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STOCHASTIC HEAT EQUATION WITH INFINITE DIMENSIONAL FRACTIONAL NOISE: L_2-THEORY

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ABSTRACT. In this article we consider the stochastic heat equation in $[0,T]\times\mathbb{R}^d$, driven by a sequence $(\beta^k)_k$ of i.i.d. fractional Brownian motions of index $H>1/2$ and random multiplication functions $(g^k)_k$. The stochastic integrals are of Hitsuda-Skorohod type and the solution is interpreted in the weak sense. Using Malliavin calculus techniques, we prove the existence and uniqueness of the solution in a certain space of random processes. Our result is similar to the one obtained in [18] for the stochastic heat equation driven by a sequence $(w^k)_k$ of i.i.d. Brownian motions, in which case the stochastic integrals are interpreted in the Itô sense.

1. Introduction

There is now a very rich theory dedicated to the study of stochastic partial differential equations (s.p.d.e.), which has been developed continuously during the past three decades, one of its goals being to offer a solid mathematical explanation for phenomena which evolve over time and are influenced by randomness.

Traditionally, the temporal structure of the noise perturbing such an equation was that of a Brownian motion. In the recent years, there has been an increased interest in looking into the possibility of replacing this structure with that of a fractional Brownian motion (fBm), which allows to build more flexibility into the time component of the noise, (depending on the value of the Hurst parameter H of the fBm), and increases the potential for applications.

The aim of the present article is to analyze the stochastic heat equation, in a space of square-integrable functions, when the driving noise bears the structure of the fBm with index $H>1/2$. Such a theory exists for equations whose noise terms behave like the Brownian motion, in spaces of arbitrary summability exponent $p\geq 2$ (the fundamental contribution is [17]; see [18] for more details), but has never been developed in the fBm case.

We recall that the *fBm with Hurst parameter* $H\in(0,1)$ is a zero-mean Gaussian process $\beta=(\beta_t)_{t\in[0,T]}$, with covariance

$$E(\beta_t\beta_s)=R_H(t,s):=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}).$$

(The Brownian motion is a fBm with parameter $H=1/2$.) The major difficulty is the fact that the fBm is not a semimartingale, and hence one can not use the Itô

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integral and its associated stochastic calculus. (We refer the reader to [30], [15] for more details on the fBm, and to [35] for a careful analysis of various integration questions related to the fBm.)

To the best of our knowledge, three methods have been developed to circumvent this difficulty, and they all perform well, especially in the regular case when $H > 1/2$, which is the one considered in the present paper. One method uses the Hitsuda-Skorohod integral and the associated Malliavin calculus (see [9], [10], [1], [2], [5] for some of the original developments). This method is used in the present article. Another method exploits the temporal "smoothness" of the noise, as opposed to the "roughness" of the Brownian path, and considers a pathwise generalized Stieltjes integral and its associated fractional calculus, as a replacement for the Itô-Stratonivich integral: the recent contributions [34], [37] show the full power of this method at work; background reading on fractional integration are [36], [40]. A third method relies on a re-thinking of the concept of noise, and the analysis of the infinite-dimensional "rough paths", as it was originally developed in [25], [26]. So far, this last method has been used mostly for the study of stochastic differential equations, the recent article [13] being among the first attempts to analyze s.p.d.e.'s driven by rough paths.

On the other hand, there are many different approaches to the theory of s.p.d.e.'s in the literature, speaking only of equations whose noise term bears the temporal structure of the Brownian motion, i.e. it is "white in time". These approaches have been developed in parallel, each of them being fruitful in its own way. A significant amount of effort is dedicated to unify to some extent these approaches.

To explain the contribution of the present paper, we need to recall briefly the salient features of these approaches, without aiming at exhausting the whole list of references in this active area of research. The vehicle that we choose for this overview of the literature is the stochastic heat equation on $[0, T] \times \mathbb{R}^d$, which is the focus of investigation in the present article.

In the stochastic approach (also called the "Walsh approach", due to [39]), when $H = 1/2$, this equation is written as:

$$\frac{\partial u}{\partial t} = \Delta u + f(t,x) + h(t,x)W(dt,dx), \quad t \in [0,T],\ x \in \mathbb{R}^d. \tag{1.1}$$

where f, h are predictable functions, the noise W is defined as a collection $\{W_t(\varphi)\}$ of Gaussian random variables with $E(W_t(\varphi)W_s(\psi)) = (t \wedge s)\langle \varphi, \psi \rangle_U$, and U is a Hilbert space of functions on $\mathbb{R}^d$. Initially, it was assumed that $U = L_2(\mathbb{R}^d)$. Later, U became a particular Hilbert space of distributions in $\mathbb{R}^d$, which is introduced via a kernel measure Γ (see [28], [29], [6], [7], [33], [3]). The noise W is said to be white in time, and "colored" in space. In short, the space U gives the color in space. The Walsh approach has been extended to cover the fBm case, by replacing W with a "fractional-colored" noise B with covariance $E(B_t(\varphi)B_s(\psi)) = R_H(t,s)\langle \varphi, \psi \rangle_U$ (see [32], [23], [4]).

A broader color-spectrum in space can be achieved via the semigroup approach, treated comprehensively in [8], in the case $H = 1/2$ (see also [16], [22], [24]). When using this approach, one works with an arbitrary Hilbert space U, and an infinite-dimensional Brownian motion $(\overline{W}_t)_{t\in[0,T]}$, whose covariance is a nuclear operator

Q. In this formulation, the equation is written as:

$$dX_t = [\Delta X_t + F(t)]dt + H(t)d\overline{W}_t, \quad t \in [0,T]. \tag{1.2}$$

Generally speaking, F, H are predictable functions with values in a suitable Hilbert space V of functions on $\mathbb{R}^d$, respectively in the space $L_2(U,V)$ of Hilbert-Schmidt operators from U to V. (If U is the particular Hilbert space of distributions on $\mathbb{R}^d$ considered in the Walsh approach, corresponding to a measure Γ, and $\overline{W}$ is a cylindrical Brownian motion on U, then $W_t(\varphi) := \langle \overline{W}_t, \varphi \rangle_U$ is a noise which is colored in space, and (1.1) can be rewritten in form (1.2) with $X_t = u(t,\cdot)$, $F(t) = f(t,\cdot)$, $H(t)e_k = h(t,\cdot)\bar{e}_k$ and $\bar{e}_k(x) = \int_{\mathbb{R}^d} e_k(x-y)\Gamma(dy)$, provided that $e_k \mapsto h(t,\cdot)\bar{e}_k$ lies in $L_2(U,V)$.) The semigroup approach has been successfully used in the fBm context, by considering an infinite-dimensional fBm $(\overline{B}_t)_{t\in[0,T]}$. Stochastic evolution equations with this type of noise have been analyzed in [38], [27], using the Hitsuda-Skorohod integral method, respectively the pathwise integral method; see also [14], [11] for some earlier developments.

Finally, in the L_p-theory approach, the equation is written as: (see [18])

$$du = [\Delta u + f(t,x)]dt + \sum_{k=1}^{\infty} g^k(t,x)dw_t^k, \quad t \in [0,T]. \tag{1.3}$$

Here f and $g = (g^k)_k$ are predictable functions with values in the Sobolev space $H_p^{n-1}(\mathbb{R}^d)$, respectively $H_p^{n+1}(\mathbb{R}^d, l_2)$, and the solution u lies in a subspace of $L_p(\Omega \times [0,T]; H_p^n(\mathbb{R}^d))$, speaking only of the main properties (n is not necessarily an integer). One of the appealing features of this approach is the relatively simple structure of the noise, given by a sequence $(w^k)_k$ of i.i.d. Brownian motions. However, a closer look reveals that in fact the multiplication functions g^k incorporate some of the spatial color of the noise, through the orthonormal basis $(e_k)_k$ of the space U (in the Walsh formulation). More precisely, by taking $W_t(\varphi) = \sum_k w_t^k \langle \varphi, e_k \rangle_U$, we obtain a noise which is colored in space, and (1.1) can be rewritten in form (1.3), with $g^k(t,x) = h(t,x)\bar{e}_k(x)$ (see [12]).

As mentioned earlier, this last approach has not been considered yet in the context of the fBm. The present paper is the first attempt to fill this gap, by using the Hitsuda-Skorohod integral and the Malliavin calculus, as a replacement for the Itô integral.

This article is organized as follows. In Section 2, we give the background material on the Malliavin calculus with respect to the fBm.

In Section 3, we develop some special Malliavin calculus techniques, suitable for treating H_2^n-valued random variables. A fundamental property which is used in the present paper is that for an arbitrary function $g \in \mathbb{D}_\beta^{1,2}(|\mathcal{H}_{H_2^n}|)$, the action of the Gross-Malliavin derivative $D^\beta g$ on a test function $\phi \in C_0^\infty$, coincides with the Gross-Malliavin derivative of the action of g on ϕ.

Section 4 contains a generalization of the second-moment maximal inequality for the Hitsuda-Skorohod integral with respect to the fBm (due to [2]), to the case of an infinite sequence of integrals with respect to some i.i.d. fBm's. This inequality is of crucial importance in the present article, being the replacement of the Burkholder-Davis-Gundy inequality, which is used in the Itô calculus. Although the result of

[2] (in the case of a single fBm) has been proved for an arbitrary moment of order $p > 1/H$, its generalization to the case of a sequence of fBm's becomes more complex. The inequality of [2], which lies at the origin of our developments, has been obtained only in the case $H > 1/2$. As far as we know, a similar inequality does not exist for the case $H < 1/2$. This is the reason the case $H < 1/2$ is not treated in the present article.

In Section 5, we introduce the concept of solution and we examine the solution space $\mathcal{H}^n_{2,H}$. Our definition of the solution space can be compared with Definition 3.1, [18], which introduces the solution space for a very general second-order s.p.d.e.'s of parabolic type (in particular the stochastic heat equation), whose noise is given by a sequence of Brownian motions. However, there are two essential differences between these two definitions. One comes from the fact that in the fBm's case, the coefficients f and g^k are jointly measurable in (ω, t), but not necessarily predictable; this can be viewed as a relaxation. The trade-off is that the coefficients g^k multiplying the fractional noise has an additional "differentiability" property in ω (rigorously defined via the Malliavin calculus techniques), which is not needed in the case of the Brownian noise.

Section 6 contains the result about the existence and uniqueness of the solution to the stochastic heat equation. The proof of this result is based on some preliminary estimates of the difference between the solution of the stochastic equation and the solution of the "deterministic" equation.

