n-1}1, 1\rangle &= \langle (X-\alpha I)^{n-1}1, a^-1\rangle \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \langle (a^0 - \alpha I)(X-\alpha I)^{n-1}1, 1\rangle &= \langle (X-\alpha I)^{n-1}1, (a^0 - \alpha I)1\rangle \\ &= 0. \end{aligned}$$

Thus we have:

$$E[(X-\alpha)^n] \;=\; \langle a^-(X-\alpha I)^{n-1}1, 1\rangle.$$

We swap now a^- and $(X-\alpha I)^n$. Since after the swap the annihilation operator a^- kills the vacuum polynomial 1, we get:

$$E[(X-\alpha)^n] \;=\; \langle [a^-, (X-\alpha I)^{n-1}]1, 1\rangle.$$

Thus we obtain:

$$\begin{aligned}
E[(X-\alpha)^n] &= \langle [a^-, (X-\alpha I)^{n-1}]1, 1\rangle \\
&= \sum_{j=0}^{n-2} \langle (X-\alpha I)^{n-2-j}[a^-, X-\alpha I](X-\alpha I)^j 1, 1\rangle \\
&= \sum_{j=0}^{n-2} \langle (X-\alpha I)^{n-2-j}[a^-, a^+ + a^0 + a^- - \alpha I](X-\alpha I)^j 1, 1\rangle \\
&= \sum_{j=0}^{n-2} \langle (X-\alpha I)^{n-2-j}[a^-, a^+](X-\alpha I)^j 1, 1\rangle \\
&+ \sum_{j=0}^{n-2} \langle (X-\alpha I)^{n-2-j}[a^-, a^0](X-\alpha I)^j 1, 1\rangle \\
&= \sum_{j=0}^{n-2} \langle (X-\alpha I)^{n-2-j}[bP_0 + (c-b)P_1](X-\alpha I)^j 1, 1\rangle \\
&- \alpha \sum_{j=0}^{n-2} \langle (X-\alpha I)^{n-2-j} a^- P_1 (X-\alpha I)^j 1, 1\rangle.
\end{aligned}$$

Since $P_0(X-\alpha)^j = E[(X-\alpha)^j]1$ and $P_1(X-\alpha)^j = (1/b)E[(X-\alpha)^{j+1}](X-\alpha)$, we obtain the following recursive formula:

$$\begin{aligned}
E[(X-\alpha)^n] &= b\sum_{j=0}^{n-2} E[(X-\alpha)^j]E[(X-\alpha)^{n-2-j}] \\
&+ \frac{c-b}{b}\sum_{j=0}^{n-2} E[(X-\alpha)^{j+1}]E[(X-\alpha)^{n-1-j}] \\
&- \alpha\sum_{j=0}^{n-2} E[(X-\alpha)^{j+1}]E[(X-\alpha)^{n-2-j}],
\end{aligned}$$

for all $n \geq 1$. Multiplying both sides of this recursive relation by t^n and then summing up from $n = 1$ to infinity, we obtain that the function $\varphi(t) = E[1/(1 - t(X-\alpha))]$ satisfies the following equation:

$$\varphi(t) - 1 = bt^2\varphi^2(t) + \frac{c-b}{b}[\varphi(t)-1]^2 - \alpha t\varphi(t)[\varphi(t)-1],$$

for all t in a neighborhood of 0. It must be observed, that in deriving this formula we interchanged the summation with the expectation, which is possible for the small values of t, due to the inequality (3.5). This relation is equivalent to the quadratic equation in $\varphi(t)$:

$$(bt^2 - \alpha t + p - 1)\varphi^2(t) + (\alpha t - 2p + 1)\varphi(t) + p = 0, \tag{3.10}$$

where $p := c/b$. Using the quadratic formula, we get:

$$\begin{aligned}\varphi(t) &= \frac{-\alpha t + 2p - 1 \pm \sqrt{(\alpha t+1)^2 - 4pbt^2}}{2(bt^2 - \alpha t + p - 1)} \\ &= \frac{-\alpha t + 2p - 1 \pm \sqrt{(\alpha t+1)^2 - 4pbt^2}}{2(bt^2 - \alpha t + p - 1)} \\ &\times \frac{-\alpha t + 2p - 1 \mp \sqrt{(\alpha t+1)^2 - 4pbt^2}}{-\alpha t + 2p - 1 \mp \sqrt{(\alpha t+1)^2 - 4pbt^2}} \\ &= \frac{4p\left(bt^2 - \alpha t + p - 1\right)}{2\left(bt^2 - \alpha t + p - 1\right)\left(-\alpha t + 2p - 1 \mp \sqrt{(\alpha t+1)^2 - 4pbt^2}\right)} \\ &= \frac{2p}{-\alpha t + 2p - 1 \mp \sqrt{(\alpha t+1)^2 - 4pbt^2}}.\end{aligned}$$