2. Malliavin Calculus Preliminaries

In this section we introduce the basic facts of Malliavin calculus with respect to the fBm of index $H > 1/2$. We refer the reader to [30], [31], [20], [21].

We begin by introducing various Hilbert spaces of deterministic functions, which are used in the present article.

If V is an arbitrary Hilbert space, let $\mathcal{E}_V$ be the class of all elementary functions $\phi : [0,T] \to V$ of the form $\phi(t) = \sum_{i=1}^m 1_{(t_i,t_{i+1}]}(t) v_i$ with $0 \le t_1 < \ldots < t_m \le T$ and $v_i \in V$, and $\mathcal{H}_V$ be the completion of $\mathcal{E}_V$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_V}$ defined by:

$$\langle \phi, \psi \rangle_{\mathcal{H}_V} := \alpha_H \int_0^T \int_0^T \langle \phi(t), \psi(s) \rangle_V |t-s|^{2H-2} ds dt, \quad \alpha_H = H(2H-1).$$

Let $|\mathcal{H}_V|$ be the space of all strongly measurable functions $\phi : [0,T] \to V$ with $\|\phi\|_{|\mathcal{H}_V|} < \infty$, where

$$\|\phi\|^2_{|\mathcal{H}_V|} := \alpha_H \int_0^T \int_0^T \|\phi(t)\|_V \ \|\phi(s)\|_V \ |t-s|^{2H-2} dt ds.$$

Then $\mathcal{E}_V$ is dense in $|\mathcal{H}_V|$ with respect to the norm $\|\cdot\|_{|\mathcal{H}_V|}$. We have:

$$\|\phi\|_{\mathcal{H}_V} \le \|\phi\|_{|\mathcal{H}_V|} \le b_H \|\phi\|_{L_{1/H}([0,T];V)} \le b_H \|\phi\|_{L_2([0,T];V)}, \tag{2.1}$$

for a constant $b_H > 0$, and $L_2([0,T];V) \subset L_{1/H}([0,T];V) \subset |\mathcal{H}_V| \subset \mathcal{H}_V$. If $V = \mathbb{R}$, we let $\mathcal{E}_V = \mathcal{E}$, $\mathcal{H}_V = \mathcal{H}$ and $|\mathcal{H}_V| = |\mathcal{H}|$. Note that $\mathcal{H}_V \simeq \mathcal{H} \otimes V$.

We denote by $\mathcal{H}\otimes|\mathcal{H}_V|$ the space of all strongly measurable functions $\phi : [0,T]^2 \to V$ with $\|\phi\|_{\mathcal{H}\otimes|\mathcal{H}_V|} < \infty$, where

$$\|\phi\|^2_{\mathcal{H}\otimes|\mathcal{H}_V|} := \alpha_H^2 \int_{[0,T]^4} \|\phi(t,\theta)\|_V \ \|\phi(s,\eta)\|_V \ |t-s|^{2H-2} \ |\theta-\eta|^{2H-2} d\theta d\eta ds dt.$$

In particular,

$$\langle \phi, \psi\rangle_{\mathcal{H}\otimes\mathcal{H}} := \alpha_H^2 \int_{[0,T]^4} \phi(t,\theta)\psi(s,\eta)|t-s|^{2H-2}||\theta-\eta|^{2H-2} d\theta d\eta ds dt,$$

and we hav:e $L_2([0,T]^2) \subset L_{1/H}([0,T]^2) \subset \mathcal{H}\otimes|\mathcal{H}| \subset \mathcal{H}\otimes\mathcal{H}$, with

$$\|\phi\|_{\mathcal{H}\otimes\mathcal{H}} \le \|\phi\|_{\mathcal{H}\otimes|\mathcal{H}|} \le b_H\|\phi\|_{L_{1/H}([0,T]^2)} \le b_H\|\phi\|_{L_2([0,T]^2)}. \tag{2.2}$$

We are now ready to introduce the main ingredients of the Malliavin calculus. Let $\beta = (\beta_t)_{t\in[0,T]}$ be a fBm of Hurst index $H > 1/2$.

One can see that $\mathcal{H}$ is the completion of $\mathcal{E}$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

The map $t \mapsto \beta_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with β. We denote this isometry by $\phi \mapsto \beta(\phi)$.

Let $\mathcal{S}_\beta := \{F = f(\beta(\phi_1),\ldots,\beta(\phi_n)); f \in C_b^\infty(\mathbb{R}^n), \phi_i \in \mathcal{H}, n \ge 1\} \subset L_2(\Omega)$ be the space of all "smooth cylindrical" random variables, where $C_b^\infty(\mathbb{R}^d)$ denotes the class of all bounded infinitely differentiable functions on $\mathbb{R}^n$, whose partial derivatives are also bounded.

The **Gross-Malliavin derivative** of an element $F = f(\beta(\phi_1,\ldots,\beta(\phi_n))) \in \mathcal{S}_\beta$, with respect to β, is defined by:

$$D^\beta F := \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\beta(\phi_1),\ldots,\beta(\phi_n))\phi_i \in L_2(\Omega;\mathcal{H}).$$

We endow $\mathcal{S}_\beta$ with the norm: $\|F\|^2_{\mathbb{D}^{1,2}_\beta} := E|F|^2 + E\|D^\beta F\|^2_{\mathcal{H}}$, and let $\mathbb{D}^{1,2}_\beta$ be the completion of $\mathcal{S}_\beta$ with respect to this norm. The operator D^β can be extended to $\mathbb{D}^{1,2}_\beta$. The adjoint $\delta^\beta : \mathrm{Dom}\ \delta^\beta \subset L_2(\Omega;\mathcal{H}) \to L_2(\Omega)$ of the operator D^β, is called the **Hitsuda-Skorohod integral** with respect to β. The operator δ^β is uniquely defined by the following relation:

$$E(F\delta^\beta(u)) = E\langle D^\beta F, u\rangle_{\mathcal{H}}, \quad \forall F \in \mathbb{D}^{1,2}_\beta.$$

We use the notation $\delta^\beta(u) = \int_0^T u_s \delta\beta_s$. Note that $E(\delta^\beta(u)) = 0, \forall u \in \mathrm{Dom}\ \delta^\beta$.

If V is a Hilbert space, let $\mathcal{S}_\beta(V)$ be the class of all "smooth cylindrical" V-valued random variables:

$$\mathcal{S}_\beta(V) := \{u = \sum_{j=1}^m F_j\phi_j; F_j \in \mathcal{S}_\beta, \phi_j \in V, m \ge 1\} \subset L_2(\Omega;V).$$

The Gross-Malliavin derivative of $u = \sum_{j=1}^m F_j\phi_j \in \mathcal{S}_B(V)$ is:

$$D^\beta u := \sum_{j=1}^m (D^\beta F_j)\phi_j \in L_2(\Omega;\mathcal{H}\otimes V).$$

We endow $\mathcal{S}_B(V)$ with the norm:

$$\|u\|^2_{\mathbb{D}^{1,2}_\beta(V)} := E\|u\|^2_V + E\|D^\beta u\|^2_{\mathcal{H}\otimes V}, \tag{2.3}$$

and let $\mathbb{D}^{1,2}_\beta(V)$ be the completion of $\mathcal{S}_\beta(V)$ with respect to this norm. The operator D^β can be extended to $\mathbb{D}^{1,2}_\beta(V)$.

In particular, if $V = \mathcal{H}$, then $\mathbb{D}^{1,2}_\beta(\mathcal{H}) \subset \text{Dom } \delta^\beta$, and

$$\begin{aligned} E|\delta^\beta(u)|^2 &= E\|u\|^2_{\mathcal{H}} + E(\langle D^\beta u, (D^\beta u)^* \rangle_{\mathcal{H}\otimes\mathcal{H})}) \\ &\leq E\|u\|^2_{\mathcal{H}} + E\|D^\beta u\|^2_{\mathcal{H}\otimes\mathcal{H}} = \|u\|^2_{\mathbb{D}^{1,2}_\beta(\mathcal{H})}, \quad \forall u \in \mathbb{D}^{1,2}_\beta(\mathcal{H}) \end{aligned} \tag{2.4}$$

where $(D^\beta u)^*$ is the adjoint of $D^\beta u$ in $\mathcal{H}\otimes\mathcal{H}$. If $u \in \mathbb{D}^{1,2}_\beta(\mathcal{H})$ then $D^\beta u \in L_2(\Omega; \mathcal{H}\otimes\mathcal{H})$. By abuse of notation, we write $D^\beta u = (D^\beta_t u_s)_{s,t\in[0,T]}$, even if $D^\beta u$ is not a function in s, t. As in [2], we introduce the following subspaces of $\mathbb{D}^{1,2}_\beta(\mathcal{H})$:

$$\begin{aligned} \mathbb{D}^{1,2}_\beta(|\mathcal{H}|) &:= \{u \in \mathbb{D}^{1,2}_\beta(\mathcal{H}); u \in |\mathcal{H}| \text{ a.s., } D^\beta u \in \mathcal{H}\otimes|\mathcal{H}| \text{ a.s., and} \\ &\qquad \|u\|_{\mathbb{D}^{1,2}_\beta(|\mathcal{H}|)} < \infty\}, \\ \mathbb{L}^{1,2}_{H,\beta} &:= \{u \in \mathbb{D}^{1,2}_\beta(|\mathcal{H}|); \|u\|_{\mathbb{L}^{1,2}_{H,\beta}} < \infty\}, \end{aligned}$$

where $\|u\|^2_{\mathbb{D}^{1,2}_\beta(|\mathcal{H}|)} := E\|u\|^2_{|\mathcal{H}|} + E\|D^\beta u\|^2_{\mathcal{H}\otimes|\mathcal{H}|}$ and

$$\|u\|^2_{\mathbb{L}^{1,2}_{H,\beta}} := E\int_0^T u_s^2 ds + E\int_0^T \left(\int_0^T |D^\beta_t u_s|^{1/H} dt\right)^{2H} ds.$$

From (2.1) and (2.2), we have:

$$\|u\|_{\mathbb{D}^{1,2}_\beta(|\mathcal{H}|)} \leq b_H \|u\|_{\mathbb{L}^{1,2}_{H,\beta}}, \quad \forall u \in \mathbb{L}^{1,2}_{H,\beta}. \tag{2.5}$$

From (2.4), (2.1) and (2.2), it follows that:

$$E|\delta^\beta(u)|^2 \leq \|u\|^2_{\mathbb{D}^{1,2}_\beta(|\mathcal{H}|)}, \quad \forall u \in \mathbb{D}^{1,2}_\beta(|\mathcal{H}|), \tag{2.6}$$

$$E|\delta^\beta(u)|^2 \leq b_H^2 (E\|u\|^2_{L_{1/H}([0,T])} + E\|D^\beta u\|^2_{L_{1/H}([0,T]^2)}), \quad \forall u \in \mathbb{L}^{1,2}_{H,\beta} \tag{2.7}$$

If $u \in \mathbb{L}^{1,2}_{H,\beta}$ then $u1_{[0,t]} \in \mathbb{D}^{1,2}_\beta(|\mathcal{H}|)$ for all $t \in [0,T]$ and we denote $\delta(u1_{[0,t]}) = \int_0^t u_s \delta\beta_s$. The following maximal inequality has been proved in [2]:

$$E \sup_{t\leq T} \left| \int_0^t u_s \delta\beta_s \right|^2 \leq C_{H,T} \|u\|^2_{\mathbb{L}^{1,2}_{H,\beta}}, \quad \forall u \in \mathbb{L}^{1,2}_{H,\beta} \tag{2.8}$$

where $C_{H,T}$ is a constant which depends on H and T.

Note that $\mathbb{L}^{1,2}_{H,\beta}$ may not be a Banach space with respect to the norm $\|\cdot\|_{L^{1,2}_{H,\beta}}$. The following definition introduces a complete subspace of $\mathbb{L}^{1,2}_{H,\beta}$.

Definition 2.1. Let $\mathcal{S}_\beta(\mathcal{E})$ be the class of processes $u_t = \sum_{i=1}^m F_i 1_{(t_{i-1},t_i]}(t), t \in [0,T]$, with $F_i \in \mathcal{S}_\beta$ and $0 \leq t_0 < \ldots < t_m \leq T$. We denote by $\widetilde{\mathbb{L}}^{1,2}_{H,\beta}$ the completion of $\mathcal{S}_\beta(\mathcal{E})$ with respect to the norm $\|\cdot\|_{\mathbb{L}^{1,2}_{H,\beta}}$.

In summary, we have:

$$\widetilde{\mathbb{L}}^{1,2}_{H,\beta} \subset \mathbb{L}^{1,2}_{H,\beta} \subset \mathbb{D}^{1,2}_{\beta}(|\mathcal{H}|) \subset \mathbb{D}^{1,2}_{\beta}(\mathcal{H}) \subset \text{Dom } \delta^{\beta} \subset L_2(\Omega;\mathcal{H}). \tag{2.9}$$

3. Malliavin Calculus for H_2^n-Valued Variables

We let C_0^{∞} be the space of infinitely differentiable functions on $\mathbb{R}^d$, with compact support, $\mathcal{D}$ be the space of real-valued Schwartz distributions on C_0^{∞}, and L_2 be the space of all square-integrable functions on $\mathbb{R}^d$. Let $n \in \mathbb{R}$ be arbitrary (*not necessarily an integer*). The **fractional Sobolev space** of index n is:

$$H_2^n := \{u \in \mathcal{D}; (1-\Delta)^{n/2}u \in L_2\},$$

with the norm given by: $\|u\|_{H_2^n} := \|(1-\Delta)^{n/2}u\|_{L_2}$. (See e.g. p. 187, [18] for the definition of $(1-\Delta)^{n/2}$). For any $u \in H_2^n$ and $\phi \in C_0^{\infty}$, we set:

$$(u,\phi) := \int_{\mathbb{R}^d} [(1-\Delta)^{n/2}u](x) \cdot [(1-\Delta)^{-n/2}\phi](x)dx.$$

By the Cauchy-Schwartz inequality, we have:

$$|(u,\phi)| \leq N\|u\|_{H_2^n}, \tag{3.1}$$

where $N = N_{n,\phi} = \|(1-\Delta)^{-n/2}\phi\|_{L_2}$ is a constant depending on n and ϕ.

Let $\beta = (\beta_t)_{t\in[0,T]}$ be a fractional Brownian motion of Hurst index $H > 1/2$, defined on a probability space $(\Omega,\mathcal{F},P)$. In the present work, we introduce an analogue of the space $\mathbb{L}^{1,2}_{H,\beta}$ for H_2^n-valued functions.

Definition 3.1. If U is an arbitrary Hilbert space, we let

$$\begin{aligned}
\mathbb{D}^{1,2}_{\beta}(|\mathcal{H}_U|) &:= \{g \in \mathbb{D}^{1,2}_{\beta}(\mathcal{H}_U); g \in |\mathcal{H}_U| \text{ a.s.}, D^{\beta}g \in \mathcal{H}\otimes|\mathcal{H}_U| \text{ a.s., and} \\
&\qquad \|g\|_{\mathbb{D}^{1,2}_{\beta}(|\mathcal{H}_U|)} < \infty\}, \\
\mathbb{L}^{1,2}_{H,\beta}(U) &:= \{g \in \mathbb{D}^{1,2}_{\beta}(|\mathcal{H}_U|); \|g\|_{\mathbb{L}^{1,2}_{H,\beta}(U)} < \infty\},
\end{aligned}$$

where $\|g\|^2_{\mathbb{D}^{1,2}_{\beta}(|\mathcal{H}_U|)} := E\|g\|^2_{|\mathcal{H}_U|} + E\|D^{\beta}g\|^2_{\mathcal{H}\otimes|\mathcal{H}_U|}$ and

$$\|g\|^2_{\mathbb{L}^{1,2}_{H,\beta}(U)} := E\int_0^T \|g_s\|_U^2 ds + E\int_0^T \left(\int_0^T \|D_t^{\beta}g_s\|_U^{1/H}dt\right)^{2H} ds. \tag{3.2}$$

Note that if $U = \mathbb{R}$, then $\mathbb{D}^{1,2}_{\beta}(|\mathcal{H}_U|) = \mathbb{D}^{1,2}_{\beta}(|\mathcal{H}|)$ and $\mathbb{L}^{1,2}_{H,\beta}(U) = \mathbb{L}^{1,2}_{H,\beta}$.

Let $\mathcal{S}_{\beta}(\mathcal{E}_U)$ be the space of all processes $g(t,\cdot) = \sum_{i=1}^m F_i 1_{(t_{i-1},t_i]}(t)v_i, t \in [0,T]$, with $F_i \in \mathcal{S}_{\beta}$, $0 \leq t_0 < \ldots < t_m \leq T, v_i \in U$. Note that $\mathcal{S}_{\beta}(\mathcal{E}_U)$ is dense in $\mathbb{D}^{1,2}_{\beta}(|\mathcal{H}_U|)$ and

$$\|u\|_{\mathbb{D}^{1,2}_{\beta}(|\mathcal{H}_U|)} \leq b_H\|u\|_{\mathbb{L}^{1,2}_{H,\beta}(U)}, \quad \forall u \in \mathbb{L}^{1,2}_{H,\beta}(U). \tag{3.3}$$

In the present article, we work with the space $\mathbb{L}^{1,2}_{H,\beta}(H_2^n)$ Let $\mathcal{S}_{\beta}(\mathcal{E}_{C_0^{\infty}})$ be the class of processes $g(t,\cdot) = \sum_{i=1}^m F_i 1_{(t_{i-1},t_i]}(t)\phi_i(\cdot), t \in [0,T]$, with $F_i \in \mathcal{S}_{\beta}$, $0 \leq t_0 < \ldots < t_m \leq T$ and $\phi_i \in C_0^{\infty}$. Note that $\mathcal{S}_{\beta}(\mathcal{E}_{C_0^{\infty}})$ is dense in $\mathbb{D}^{1,2}_{\beta}(|\mathcal{H}_{H_2^n}|)$.

Remark 3.2. Note that $g \in \mathbb{L}^{1,2}_{H,\beta}(H^n_2)$ implies that $g \in \mathbb{H}^n_2$ and $\mathbb{D}^\beta g \in \mathbb{H}^n_{2,H}$, where

$$\begin{aligned}\mathbb{H}^n_2 &:= L_2(\Omega \times [0,T], \mathcal{F} \times \mathcal{B}([0,T]); H^n_2)\\ \mathbb{H}^n_{2,H} &:= L_2(\Omega \times [0,T], \mathcal{F} \times \mathcal{B}([0,T]); L_{1/H}([0,T]; H^n_2))\end{aligned}$$

are stochastic spaces of Sobolev type. (We should emphasize that our definition for the space $\mathbb{H}^n_2$ is different than the one found in [18], since we are using the product σ-field $\mathcal{F} \times \mathcal{B}([0,T])$ instead of the *predictable* σ-field $\mathcal{P}$.) Moreover,

$$\begin{aligned}\|g\|^2_{\mathbb{L}^{1,2}_{H,\beta}(H^n_2)} &= E\int_0^T \|g_s\|^2_{H^n_2} ds + E\int_0^T \left(\int_0^T \|D^\beta_t g_s\|^{1/H}_{H^n_2} dt\right)^{2H} ds\\ &= \|g\|^2_{\mathbb{H}^n_2} + \|D^\beta g\|^2_{\mathbb{H}^n_{2,H}}. \qquad (3.4)\end{aligned}$$

In what follows, we examine some of the properties of a random function $g \in \mathbb{D}^{1,2}_\beta(|\mathcal{H}_{H^n_2}|)$. To simplify the writing, we denote by $*$ the missing t variable of such a function, to distinguish it from the missing x variable, denoted by $\cdot$.

We have the following preliminary estimates.

Lemma 3.3. *If $g \in \mathbb{D}^{1,2}_\beta(|\mathcal{H}_{H^n_2}|)$, then for any $\phi \in C^\infty_0$, we have:*

$$E\|(g(*,\cdot),\phi)\|^2_{|\mathcal{H}|} \leq N^2 E\|g\|^2_{|\mathcal{H}_{H^n_2}|} \qquad (3.5)$$

$$E\|(D^\beta g(*,\cdot),\phi)\|^2_{\mathcal{H}\otimes|\mathcal{H}|} \leq N^2 E\|D^\beta g\|^2_{\mathcal{H}\otimes|\mathcal{H}_{H^n_2}|}, \qquad (3.6)$$

where $N = N_{n,\phi} = \|(1-\Delta)^{-n/2}\phi\|_{L_2}$ is a constant depending on n and ϕ.

Proof. The result follows by (3.1). □

The next result shows that for an arbitrary function $g \in \mathbb{D}^{1,2}_\beta(|\mathcal{H}_{H^n_2}|)$, the Gross derivative commutes with the action of a test function $\phi \in C^\infty_0$.