Since, $\varphi(0) = E[1] = 1$, we get:

$$\varphi(t) = \frac{2p}{-\alpha t + 2p - 1 + \sqrt{(\alpha t+1)^2 - 4pbt^2}},$$

for all t in a neighborhood of 0. Thus we get

$$E\left[\frac{1}{1 - t(X-\alpha)}\right] = \frac{2p}{-\alpha t + 2p - 1 + \sqrt{(\alpha t+1)^2 - 4pbt^2}}. \tag{3.11}$$

Replacing t by $1/t$, we obtain that the Cauchy–Stieltjes transform of X is:

$$E\left[\frac{1}{t - (X-\alpha)}\right] = \frac{2p}{-\alpha + (2p-1)t + s(t)\sqrt{(t+\alpha)^2 - 4pb}}, \tag{3.12}$$

for all t away from 0, where $s(t)$ denotes the sign function of t, i.e., $s(t) = t/|t|$.

We can invert the Cauchy–Stieltjes transform to find the probability distribution of $X - \alpha$ first, and then of X. We are not going over this computation, but the interested reader can read Theorem 5.3 from [7].

4. The $\mathcal{R}$–transform

We will close the paper, by giving a new proof of a theorem by Voiculescu concerning the analytic function theory tools for computing the $\mathcal{R}$–transform. We will briefly explain this transform, following the concepts from [10].

Let $H = \mathbb{C}e$ be a one–dimensional Hilbert space, where $\{e\}$ is an orthonormal basis of H. Let $\Gamma(H)$ be the *full Fock space* generated by H, that means:

$$\Gamma(H) := \mathbb{C}1 \oplus H \oplus H^{\otimes 2} \oplus H^{\otimes 3} \oplus \cdots,$$

where "$\oplus$" means orthogonal direct sum. We define a *left creation operator* a^+ (denoted by l in [10]) on $\Gamma(H)$ in the following way:

$$a^+\tau = \begin{cases} e & \text{if} \quad \tau = 1 \\ e \otimes \tau & \text{if} \quad \tau \in \Gamma(H) \ominus \mathbb{C}1, \end{cases} \tag{4.1}$$

where "$\ominus$" denotes the orthogonal complement. The adjoint of this operator is the *left annihilation operator* a^- (denoted by l^* in [10]) and is defined as:

$$a^-(k_1 \otimes k_2 \otimes \cdots \otimes k_n) \quad = \quad \begin{cases} \langle k_1, e\rangle k_2 \otimes \cdots \otimes k_n & \text{if} \quad n \geq 1 \\ 0 & \text{if} \quad n = 0, \end{cases} \tag{4.2}$$

where for $n = 0$, $k_1 \otimes k_2 \otimes \cdots \otimes k_n$ is understood to be a complex multiple of 1 (that means an element of $\mathbb{C}1$), and $\langle \cdot, \cdot\rangle$ denotes the inner product of H.

It is not hard to see that the commutator of the left creation and annihilation operators is:

$$[a^-, a^+] \quad = \quad P_0, \tag{4.3}$$

where P_0 denotes the orthogonal projection of $\Gamma(H)$ onto the vacuum space $\mathbb{C}1$. Moreover, $a^-a^+ = I$, where I denotes the identity operator of $\Gamma(H)$.

Definition 4.1. A *noncommutative probability space* is a unital algebra, A over $\mathbb{C}$ together with a linear functional, $\phi : A \to \mathbb{C}$, such that $\phi(1) = 1$.

Every element f in A is called a *random variable*, and $\phi(f)$ is called the *expectation* of f. For this reason we will replace the letter "ϕ" from [10] by "E". Every random variable f from A generates a *distribution* μ_f on the algebra of complex polynomials in one variable $\mathbb{C}[X]$, i.e., a linear functional from $\mathbb{C}[X]$ to $\mathbb{C}$ that maps the constant polynomial 1 into the complex number 1. It is defined by the formula:

$$\mu_f(P[X]) \quad := \quad E[P(f)], \tag{4.4}$$

for all $P[X] \in \mathbb{C}[X]$. Let Σ denote the set of all linear functionals μ, on $\mathbb{C}[X]$, such that $\mu(1) = 1$.