Proposition 3.4. *If $g \in \mathbb{D}^{1,2}_\beta(|\mathcal{H}_{H^n_2}|)$, then for any $\phi \in C^\infty_0$, we have $(g(*,\cdot),\phi) \in \mathbb{D}^{1,2}_\beta(|\mathcal{H}|)$, and*

$$D^\beta(g(*,\cdot),\phi) = (D^\beta g(*,\cdot),\phi), \qquad (3.7)$$

$$\|(g(*,\cdot),\phi)\|_{\mathbb{D}^{1,2}_\beta(|\mathcal{H}|)} \leq N\|g\|_{\mathbb{D}^{1,2}_\beta(|\mathcal{H}_{H^n_2}|)}, \qquad (3.8)$$

where $N = N_{n,\phi} = \|(1-\Delta)^{-n/2}\phi\|_{L_2}$ is a constant depending on n and ϕ.

Proof. Case 1. Suppose that $g \in \mathcal{S}_\beta(\mathcal{E}_{C^\infty_0})$. Say $g(t,\cdot) = \sum_{i=1}^m F_i 1_{(t_i,t_{i+1}]}(t)\phi_i$ with $F_i \in \mathcal{S}_\beta$, $0 \leq t_1 < \ldots < t_{m+1} \leq T$ and $\phi_i \in C^\infty_0$. Denote $\Psi_i(t,\cdot) = 1_{(t_i,t_{i+1}]}(t)\phi_i(\cdot)$. Let $\phi \in C^\infty_0$ be arbitrary. Clearly $(\Psi^i(*,\cdot),\phi) \in \mathcal{E}$, and hence $(g(*,\cdot),\phi) = \sum_{i=1}^m F_i(\Psi_i(*,\cdot),\phi) \in \mathcal{S}_\beta(\mathcal{E}) \subset \mathbb{D}^{1,2}_\beta(|\mathcal{H}|)$. Due to the linearity of D^β, we have: $D^\beta_t(g(s,\cdot),\phi) = \sum_{i=1}^m (D^\beta_t F_i)(\Psi_i(s,\cdot),\phi) = (D^\beta_t g(s,\cdot),\phi)$. Finally, (3.8) follows from (3.7) and the preliminary estimates (3.5), (3.6):

$$\begin{aligned}\|(g(*,\cdot),\phi)\|^2_{\mathbb{D}^{1,2}_\beta(|\mathcal{H}|)} &= E\|(g(*,\cdot),\phi)\|^2_{|\mathcal{H}|} + E\|D^\beta(g(*,\cdot),\phi)\|^2_{\mathcal{H}\otimes|\mathcal{H}|}\\ &= E\|(g(*,\cdot),\phi)\|^2_{|\mathcal{H}|} + E\|(D^\beta g(*,\cdot),\phi)\|^2_{\mathcal{H}\otimes|\mathcal{H}|}\\ &\leq N^2 E\|g\|^2_{|\mathcal{H}_{H^n_2}|} + N^2 E\|D^\beta g\|^2_{\mathcal{H}\otimes|\mathcal{H}_{H^n_2}|}\\ &= N^2\|g\|^2_{\mathbb{D}^{1,2}_\beta(|\mathcal{H}_{H^n_2}|)}.\end{aligned}$$

Case 2. Suppose that $g \in \mathbb{D}_\beta^{1,2}(|\mathcal{H}_{H_2^n}|)$ is arbitrary. Then, there exists a sequence $(g_j)_j \subset \mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ such that $\|g_j - g\|_{\mathbb{D}_\beta^{1,2}(|\mathcal{H}_{H_2^n}|)} \to 0$ as $j \to \infty$, i.e. $E\|g_j - g\|^2_{|\mathcal{H}_{H_2^n}|} \to 0$ and $E\|D^\beta g_j - D^\beta g\|^2_{\mathcal{H}\otimes|\mathcal{H}_{H_2^n}|} \to 0$.

From *Case 1*, it follows that for any $\phi \in C_0^\infty$, we have:

$$D^\beta(g_j(*,\cdot),\phi) = (D^\beta g_j(*,\cdot),\phi). \tag{3.9}$$

On one hand, due to the estimates (3.5) and (3.6), we have:

$$E\|(g_j(*,\cdot),\phi) - (g(*,\cdot),\phi)\|^2_{|\mathcal{H}|} \leq N^2 E\|g_j - g\|^2_{|\mathcal{H}_{H_2^n}|} \to 0 \tag{3.10}$$

$$E\|(D^\beta g_j(*,\cdot) - D^\beta g(*,\cdot),\phi)\|^2_{\mathcal{H}\otimes|\mathcal{H}|} \leq N^2 E\|D^\beta g_j - D^\beta g\|^2_{\mathcal{H}\otimes|\mathcal{H}_{H_2^n}|} \to 0. \tag{3.11}$$

On the other hand, due to the estimate (3.8) obtained in *Case 1*, it follows that $\{(g_j(*,\cdot),\phi)\}_j$ is a Cauchy sequence in $\mathbb{D}_\beta^{1,2}(|\mathcal{H}|)$. Hence, there exists $h_\phi \in \mathbb{D}_\beta^{1,2}(|\mathcal{H}|)$ such that $\|(g_j(*,\cdot),\phi) - h_\phi\|_{\mathbb{D}_\beta^{1,2}(|\mathcal{H}|)} \to 0$, i.e.

$$E\|(g_j(*,\cdot),\phi) - h_\phi\|^2_{|\mathcal{H}|} \to 0, \quad \text{and} \tag{3.12}$$

$$E\|D^\beta(g_j(*,\cdot),\phi) - D^\beta h_\phi\|^2_{\mathcal{H}\otimes|\mathcal{H}|} \to 0 \tag{3.13}$$

From (3.10) and (3.12), it follows that $(g(*,\cdot),\phi) = h_\phi \in \mathbb{D}_\beta^{1,2}(|\mathcal{H}|)$. From (3.9), (3.11) and (3.13), we conclude that $(D^\beta g(*,\cdot),\phi) = D^\beta h_\phi = D^\beta(g(*,\cdot),\phi)$, i.e. (3.7) holds. Based on (3.7), one deduces the estimate (3.8) as in *Case 1*. □

The next result is an immediate consequence of Proposition 3.4 and (3.1).

Corollary 3.5. *If $g \in \mathbb{L}_{H,\beta}^{1,2}(H_2^n)$, then for any $\phi \in C_0^\infty$, $(g(*,\cdot),\phi) \in \mathbb{L}_{H,\beta}^{1,2}$, and*

$$\|(g(*,\cdot),\phi)\|_{\mathbb{L}_{H,\beta}^{1,2}} \leq N\|g\|_{\mathbb{L}_{H,\beta}^{1,2}(H_2^n)}, \tag{3.14}$$

where $N = N_{n,\phi} = \|(1-\Delta)^{-n/2}\phi\|_{L_2}$ is a constant depending on n and ϕ.

The following definition introduces a complete subspace of $\|_{\mathbb{L}_{H,\beta}^{1,2}(H_2^n)}$.

Definition 3.6. We let $\widetilde{\mathbb{L}}_{H,\beta}^{1,2}(H_2^n)$ be the completion of $\mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ with respect to the norm $\|\cdot\|_{L_{H,\beta}^{1,2}(H_2^n)}$.

To summarize, here are the spaces introduced in this section:

$$\widetilde{\mathbb{L}}_{H,\beta}^{1,2}(H_2^n) \subset \mathbb{L}_{H,\beta}^{1,2}(H_2^n) \subset \mathbb{D}_\beta^{1,2}(|\mathcal{H}_{H_2^n}|) \subset \mathbb{D}_\beta^{1,2}(\mathcal{H}_{H_2^n}) \subset L_2(\Omega;\mathcal{H}_{H_2^n}).$$

4. The Infinite Dimensional Noise

Let us now consider a sequence $\beta^k = (\beta_t^k)_{t\in[0,T]}, k \geq 1$ of i.i.d. fBm's with Hurst index $H > 1/2$, defined on the same probability space $(\Omega,\mathcal{F},P)$.

The following result generalizes the second-moment maximal inequality (2.8) to an infinite sequence of i.i.d. fBm's.

Theorem 4.1. *Let* $u^k \in \mathbb{L}^{1,2}_{H,\beta^k}$ *be such that* $\sum_{k=1}^{\infty} \|u^k\|^2_{\mathbb{L}^{1,2}_{H,\beta^k}} < \infty$. *Then*

$$E \sup_{t \leq T} \left| \sum_{k=1}^{\infty} \int_0^t u_s^k \delta\beta_s^k \right|^2 \leq C_{H,T} \sum_{k=1}^{\infty} \|u^k\|^2_{\mathbb{L}^{1,2}_{H,\beta^k}},$$

where $C_{H,T}$ *is a constant depending on* H *and* T.

Proof. As in the proof of Theorem 4, [2], we let $\alpha = 1/2 - \varepsilon$ with $\varepsilon \in (0, H - 1/2)$ and we use the fact that $\int_0^t u_s^k \delta\beta_s^k = c_\alpha \int_0^t (t-r)^{-\alpha} \left(\int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right) dr$. Using Cauchy-Schwartz inequality and the fact that $2\alpha < 1$, we obtain

$$\begin{aligned}
\left| \sum_{k=1}^{\infty} \int_0^t u_s^k \delta\beta_s^k \right|^2 &= c_\alpha^2 \left| \int_0^t (t-r)^{-\alpha} \left(\sum_{k=1}^{\infty} \int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right) dr \right|^2 \\
&\leq c_\alpha^2 \left(\int_0^t (t-r)^{-2\alpha} dr \right) \int_0^t \left| \sum_{k=1}^{\infty} \int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right|^2 dr \\
&= c'_\alpha \int_0^t \left| \sum_{k=1}^{\infty} \int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right|^2 dr, \quad \forall t \leq T.
\end{aligned}$$

Therefore,

$$\sup_{t \leq T} \left| \sum_{k=1}^{\infty} \int_0^t u_s^k \delta\beta_s^k \right|^2 \leq c'_\alpha \int_0^T \left| \sum_{k=1}^{\infty} \int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right|^2 dr. \quad (4.1)$$

Let $v_s^k = u_s^k (r-s)^{\alpha-1}, s \in [0,T]$ and note that $v^k \in \mathbb{L}^{1,2}_{H,\beta^k}$. Since $(\beta^k)_k$ are independent fBm's and each $v^k \in \mathbb{L}^{1,2}_{H,\beta^k}$, it follows that the random variables $X_k = \int_0^r v_s^k \delta\beta_s^k, k \geq 1$ are independent. Moreover, $E(X_k) = 0$ for all k. Hence

$$E \left| \sum_{k=1}^{n} \int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right|^2 = \sum_{k=1}^{n} E \left| \int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right|^2. \quad (4.2)$$

Using the Fatou's lemma, we infer that

$$E \left| \sum_{k=1}^{\infty} \int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right|^2 \leq \sum_{k=1}^{\infty} E \left| \int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right|^2. \quad (4.3)$$

From (4.1) and (4.3), we get:

$$E \sup_{t \leq T} \left| \sum_{k=1}^{\infty} \int_0^t u_s^k \delta\beta_s^k \right|^2 \leq c'_\alpha \sum_{k=1}^{\infty} \int_0^T E \left| \int_0^r u_s^k (r-s)^{\alpha-1} \delta\beta_s^k \right|^2 dr.$$

Using (2.7), we get

$$E \sup_{t \leq T} \left| \sum_{k=1}^{\infty} \int_0^t u_s^k \delta\beta_s^k \right|^2 \leq c'_\alpha b_H^2 \left\{ \sum_{k=1}^{\infty} \int_0^T E \left(\int_0^r |u_s^k|^{1/H} (r-s)^{(\alpha-1)/H} ds \right)^{2H} dr + \right.$$

$$\sum_{k=1}^{\infty}\int_0^T E\left(\int_0^r\int_0^T |D_\theta^{\beta^k}u_s^k|^{1/H}(r-s)^{(\alpha-1)/H}d\theta ds\right)^{2H} dr\Bigg\} := c'_\alpha b_H^2(I_1+I_2).$$

Using Holder's inequality with $p=2H$ and $q=2H/(2H-1)$, we get:

$$\left(\int_0^r |u_s^k|^{1/H}(r-s)^{(\alpha-1)/H}ds\right)^{2H} \leq c_{\alpha,H}r^{2(\alpha-1)+2H-1}\int_0^r |u_s^k|^2 ds,$$

$$\left(\int_0^r\int_0^T |D_\theta^{\beta^k}u_s^k|^{1/H}(r-s)^{(\alpha-1)/H}d\theta ds\right)^{2H} \leq$$
$$c_{\alpha,H}r^{2(\alpha-1)+2H-1}\int_0^r\left(\int_0^T |D_\theta^{\beta^k}u_s^k|^{1/H}d\theta\right)^{2H} ds,$$

and hence

$$\begin{aligned} I_1 &\leq c_{\alpha,H}\int_0^T r^{2(\alpha-1)+2H-1}\sum_{k=1}^{\infty}E\int_0^r |u_s^k|^2 dsdr \leq c_{\alpha,H,T}E\sum_{k=1}^{\infty}\int_0^T |u_s^k|^2 ds \\ I_2 &\leq c_{\alpha,H}\int_0^T r^{2(\alpha-1)+2H-1}\sum_{k=1}^{\infty}E\int_0^r\left(\int_0^T |D_\theta^{\beta^k}u_s^k|^{1/H}d\theta\right)^{2H} dsdr \\ &\leq c_{\alpha,H,T}E\sum_{k=1}^{\infty}\int_0^T\left(\int_0^T |D_\theta^{\beta^k}u_s^k|^{1/H}d\theta\right)^{2H} ds. \end{aligned}$$

□

In what follows, we let l_2 be the set of all real-valued sequences $a=(a^k)_k$ with $\sum_k |a^k|^2<\infty$.

The following space is the l_2-variant of the space $\mathbb{L}_{H,\beta}^{1,2}(H_2^n)$.

Definition 4.2. We denote by $\mathbb{L}_H^{1,2}(H_2^n,l_2)$ the set of all $g=(g^k)_k$ such that $g^k\in\mathbb{D}_{\beta^k}^{1,2}(|\mathcal{H}_{H_2^n}|)$ for all k and $\|g\|_{\mathbb{L}_H^{1,2}(H_2^n,l_2)}<\infty$, where

$$\|g\|^2_{\mathbb{L}_H^{1,2}(H_2^n,l_2)} := \sum_{k=1}^{\infty}\|g^k\|^2_{\mathbb{L}_{H,\beta^k}^{1,2}(H_2^n)}. \tag{4.4}$$

We also consider the following l_2-variants of the stochastic spaces $\mathbb{H}_2^n$ and $\mathbb{H}_{2,H}^n$, defined in Remark 3.2:

$$\begin{aligned} \mathbb{H}_2^n(l_2) &:= \{g=(g^k)_k; g^k\in\mathbb{H}_2^n\ \forall k,\ \|g\|_{\mathbb{H}_2^n(l_2)}<\infty\} \\ \mathbb{H}_{2,H}^n(l_2) &:= \{g=(g^k)_k; g^k\in\mathbb{H}_{2,H}^n\ \forall k,\ \|g\|_{\mathbb{H}_{2,H}^n(l_2)}<\infty\}, \end{aligned}$$

where

$$\|g\|^2_{\mathbb{H}_2^n(l_2)} := \sum_{k=1}^{\infty}\|g^k\|^2_{\mathbb{H}_2^n} \quad\text{and}\quad \|g\|^2_{\mathbb{H}_{2,H}^n(l_2)} := \sum_{k=1}^{\infty}\|g^k\|^2_{\mathbb{H}_{2,H}^n}.$$

Let $Dg := (D^{\beta^k} g^k)_k$ be the "Gross derivative" of $g = (g^k)_k \in \mathbb{L}_H^{1,2}(H_2^n, l_2)$. Then,

$$\|g\|^2_{\mathbb{L}_H^{1,2}(H_2^n,l_2)} = \|g\|^2_{\mathbb{H}_2^n(l_2)} + \|Dg\|^2_{\mathbb{H}^n_{2,H}(l_2)} \quad (4.5)$$

$$\sum_{k=1}^{\infty} \left[E\int_0^T \|g^k(s,\cdot)\|^2_{H_2^n} ds + E\int_0^T \left(\int_0^T \|D_t^{\beta^k} g^k(s,\cdot)\|^{1/H}_{H_2^n} dt \right)^{2H} ds \right] (4.6)$$

The following definition introduces the space in which we are allowed to pick the random coefficient $g = (g^k)_k$, multiplying the noise $(\beta^k)_k$.

Definition 4.3. We let $\widetilde{\mathbb{L}}_H^{1,2}(H_2^n, l_2)$ be the set of all $g \in \mathbb{L}_H^{1,2}(H_2^n, l_2)$ for which there exists a sequence $(g_j)_j \subset \mathbb{L}_H^{1,2}(H_2^n, l_2)$ such that $\|g_j - g\|_{\mathbb{L}_H^{1,2}(H_2^n,l_2)} \to 0$ as $j \to \infty$, $(g_j^k)_j \subset \mathcal{S}_{\beta^k}(\mathcal{E}_{C_0^\infty})$ for $k \le K_j$, and $g_j^k = 0$ for $k > K_j$.

Theorem 4.4. *Let $g \in \mathbb{L}_H^{1,2}(H_2^n, l_2)$ be arbitrary. Then $g \in \widetilde{\mathbb{L}}_H^{1,2}(H_2^n, l_2)$ if and only if $g^k \in \widetilde{\mathbb{L}}^{1,2}_{H,\beta^k}(H_2^n)$ for all k.*

Proof. The argument is standard and is omitted. □

5. The Solution Space

The following definition introduces the solution space for the stochastic heat equation, whose noise term is given by a sequence of i.i.d. fBm's.

Definition 5.1. Let $\beta^k = (\beta_t^k)_{t\in[0,T]}, k \ge 1$ be a sequence of i.i.d. fBm's with Hurst index $H > 1/2$, defined on the same probability space $(\Omega, \mathcal{F}, P)$.

Let $u = \{u(t,\cdot)\}_{t\in[0,T]}$ be a $\mathcal{D}$-valued random process defined on the probability space $(\Omega, \mathcal{F}, P)$. We write $u \in \mathcal{H}^n_{2,H}$ if:

(i) $u(0,\cdot) \in L_2(\Omega, \mathcal{F}; H_2^{n-1})$;

(ii) $u \in \mathbb{H}_2^n$, $u_{xx} \in \mathbb{H}_2^{n-2}$; and

(iii) there exist $f \in \mathbb{H}_2^{n-2}$ and $g \in \widetilde{\mathbb{L}}_H^{1,2}(H_2^{n-1}, l_2)$ such that for any $\phi \in C_0^\infty$, the equality

$$(u(t,\cdot),\phi) = (u(0,\cdot),\phi) + \int_0^t (f(s,\cdot),\phi)ds + \sum_{k=1}^{\infty} \int_0^t (g^k(s,\cdot),\phi)\delta\beta_s^k \quad (5.1)$$

holds for any $t \in [0,T]$ a.s. We define

$$\|u\|_{\mathcal{H}^n_{2,H}} = (E\|u(0,\cdot)\|^2_{H_2^{n-1}})^{1/2} + \|u_{xx}\|_{\mathbb{H}_2^{n-2}} + \|f\|_{\mathbb{H}_2^{n-2}} + \|g\|_{\mathbb{L}_H^{1,2}(H_2^{n-1},l_2)}. \quad (5.2)$$

The next lemma shows that the series of stochastic integrals in (5.1) converges uniformly in $t \in [0,T]$, in probability.