Proposition 4.2. *For all p_1, p_2, q_1, and q_2 non-negative integers, we have:*

$$(a^+)^{p_1}(a^-)^{q_1} \quad = \quad (a^+)^{p_2}(a^-)^{q_2} \tag{4.5}$$

if and only if $p_1 = p_2$ and $q_1 = q_2$.

Proof. Let us assume that $(a^+)^{p_1}(a^-)^{q_1} = (a^+)^{p_2}(a^-)^{q_2}$. Since for all $k \geq q_1$, $(a^+)^{p_1}(a^-)^{q_1}$ maps $H^{\otimes k}$ into $H^{\otimes(k+p_1-q_1)}$, and for all $k \geq q_2$, $(a^+)^{p_2}(a^-)^{q_2}$ maps $H^{\otimes k}$ into $H^{\otimes(k+p_2-q_2)}$, we conclude that:

$$p_1 - q_1 \quad = \quad p_2 - q_2. \tag{4.6}$$

Let us assume that $p_2 \geq p_1$ and thus, $m := p_2 - p_1 = q_2 - q_1 \geq 0$. By composing $(a^-)^{p_1}$ with each side of the equality (4.5), we get:

$$(a^-)^{p_1}(a^+)^{p_1}(a^-)^{q_1} \quad = \quad (a^-)^{p_1}(a^+)^{p_2}(a^-)^{q_2}.$$

Since $a^-a^+ = I$ and $p_2 \geq p_1$, we obtain:

$$(a^-)^{q_1} \quad = \quad (a^+)^{p_2-p_1}(a^-)^{q_2},$$

which means:

$$(a^-)^{q_1} \quad = \quad (a^+)^{m}(a^-)^{q_2}. \tag{4.7}$$

Let us compose now each side of the equality (4.7) with $(a^+)^{q_1}$ (to the right). We obtain:

$$(a^-)^{q_1}(a^+)^{q_1} = (a^+)^m(a^-)^{q_2}(a^+)^{q_1}.$$

Since $a^-a^+ = I$ and $q_2 \geq q_1$, we have:

$$I = (a^+)^m(a^-)^{q_2-q_1},$$

which means:

$$I = (a^+)^m(a^-)^m. \tag{4.8}$$

If $m > 0$, then the equality (4.8) is impossible since if we apply each side of it to the vacuum vector 1, we get: $I1 = 1$ while $(a^+)^m(a^-)^m 1 = 0$, because a^- kills the vacuum vector. Thus $m = 0$ and so, $p_1 = p_2$ and $q_1 = q_2$. □

Proposition 4.2 allows us to define the unital algebra $\tilde{\mathcal{E}}_1$ of formal series of the form:

$$\sum_{q=0}^{Q}\sum_{p=0}^{\infty} c_{p,q}(a^+)^p(a^-)^q, \tag{4.9}$$

where $Q \geq 0$ and $c_{p,q} \in \mathbb{C}$, for all $0 \leq q \leq Q$ and $p \geq 0$.

$\tilde{\mathcal{E}}_1$ is an algebra (i.e., closed under multiplication) due to the fact that the creation operators are always to the left of the annihilation operators, and $a^-a^+ = I$. We define the expectation E (i.e., the linear functional ϕ mapping 1 to 1), on $\tilde{\mathcal{E}}$, in the following way:

$$E\left[\sum_{q=0}^{Q}\sum_{p=0}^{\infty} c_{p,q}(a^+)^p(a^-)^q\right] := c_{0,0}. \tag{4.10}$$

Let us observe that formally, for any $f \in \tilde{\mathcal{E}}_1$, we have:

$$E[f] = \langle\langle f1, 1\rangle\rangle, \tag{4.11}$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the inner product of the Fock space $\Gamma(H)$ and 1 the vacuum vector of $\Gamma(H)$. Thus $(\tilde{\mathcal{E}}, E)$ is a noncommutative probability space.