Lemma 5.2. *Let $g \in \mathbb{L}_H^{1,2}(H_2^n, l_2)$ and $\phi \in C_0^\infty$ be arbitrary. For each $t \in [0,T]$, let $X_t^{(K)} = \sum_{k=1}^K \int_0^t (g^k(s,\cdot),\phi)\delta\beta_s^k, K \ge 1$, and $X_t = \sum_{k=1}^\infty \int_0^t (g^k(s,\cdot),\phi)\delta\beta_s^k$. Then X_t is finite a.s., and the sequence $(X^{(K)})_K$ converges in probability to X, in the sup-norm metric, i.e. $\lim_{K\to\infty} P(\sup_{t\le T} |X_t^{(K)} - X_t| \ge \varepsilon) = 0$, $\forall \varepsilon > 0$.*

Proof. Let $u^k_s = (g^k(s,\cdot),\phi)$. Note that $u^k \in \mathbb{L}^{1,2}_{H,\beta^k}$ for all k. By Theorem 4.1 and (3.14), $E(X_t^2) \leq C_{H,T}\sum_{k=1}^{\infty}\|u^k\|^2_{\mathbb{L}^{1,2}_{H,\beta^k}} \leq C_{H,T}N^2\sum_{k=1}^{\infty}\|g^k\|^2_{\mathbb{L}^{1,2}_{H,\beta^k}(H_2^n)} < \infty$. Hence, X_t is finite a.s. Using Chebyshev's inequality, Theorem 4.1, and (3.14), we get:

$$P(\sup_{t\leq T}|X_t^{(K)} - X_t| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E\sup_{t\leq T}\left|\sum_{k=K+1}^{\infty}\int_0^t u^k_s \delta\beta^k_s\right|^2 \leq$$
$$\frac{1}{\varepsilon^2}C_{H,T}\sum_{k=K+1}^{\infty}\|u^k_s\|^2_{\mathbb{L}^{1,2}_{H,\beta^k}} \leq \frac{1}{\varepsilon^2}C_{H,T}N^2\sum_{k=K+1}^{\infty}\|g^k\|^2_{\mathbb{L}^{1,2}_{H,\beta^k}(H_2^n)}.$$

The last terms converge to 0 as $K\to\infty$, since $\|g\|_{\mathbb{L}^{1,2}_H(H_2^n,l_2)} < \infty$. □

Remark 5.3. Note that by Theorem 5, [2], each process $X^{(K)}$ has an a.s. continuous modification. By invoking the previous lemma, and using a "classical" argument in probability theory (see e.g. the proof of Theorem 6.1.10, [19]), we conclude that the process X has an a.s. continuous modification. By (5.1), it follows that if $u \in \mathcal{H}^n_{2,H}$, then the process $\{(u(t,\cdot),\phi)\}_{t\in[0,T]}$ has an a.s. continuous modification, for any $\phi \in C_0^\infty$. We work with this modification.

For technical reasons, one prefers not to handle directly the elements in the space H_2^n, and work instead with the images of these elements in the nicer space L_2, via the operator $(1-\Delta)^{n/2}$. More liberty in choosing the right index n depending on the problem at hand, comes from the fact that the operator $(1-\Delta)^{m/2}$ maps isometrically H_2^n onto H_2^{n-m}, for any n and m. This property continues to hold for the stochastic spaces $\mathbb{H}_2^n$ and $\mathbb{H}_2^n(l_2)$ (see Remark 3.4, [18]). The next two results empower us with the same freedom of choice of the right index n, when working with the newly introduced spaces $\widetilde{\mathbb{L}}^{1,2}_{H,\beta}(H_2^n)$ and $\widetilde{\mathbb{L}}^{1,2}_{H}(H_2^n,l_2)$, and the solution space $\mathcal{H}^n_{2,H}$.

Proposition 5.4. *The operator* $(1-\Delta)^{m/2}$ *maps isometrically* $\widetilde{\mathbb{L}}^{1,2}_{H}(H_2^n,l_2)$ *onto* $\widetilde{\mathbb{L}}^{1,2}_{H}(H_2^{n-m},l_2)$.

Proof. Step 1. Let $\beta = (\beta_t)_{t\in[0,T]}$ be a fixed fBm. We first prove that $(1-\Delta)^{m/2}$ maps isometrically $\widetilde{\mathbb{L}}^{1,2}_{H,\beta}(H_2^n)$ onto $\widetilde{\mathbb{L}}^{1,2}_{H,\beta}(H_2^{n-m})$.

Let $g \in \mathbb{L}^{1,2}_{H,\beta}(H_2^n)$ be arbitrary. We prove that $(1-\Delta)^{m/2}g \in \mathbb{L}^{1,2}_{H,\beta}(H_2^{n-m})$. Since the Malliavin derivative commutes with the action of a test function $\phi \in C_0^\infty$ (see (3.7)), we have:

$$(D^\beta[(1-\Delta)^{m/2}g(*,\cdot)],\phi) = D^\beta((1-\Delta)^{m/2}g(*,\cdot),\phi) =$$
$$D^\beta(g(*,\cdot),(1-\Delta)^{m/2}\phi) = (D^\beta g(*,\cdot),(1-\Delta)^{m/2}\phi) =$$
$$((1-\Delta)^{m/2}[D^\beta g(*,\cdot)],\phi),$$

for any $\phi \in C_0^\infty$, i.e.

$$D_t^\beta[(1-\Delta)^{m/2}g(s,\cdot)] = (1-\Delta)^{m/2}[D_t^\beta g(s,\cdot)], \quad \forall s,t\in[0,T]. \tag{5.3}$$

Using an approximation argument and the fact that $\|u\|_{H_2^n} = \|(1-\Delta)^{m/2}u\|_{H_2^{n-m}}$ for any $u \in H_2^n$, we conclude that $(1-\Delta)^{m/2}g \in \mathbb{D}_\beta^{1,2}(|\mathcal{H}_{H_2^{n-m}}|)$. Using (3.4) and (5.3), we obtain:

$$\begin{aligned} \|(1-\Delta)^{m/2}g\|^2_{\mathbb{L}^{1,2}_{H,\beta}(H_2^{n-m})} &= \|(1-\Delta)^{m/2}g\|^2_{\mathbb{H}_2^{n-m}} + \|D^\beta[(1-\Delta)^{m/2}g]\|^2_{\mathbb{H}^{n-m}_{2,H}} \\ &= \|g\|^2_{\mathbb{H}_2^n} + \|D^\beta g\|^2_{\mathbb{H}^n_{2,H}} = \|g\|^2_{\mathbb{L}^{1,2}_{H,\beta}(H_2^n)}. \end{aligned} \tag{5.4}$$

Let $g \in \widetilde{\mathbb{L}}^{1,2}_{H,\beta}(H_2^n)$ be arbitrary and $h := (1-\Delta)^{m/2}g$. An approximation argument shows that $h \in \widetilde{\mathbb{L}}^{1,2}_{H,\beta}(H_2^{n-m})$. More precisely, we know that there exists a sequence $(g_j)_j \subset \mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ such that $\|g_j - g\|_{\mathbb{L}^{1,2}_{H,\beta}(H_2^n)} \to 0$ as $j \to \infty$. Note that $h_j := (1-\Delta)^{m/2}g_j \in \mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ for any j. By (5.4), we have $\|h_j - h\|_{\mathbb{L}^{1,2}_{H,\beta}(H_2^{n-m})} = \|g_j - g\|_{\mathbb{L}^{1,2}_{H,\beta}(H_2^n)} \to 0$ as $j \to \infty$. Hence, $h \in \widetilde{\mathbb{L}}^{1,2}_{H,\beta}(H_2^{n-m})$.

Finally, if $h \in \widetilde{\mathbb{L}}^{1,2}_{H,\beta}(H_2^{n-m})$ is arbitrary, we let $g = (1-\Delta)^{-m/2}h$; then $g \in \widetilde{\mathbb{L}}^{1,2}_{H,\beta}(H_2^n)$ and $(1-\Delta)^{m/2}g = h$. This proves that $(1-\Delta)^{m/2}$ is onto.

Step 2. Let $\beta^k = (\beta^k_t)_{t\in[0,T]}, k \geq 1$ be a sequence of i.i.d fBm's. We now prove that $(1-\Delta)^{m/2}$ maps isometrically $\mathbb{L}^{1,2}_H(H_2^n, l_2)$ onto $\mathbb{L}^{1,2}_H(H_2^{n-m}, l_2)$.

Let $g = (g^k)_k \in \mathbb{L}^{1,2}_H(H_2^n, l_2)$ be arbitrary. By *Step 1*,

$$(1-\Delta)^{m/2}g^k \in \mathbb{D}^{1,2}_{\beta^k}(|\mathcal{H}_{H_2^{n-m}}|)$$

and $\|(1-\Delta)^{m/2}g^k\|_{\mathbb{L}^{1,2}_{H,\beta^k}(H_2^{n-m})} = \|g^k\|_{\mathbb{L}^{1,2}_{H,\beta^k}(H_2^n)}$, for all k. Hence

$$\|(1-\Delta)^{m/2}g\|_{\mathbb{L}^{1,2}_H(H_2^{n-m},l_2)} = \|g\|_{\mathbb{L}^{1,2}_H(H_2^n,l_2)} < \infty,$$

and $(1-\Delta)^{m/2}g \in \mathbb{L}^{1,2}_H(H_2^{n-m}, l_2)$. The fact that $(1-\Delta)^{m/2}$ is onto follows by the same principles as in *Step 1*.

Step 3. Finally, let $g = (g^k)_k \in \widetilde{\mathbb{L}}^{1,2}_H(H_2^n, l_2)$ be arbitrary. Then, there exists a sequence $(g_j)_j \subset \mathbb{L}^{1,2}_H(H_2^n, l_2)$ such that $\|g_j - g\|_{\mathbb{L}^{1,2}_H(H_2^n,l_2)} \to 0$ and $(g^k_j)_j \subset \mathcal{S}_{\beta^k}(\mathcal{E}_{C_0^\infty})$ for each k. Using *Step 2*, $\|(1-\Delta)^{m/2}g_j - (1-\Delta)^{m/2}g\|_{\mathbb{L}^{1,2}_H(H_2^{n-m},l_2)} = \|g_j - g\|_{\mathbb{L}^{1,2}_H(H_2^n,l_2)} \to 0$. Since $\{(1-\Delta)^{m/2}g^k_j\}_j \subset \mathcal{S}_{\beta^k}(\mathcal{E}_{C_0^\infty})$ for each k, it follows that $(1-\Delta)^{m/2}g \in \widetilde{\mathbb{L}}^{1,2}_H(H_2^{n-m}, l_2)$. □

Corollary 5.5. *The operator* $(1-\Delta)^{m/2}$ *maps isometrically* $\mathcal{H}^n_{2,H}$ *onto* $\mathcal{H}^{n-m}_{2,H}$.

Proof. The argument is the same as in Remark 3.4, [18], and is omitted. □

The following definition introduces the deterministic and stochastic components of a solution process u.

Definition 5.6. If $u \in \mathcal{H}^n_{2,H}$ and (5.1) holds for some $f \in \mathbb{H}_2^{n-2}$ and $g = (g^k)_k \in \widetilde{\mathbb{L}}^{1,2}_H(H_2^{n-1}, l_2)$, then we write $du = f dt + \sum_{k=1}^\infty g^k \delta\beta^k_t, t \in [0,T]$. We say that $\mathbf{D}u := f$ is the *deterministic* part of u, and $\mathbf{S}u = (\mathbf{S}^k u)_k := g$ is the *stochastic* part of u.

Remark 5.7. The operators $\mathbf{D}:\mathcal{H}^n_{2,H}\to\mathbb{H}^{n-2}_2$ and $\mathbf{S}:\mathcal{H}^n_{2,H}\to\widetilde{\mathbb{L}}^{1,2}_H(H^{n-1}_2,l_2)$ are continuous, by the definition of the norm in $\mathcal{H}^n_{2,H}$.

The next theorem is the analogue of Theorem 3.7, [18], whose proof we follow very closely. The essential difference is that we use the maximal inequality given by Theorem 4.1, instead of the Burkholder-Davis-Gundy inequality.

Theorem 5.8. *(a) For any $u\in\mathcal{H}^n_{2,H}$, we have*

$$E\sup_{t\le T}\|u(t,\cdot)\|^2_{H^{n-2}_2}\le N\|u\|^2_{\mathcal{H}^n_{2,H}},\ \text{and} \tag{5.5}$$

$$\|u\|_{\mathbb{H}^n_2}\le N\|u\|_{\mathcal{H}^n_{2,H}}, \tag{5.6}$$

where N is a constant which depends on d,T and H.

(b) The space $\mathcal{H}^n_{2,H}$ is a Banach space with the norm (5.2).

Proof. We refer the reader to the proof of Theorem 3.7, [18] for the details.

(a) By Proposition 5.4, it suffices to consider only the case $n=2$. We want to prove that $E\sup_{t\le T}\|u(t,\cdot)\|^2_{L_2}\le N\|u\|^2_{\mathcal{H}^2_{2,H}}$ for any $u\in\mathcal{H}^2_{2,H}$. This can be achieved via the Fatou's lemma, once we show that

$$E\sup_{t\le T}\|u^{(\varepsilon)}(t,\cdot)\|^2_{L_2}\le N\|u\|^2_{\mathcal{H}^2_{2,H}},\quad\forall\varepsilon>0 \tag{5.7}$$

$$\sup_{t\le T}\|u^{(1/m)}(t,\cdot)-u(t,\cdot)\|_{L_2}\to 0\text{ as }m\to\infty,\ \text{a.s.} \tag{5.8}$$

(Here $u^{(\varepsilon)}=u*\zeta_\varepsilon$ is the "mollification" of u using a test function $\zeta\in C_0^\infty,\zeta\ge 0$ with $\int_{\mathbb{R}^d}\zeta(x)dx=1$, and $\zeta_\varepsilon(x)=\varepsilon^{-d}\zeta(x/d)$.)

To prove (5.7), we note that $u^{(\varepsilon)}$ satisfies (5.1) with the pair $(f^{(\varepsilon)},g^{(\varepsilon)})$ in place of (f,g) (see (3.6) of [18]). The estimates for $u^{(\varepsilon)}(0,\cdot)$ and $f^{(\varepsilon)}$ are the same as in [18]. The estimate for $g^{(\varepsilon)}$ is obtained using a different technique. More precisely, using Theorem 4.1, for all $x\in\mathbb{R}^d$ we have

$$E\sup_{t\le T}\left|\sum_{k=1}^\infty\int_0^t g^{(\varepsilon)k}(s,x)\delta\beta^k_s\right|^2\le C_{H,T}\sum_{k=1}^\infty\|g^{(\varepsilon)k}(*,x)\|^2_{\mathbb{L}^{1,2}_{H,\beta^k}}.$$

We integrate with respect to x. Using Minkowksi's inequality and the fact that $\|h^{(\varepsilon)}\|_{L_2}\le\|h\|_{L_2}$ for any $h\in L_2$, we get

$$\begin{aligned}E\sup_{t\le T}\left\|\sum_{k=1}^\infty\int_0^t g^{(\varepsilon)k}(s,x)\delta\beta^k_s\right\|^2_{L_2}&\le C_{H,T}\left\{\sum_{k=1}^\infty E\int_0^T\int_{\mathbb{R}^d}|g^{(\varepsilon)k}(s,x)|^2dxds\right.\\&\left.+\sum_{k=1}^\infty E\int_0^T\int_{\mathbb{R}^d}\left(\int_0^T|D^{\beta^k}_\theta g^{(\varepsilon)k}(s,x)|^{1/H}d\theta\right)^{2H}dxds\right\}\\&\le C_{H,T}\sum_{k=1}^\infty\left[E\int_0^T\left\|g^k(s,\cdot)\right\|^2_{L_2}ds+E\int_0^T\left(\int_0^T\|D^{\beta^k}_\theta g^k(s,\cdot)\|^{1/H}_{L_2}d\theta\right)^{2H}ds\right]\\&=C_{H,T}\|g\|^2_{\mathbb{L}^{1,2}_H(L_2,l_2)}\le C_{H,T}\|g\|^2_{\mathbb{L}^{1,2}_H(H^1_2,l_2)}\le C_{H,T}\|u\|^2_{\mathcal{H}^2_{2,H}}.\end{aligned}$$

(We used the fact that $D_\theta^{\beta^k} g^{(\varepsilon)k}(s,x) = (D_\theta^{\beta^k} g^k(s,x))^{(\varepsilon)}$, which is a consequence of (3.7), since $g^{(\varepsilon)k}(s,x) = (g^k(s,\cdot), \zeta_\varepsilon(x-\cdot))$.) The arguments for proving (5.8) and (b) are similar to those of [18] and are omitted. □

6. The Existence and Uniqueness of a Solution

In this section, we consider the stochastic heat equation:

$$du(t,x) = (\Delta u(t,x) + f(t,x))dt + \sum_{k=1}^{\infty} g^k(t,x)\delta\beta_t^k, \quad t \in [0,T]. \tag{6.1}$$

This equation is interpreted in the sense of Definition 5.6. More precisely, we say that $u \in \mathcal{H}_{2,H}^n$ is a *solution* of (6.1) if $\mathbf{D}u = \Delta u + f$ and $\mathbf{S}u = g$.

The next theorem is the main result of the present paper, which can be viewed as an analogue of Theorem 4.2, [18].

Theorem 6.1. *Let $n \in \mathbb{R}$ be arbitrary. Let*

$$f \in \mathbb{H}_2^n \quad \textit{and} \quad g \in \widetilde{\mathbb{L}}_H^{1,2}(H_2^{n+1}, l_2).$$

Then, equation (6.1) with zero initial condition has a unique solution $u \in \mathcal{H}_{2,H}^{n+2}$. For this solution, we have

$$\|u_{xx}\|_{\mathbb{H}_2^n} \le N(\|f\|_{\mathbb{H}_2^n} + \|g\|_{\mathbb{L}_H^{1,2}(H_2^{n+1},l_2)}), \ \textit{and} \tag{6.2}$$

$$\|u\|_{\mathcal{H}_{2,H}^{n+2}} \le N(\|f\|_{\mathbb{H}_2^n} + \|g\|_{\mathbb{L}_H^{1,2}(H_2^{n+1},l_2)}), \tag{6.3}$$

where N is a constant depending on d, T and H.

Some preliminaries are needed before we can give the proof of this result. Recall that if $f(t,x)$ and $u_0(x)$ are deterministic functions, then a solution of: $u_t = \Delta u + f$, $u(0,\cdot) = u_0$, is given by $u(t,x) = T_t u_0(x) + \int_0^t T_{t-s}[f(s,\cdot)](x)ds$, where $T_t h(x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} h(y) e^{-|x-y|^2/(4t)} dy$.

The following result is due to Doyoon Kim (personal communication).

Lemma 6.2. *We have: $\int_r^t T_{t-s}(\Delta\phi)(x)ds = T_{t-r}\phi(x) - \phi(x)$, for all $\phi \in C_0^\infty$.*

Proof. Let z be the solution of $z_t = \Delta z$, $z(0,\cdot) = \phi(\cdot)$, and $\hat{z}$ be the solution of $\hat{z}_t = \Delta\hat{z}$, $\hat{z}(0,\cdot) = \Delta\phi(\cdot)$. Then $z(t,x) = T_t\phi(x)$ and $\hat{z}(t,x) = T_t(\Delta\phi)(x) = \Delta[T_t\phi(x)] = \Delta z(t,x)$. Hence

$$\begin{aligned}\int_r^t T_{t-s}(\Delta\phi)(x)ds &= \int_r^t \hat{z}(t-s,x)ds = \int_r^t \Delta z(t-s,x)ds = \int_r^t z_t(t-s,x)ds \\ &= z(t-r,x) - z(0,x) = T_{t-r}\phi(x) - \phi(x).\end{aligned}$$

□

The first idea of the proof is to treat separately the particular case when the g^k's are smooth elementary processes (in which case the solution can be written in closed form), and then apply an approximation argument.

The second idea is to evaluate (in norm) the difference between the solution u of the original equation (6.1) and the solution u_1 of the "deterministic" equation (i.e. equation (6.1) with $g_k = 0$ for all k), having in mind that bounds for u_1 are available from the PDE theory. This is achieved by the following proposition.

Proposition 6.3. *Let* $f \in \mathbb{H}_2^{-1}$ *and* $u_1(t,x) = \int_0^t T_{t-s}[f(s,\cdot)](x)ds$. *Let* $g^k \in \mathcal{S}_{\beta^k}(\mathcal{E}_{C_0^\infty})$ *for* $k \le K$, *and* $g^k = 0$ *for* $k > K$. *Let* $u(t,x) = v(t,x) + \int_0^t T_{t-s}[(\Delta v + f)(s,\cdot)](x)ds$, *where* $v(t,x) = \sum_{k=1}^\infty \int_0^t g^k(s,x)\delta\beta_s^k$. *Then*

$$\|u - u_1\|_{\mathbb{H}_2^0} \le N\|g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)} \tag{6.4}$$

$$\|u_x - u_{1x}\|_{\mathbb{H}_2^0} \le N\|g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)} \tag{6.5}$$

$$\|u_{xx} - u_{1xx}\|_{\mathbb{H}_2^{-1}} \le N\|g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)}, \tag{6.6}$$

where N *is a constant depending on* d, T *and* H.

Proof. Let $g^k(t,\cdot) = \sum_{i=1}^{m_k} F_i^k 1_{(t_{i-1}^k, t_i^k]}(t) g_i^k(\cdot)$, with $F_i^k \in \mathcal{S}_{\beta^k}$, $0 \le t_0^k < \ldots < t_{m_k}^k \le T$ (non-random) and $g_i^k \in C_0^\infty$.