Voiculescu proved (see [10]) that, for every $\mu \in \Sigma$ (let us remember that Σ denotes the set of all linear functionals μ, on $\mathbb{C}[X]$, such that $\mu(1) = 1$), there exists a unique random variable, T_μ, of the form $a^- + \sum_{k=0}^{\infty} \alpha_{k+1}(a^+)^k$ in $\tilde{\mathcal{E}}_1$, whose distribution in $(\tilde{\mathcal{E}}, E)$ is μ. Here the numbers $\alpha_1, \alpha_2, \dots$, represent arbitrary coefficients and have nothing to do with the Szegö–Jacobi parameters. T_μ is called the *canonical random variable* of μ. We define the $\mathcal{R}$*–transform* of μ to be the formal power series:

$$\mathcal{R}_\mu = \sum_{k=0}^{\infty} \alpha_{k+1}x^k. \tag{4.12}$$

We will give a proof of the following theorem (Theorem 3.3.1. from [10]), using both our commutator and dual commutator method.

Theorem 4.3. *Let μ be a distribution on $\mathbb{C}[X]$, with $\mathcal{R}$–transform*

$$\mathcal{R}_\mu(z) \quad = \quad \sum_{k=0}^{\infty} \alpha_{k+1} z^k. \tag{4.13}$$

Then denoting by μ_k the kth moment of μ, $\mu(X^k)$, we have that the formal power series

$$G(w) \quad = \quad w^{-1} + \sum_{k=1}^{\infty} \mu_k w^{-k-1} \tag{4.14}$$

and

$$K(z) \quad = \quad \frac{1}{z} + \mathcal{R}_\mu(z) \tag{4.15}$$

are inverses with respect to composition.

Proof. Let $T_\mu := a^- + \sum_{k=0}^{\infty} \alpha_{k+1}(a^+)^k \in \tilde{\mathcal{E}}_1$ be the canonical random variable of μ. Let us compute the moments of μ, or equivalently of T_μ, using our commutator method. For all $n \geq 1$, we have:

$$\begin{aligned}
\mu_n &= E\left[T_\mu^n\right] \\
&= \langle\langle T_\mu^n 1, 1\rangle\rangle \\
&= \langle\langle \left[a^- + \sum_{k=0}^{\infty} \alpha_{k+1}(a^+)^k\right] T_\mu^{n-1} 1, 1\rangle\rangle \\
&= \langle\langle a^- T_\mu^{n-1} 1, 1\rangle\rangle + \alpha_1 \langle\langle T_\mu^{n-1} 1, 1\rangle\rangle + \sum_{k=1}^{\infty} \alpha_{k+1} \langle\langle (a^+)^k T_\mu^{n-1} 1, 1\rangle\rangle.
\end{aligned}$$

For all $k \geq 1$, $(a^+)^k T_\mu^{n-1} 1 \in \Gamma(H) \ominus \mathbb{C}1$, and thus, $\langle\langle (a^+)^k T_\mu^{n-1} 1, 1\rangle\rangle = 0$. Hence, we obtain:

$$\begin{aligned}
\mu_n &= \langle\langle a^- T_\mu^{n-1} 1, 1\rangle\rangle + \alpha_1 \mu_{n-1} \\
&= \langle\langle [a^-, T_\mu^{n-1}] 1, 1\rangle\rangle + \langle\langle T_\mu^{n-1} a^- 1, 1\rangle\rangle + \alpha_1 \mu_{n-1} \\
&= \sum_{j=0}^{n-2} \langle\langle T_\mu^{n-2-j} [a^-, T_\mu] T_\mu^j 1, 1\rangle\rangle + \alpha_1 \mu_{n-1}.
\end{aligned}$$

We have:

$$\begin{aligned}
[a^-, T_\mu] &= [a^-, a^- + \alpha_1 I + \sum_{k=1}^{\infty} \alpha_{k+1}(a^+)^k] \\
&= \sum_{k=1}^{\infty} \alpha_{k+1} [a^-, (a^+)^k] \\
&= \sum_{k=1}^{\infty} \alpha_{k+1} \sum_{r=0}^{k-1} (a^+)^{k-1-r} [a^-, a^+] (a^+)^r \\
&= \sum_{k=1}^{\infty} \alpha_{k+1} \sum_{r=0}^{k-1} (a^+)^{k-1-r} P_0 (a^+)^r.
\end{aligned}$$

Now, we make the crucial observation that, for all $r \geq 1$, $P_0(a^+)^r = 0$, due to the fact that the range of $(a^+)^r$ is $H^{\otimes r} \oplus H^{\otimes(r+1)} \otimes \cdots$ which is orthogonal to the vacuum space $\mathbb{C}1$ (the range of P_0). Thus in the sum $\sum_{r=0}^{k-1}(a^+)^{k-1-r}P_0(a^+)^r$, from the commutator $[a^-, T_\mu]$, only the term corresponding to $r = 0$ survives. Therefore, we get:

$$[a^-, T_\mu] = \sum_{k=1}^{\infty} \alpha_{k+1}(a^+)^{k-1}P_0.$$

It follows now, that:

$$\begin{aligned}
\mu_n &= \sum_{j=0}^{n-2}\langle\langle T_\mu^{n-2-j}[a^-, T_\mu]T_\mu^j 1, 1\rangle\rangle + \alpha_1\mu_{n-1} \\
&= \alpha_1\mu_{n-1} + \sum_{j=0}^{n-2}\sum_{k=1}^{\infty}\alpha_{k+1}\langle\langle T_\mu^{n-2-j}(a^+)^{k-1}P_0T_\mu^j 1, 1\rangle\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
P_0T_\mu^j 1 &= \langle\langle T_\mu^j 1, 1\rangle\rangle 1 \\
&= E[T_\mu^j]1 \\
&= \mu_j 1,
\end{aligned}$$

we obtain:

$$\begin{aligned}
\mu_n &= \alpha_1\mu_{n-1} + \sum_{j=0}^{n-2}\mu_j\sum_{k=1}^{\infty}\alpha_{k+1}\langle\langle T_\mu^{n-2-j}(a^+)^{k-1}1, 1\rangle\rangle \\
&= \alpha_1\mu_{n-1} + \alpha_2\sum_{j=0}^{n-2}\mu_j\langle\langle T_\mu^{n-2-j}1, 1\rangle\rangle \\
&+ \sum_{j=0}^{n-2}\mu_j\sum_{k=2}^{\infty}\alpha_{k+1}\langle\langle T_\mu^{n-2-j}(a^+)^{k-1}1, 1\rangle\rangle \\
&= \alpha_1\mu_{n-1} + \alpha_2\sum_{j=0}^{n-2}\mu_j\mu_{n-2-j} \\
&+ \sum_{j=0}^{n-2}\mu_j\sum_{k=1}^{\infty}\alpha_{k+2}\langle\langle T_\mu^{n-2-j}(a^+)^k 1, 1\rangle\rangle.
\end{aligned}$$

In the last sum:

$$\sum_{j=0}^{n-2}\mu_j\sum_{k=1}^{\infty}\alpha_{k+2}\langle\langle T_\mu^{n-2-j}(a^+)^k 1, 1\rangle\rangle,$$

j is actually running from 0 to $n-3$, since for $j = n-2$, we have:

$$\begin{aligned}
\langle\langle T_\mu^{n-2-j}(a^+)^k 1, 1\rangle\rangle &= \langle\langle (a^+)^k 1, 1\rangle\rangle \\
&= \langle\langle (a^+)^{k-1}1, a^-1\rangle\rangle \\
&= 0,
\end{aligned}$$

for all $k \geq 1$.

We will now use the dual commutator method, to bring the creation operators from right to left. In the last sum:

$$\sum_{j=0}^{n-3} \mu_j \sum_{k=1}^{\infty} \alpha_{k+2} \langle\langle T_\mu^{n-2-j} (a^+)^k 1, 1 \rangle\rangle,$$

we swap T_μ^{n-2-j} and $(a^+)^k$ using the commutator formula:

$$\begin{aligned}
[T_\mu^{n-2-j}, (a^+)^k] 1 &= \sum_{i=0}^{n-3-j} T_\mu^{n-3-j-i} [T_\mu, (a^+)^k] T_\mu^i 1 \\
&= \sum_{i=0}^{n-3-j} T_\mu^{n-3-j-i} [a^-, (a^+)^k] T_\mu^i 1 \\
&= \sum_{i=0}^{n-3-j} T_\mu^{n-3-j-i} (a^+)^{k-1} P_0 T_\mu^i 1 \\
&= \sum_{i=0}^{n-3-j} \mu_i T_\mu^{n-3-j-i} (a^+)^{k-1} 1.
\end{aligned}$$

Thus, since after the swap $\langle\langle (a^+)^k T_\mu^{n-2-j} 1,\, 1 \rangle\rangle = 0$, we obtain:

$$\begin{aligned}
\mu_n &= \alpha_1 \mu_{n-1} + \alpha_2 \sum_{j=0}^{n-2} \mu_j \mu_{n-2-j} \\
&+ \sum_{j=0}^{n-3} \mu_j \sum_{k=1}^{\infty} \alpha_{k+2} \sum_{i=0}^{n-3-j} \mu_i \langle\langle T_\mu^{n-3-j-i} (a^+)^{k-1} 1, 1 \rangle\rangle \\
&= \alpha_1 \mu_{n-1} + \alpha_2 \sum_{j=0}^{n-2} \mu_j \mu_{n-2-j} \\
&+ \sum_{j=0}^{n-3} \mu_j \alpha_3 \sum_{i=0}^{n-3-j} \mu_i \langle\langle T_\mu^{n-3-j-i} 1, 1 \rangle\rangle \\
&+ \sum_{j=0}^{n-3} \mu_j \sum_{k=2}^{\infty} \alpha_{k+2} \sum_{i=0}^{n-3-j} \mu_i \langle\langle T_\mu^{n-3-j-i} (a^+)^{k-1} 1, 1 \rangle\rangle \\
&= \alpha_1 \mu_{n-1} + \alpha_2 \sum_{j=0}^{n-2} \mu_j \mu_{n-2-j} \\
&+ \alpha_3 \sum_{j=0}^{n-3} \sum_{i=0}^{n-3-j} \mu_j \mu_i \mu_{n-3-j-i} \\
&+ \sum_{j=0}^{n-3} \mu_j \sum_{k=1}^{\infty} \alpha_{k+3} \sum_{i=0}^{n-3-j} \mu_i \langle\langle T_\mu^{n-3-j-i} (a^+)^k 1, 1 \rangle\rangle.
\end{aligned}$$

We observe, as before, that in the last sum:

$$\sum_{j=0}^{n-3}\mu_j\sum_{k=1}^{\infty}\alpha_{k+3}\sum_{i=0}^{n-3-j}\mu_i\langle\langle T_{\mu}^{n-3-j-i}(a^+)^k 1, 1\rangle\rangle,$$

j is actually running from 0 to $n-4$, and i from 0 to $n-4-j$. We repeat this procedure swapping now $T_{\mu}^{n-3-j-i}$ and $(a^+)^k$, and so on, each time reducing the running interval for j, until this interval disappears. It is now clear that in the end, we get:

$$\begin{aligned}\mu_n &= \alpha_1\mu_{n-1}+\alpha_2\sum_{j_1+j_2=n-2}\mu_{j_1}\mu_{j_2}+\alpha_3\sum_{j_1+j_2+j_3=n-3}\mu_{j_1}\mu_{j_2}\mu_{j_3}+\cdots\\ &+ \alpha_n\sum_{j_1+j_2+\cdots+j_n=0}\mu_{j_1}\mu_{j_2}\cdots\mu_{j_n}, \qquad (4.16)\end{aligned}$$

for all $n\geq 1$. Formula (4.16) is very interesting and easy to memorize.

Dividing first both sides of formula (4.16) by w^{n+1}, and then summing up from $n=1$ to ∞, we get:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mu_n}{w^{n+1}} &= \frac{1}{w}\alpha_1\sum_{n=1}^{\infty}\frac{\mu_{n-1}}{w^n}\\ &+ \frac{1}{w}\alpha_2\left[\sum_{n=1}^{\infty}\frac{\mu_{n-1}}{w^n}\right]^2\\ &+ \frac{1}{w}\alpha_3\left[\sum_{n=1}^{\infty}\frac{\mu_{n-1}}{w^n}\right]^3\\ &\cdots \quad \cdots .\end{aligned}$$

Since $\mu_0=1$, this means:

$$\begin{aligned}G(w)-\frac{1}{w} &= \frac{G(w)}{w}\sum_{k=0}^{\infty}\alpha_{k+1}\left[G(w)\right]^k\\ &= \frac{G(w)}{w}\mathcal{R}(G(w)).\end{aligned}$$

This is equivalent to:

$$wG(w) = G(w)\mathcal{R}(G(w))+1,$$

which means:

$$\begin{aligned}w &= \mathcal{R}(G(w))+\frac{1}{G(w)}\\ &= K(G(w)).\end{aligned}$$

Thus $G(w)$ and $K(z)$ are inverses with respect to composition. □

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