We begin with the proof of (6.4). By definition, $u(t,x) - u_1(t,x) = v(t,x) + \int_0^t T_{t-s}[\Delta v(s,\cdot)](x)ds$. Note that

$$v(s,x) = \sum_{k=1}^\infty \int_0^s g^k(r,x)\delta\beta_r^k = \sum_{k=1}^\infty \sum_{i=1}^{m_k} g_i^k(x) \int_0^s F_i^k 1_{(t_{i-1}^k, t_i^k]}(r)\delta\beta_r^k, \tag{6.7}$$

and hence $T_{t-s}[\Delta v(s,\cdot)](x) = \sum_{k=1}^\infty \sum_{i=1}^{m_k} T_{t-s}(\Delta g_i^k)(x) \int_0^s F_i^k 1_{(t_{i-1}^k, t_i^k]}(r)\delta\beta_r^k$. By the stochastic Fubini's theorem and Lemma 6.2, it follows that:

$$\begin{aligned} u(t,x) - u_1(t,x) &= v(t,x) + \sum_{k=1}^\infty \sum_{i=1}^{m_k} \int_0^t T_{t-s}(\Delta g_i^k)(x) \int_0^s F_i^k 1_{(t_{i-1}^k, t_i^k]}(r)\delta\beta_r^k ds \\ &= v(t,x) + \sum_{k=1}^\infty \sum_{i=1}^{m_k} \int_0^t F_i^k 1_{(t_{i-1}^k, t_i^k]}(r) \int_r^t T_{t-s}(\Delta g_i^k)(x) ds \delta\beta_r^k \\ &= v(t,x) + \sum_{k=1}^\infty \sum_{i=1}^{m_k} \int_0^t F_i^k 1_{(t_{i-1}^k, t_i^k]}(r)(T_{t-r} g_i^k(x) - g_i^k(x))\delta\beta_r^k. \end{aligned}$$

Using (6.7) and the fact that

$$T_{t-r}[g^k(r,\cdot)](x) = \sum_{i=1}^{m_k} F_i^k 1_{(t_{i-1}^k, t_i^k]}(r) T_{t-r} g_i^k(x), \tag{6.8}$$

we obtain that: $u(t,x) - u_1(t,x) = \sum_{k=1}^\infty \int_0^t T_{t-r}[g^k(r,\cdot)](x)\delta\beta_r^k$. From here, using Theorem 4.1 and Fubini's theorem, we get:

$$\begin{aligned} \|u - u_1\|_{\mathbb{H}_2^0}^2 &= \int_0^T \int_{\mathbb{R}^d} E\left|\sum_{k=1}^\infty \int_0^t T_{t-s}[g^k(s,\cdot)](x)\delta\beta_s^k\right|^2 dxdt \\ &\le C_{H,T}\left\{\sum_{k=1}^\infty E\int_0^T \int_0^t \int_{\mathbb{R}^d} \left|T_{t-s}[g^k(s,\cdot)](x)\right|^2 dxdsdt + \right. \\ &\left. \sum_{k=1}^\infty E\int_0^T \int_0^t \int_{\mathbb{R}^d} \left(\int_0^T |D_\theta^{\beta^k}\{T_{t-s}[g^k(s,\cdot)](x)\}|^{1/H} d\theta\right)^{2H} dxdsdt\right\} \\ &:= C_{H,T}(I_1 + I_2). \end{aligned}$$

We treat I_1 first. Using (6.8), we have:

$$\begin{aligned}\int_{\mathbb{R}^d}|T_{t-s}[g^k(s,\cdot)](x)|^2dx &= \sum_{i=1}^{m_k}|F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s)\int_{\mathbb{R}^d}|T_{t-s}g_i^k(x)|^2dx\\ &= \sum_{i=1}^{m_k}|F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s)\int_{\mathbb{R}^d}e^{-(t-s)|\xi|^2}|\mathcal{F}g_i^k(\xi)|^2d\xi,\end{aligned}$$

where for the second equality we used the fact that

$$\begin{aligned}\mathcal{F}(T_{t-s}g_i^k)(\xi) &= \frac{1}{[4\pi(t-s)]^{d/2}}\int_{\mathbb{R}^d}e^{-i\xi\cdot x}\int_{\mathbb{R}^d}g_i^k(y)e^{-\frac{|x-y|^2}{4(t-s)}}dydx\\ &= e^{-(t-s)|\xi|^2}\mathcal{F}g_i^k(\xi). \qquad (6.9)\end{aligned}$$

Using Fubini's theorem and the fact that

$$\int_s^T e^{-(t-s)|\xi|^2}dt \leq \int_0^T e^{-t|\xi|^2}dt = \frac{1-e^{-T|\xi|^2}}{|\xi|^2} \leq N_T \quad \forall \xi \in \mathbb{R}^d, \qquad (6.10)$$

we obtain:

$$\begin{aligned}I_1 &= \sum_{k=1}^{\infty}E\int_0^T\int_0^t\sum_{i=1}^{m_k}|F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s)\int_{\mathbb{R}^d}e^{-(t-s)|\xi|^2}|\mathcal{F}g_i^k(\xi)|^2d\xi dsdt\\ &= \sum_{k=1}^{\infty}E\int_0^T\sum_{i=1}^{m_k}|F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s)\int_{\mathbb{R}^d}\left(\int_s^T e^{-(t-s)|\xi|^2}dt\right)|\mathcal{F}g_i^k(\xi)|^2d\xi ds\\ &\leq N_T\sum_{k=1}^{\infty}E\int_0^T\sum_{i=1}^{m_k}|F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s)\int_{\mathbb{R}^d}|\mathcal{F}g_i^k(\xi)|^2d\xi ds\\ &= N_T\sum_{k=1}^{\infty}E\int_0^T\int_{\mathbb{R}^d}\sum_{i=1}^{m_k}|F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s)|g_i^k(x)|^2dxds\\ &= N_T\sum_{k=1}^{\infty}E\int_0^T\int_{\mathbb{R}^d}|g^k(s,x)|^2 = N_T\|g\|^2_{\mathbb{H}_2^0(l_2)}. \qquad (6.11)\end{aligned}$$

We treat I_2 next. Using Minkowski's inequality, we have:

$$\begin{aligned}I_2 = \sum_{k=1}^{\infty}E\int_0^T\int_0^T\int_{\mathbb{R}^d}\left(\int_0^T 1_{\{s\leq t\}}|D_\theta^{\beta^k}\{T_{t-s}[g^k(s,\cdot)](x)\}|^{1/H}d\theta\right)^{2H}dxdtds \leq\\ \sum_{k=1}^{\infty}E\int_0^T\left[\int_0^T\left(\int_s^T\int_{\mathbb{R}^d}|D_\theta^{\beta^k}\{T_{t-s}[g^k(s,\cdot)](x)\}|^2dxdt\right)^{1/(2H)}d\theta\right]^{2H}ds.\end{aligned}$$

From (6.8),

$$|D_\theta^{\beta^k}\{T_{t-s}[g^k(s,\cdot)](x)\}|^2 = \sum_{i=1}^{m_k}|D_\theta^{\beta^k}F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s)|T_{t-s}g_i^k(x)|^2,$$

and hence

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} 1_{\{s\le t\}} |D_\theta^{\beta^k}\{T_{t-s}[g^k(s,\cdot)](x)\}|^2 dxdt \\
&= \sum_{i=1}^{m_k} |D_\theta^{\beta^k} F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s)] \int_s^T \int_{\mathbb{R}^d} |\mathcal{F}(T_{t-s}g_i^k)(\xi)|^2 d\xi dt \\
&= \sum_{i=1}^{m_k} |D_\theta^{\beta^k} F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s) \int_{\mathbb{R}^d} |\mathcal{F}g_i^k(\xi)|^2 \left(\int_s^T e^{-(t-s)|\xi|^2} dt\right) d\xi \\
&\le N_T \sum_{i=1}^{m_k} |D_\theta^{\beta^k} F_i^k|^2 1_{(t_{i-1}^k,t_i^k]}(s) \int_{\mathbb{R}^d} |g_i^k(x)|^2 dx \\
&= N_T \int_{\mathbb{R}^d} \left|\sum_{i=1}^{m_k} (D_\theta^{\beta^k} F_i^k) 1_{(t_{i-1}^k,t_i^k]}(s) g_i^k(x)\right|^2 dx \\
&= N_T \int_{\mathbb{R}^d} |D_\theta^{\beta^k} g^k(s,x)|^2 dx = N_T \|D_\theta^{\beta^k} g^k(s,\cdot)\|_{L_2}^2,
\end{aligned}$$

where we used (6.9) for the second equality above, and (6.10) for the inequality. From here, we obtain that

$$I_2 \le N_T \sum_{k=1}^\infty E\int_0^T \left(\int_0^T \|D_\theta^{\beta^k} g^k(s,\cdot)\|_{L_2}^{1/H} d\theta\right)^{2H} ds = N_T \|Dg\|^2_{\mathbb{H}^0_{2,H}(l_2)}. \quad (6.12)$$

Relation (6.4) follows by taking the sum of (6.11) and (6.12), and using (4.5).

We now turn to the proof of (6.5). Note that $u_x(t,x) - u_{1x}(t,x) = v_x(t,x) + \int_0^t T_{t-s}[\Delta v_x(s,\cdot)](x)ds = \sum_{k=1}^\infty \int_0^t T_{t-r}[g_x^k(r,\cdot)](x)\delta\beta_r^k$, and

$$T_{t-s}[g_x^k(s,\cdot)](x) = \sum_{i=1}^{m_k} F_i^k 1_{(t_{i-1}^k,t_i^k]}(s) T_{t-s} g_{ix}^k(x), \quad (6.13)$$

where we use the notation $T_{t-s}g_{ix}^k(x) = (T_{t-s}g_{ix_l}^k(x))_{1\le l\le d}$ and $g_{ix_l}^k = \partial g_i^k/\partial x_l$.

By Theorem 4.1, we get:

$$\begin{aligned}
\|u_x - u_{1x}\|^2_{\mathbb{H}^0_2} &= \int_0^T \int_{\mathbb{R}^d} E\left|\sum_{k=1}^\infty \int_0^t T_{t-s}[g_x^k(s,\cdot)](x)\delta\beta_s^k\right|^2 dxdt \\
&\le C_{H,T}\Bigg\{\sum_{k=1}^\infty E\int_0^T \int_0^t \int_{\mathbb{R}^d} \left|T_{t-s}[g_x^k(s,\cdot)](x)\right|^2 dxdsdt + \\
&\sum_{k=1}^\infty E\int_0^T \int_0^t \int_{\mathbb{R}^d} \left(\int_0^T |D_\theta^{\beta^k}\{T_{t-s}[g_x^k(s,\cdot)](x)\}|^{1/H} d\theta\right)^{2H} dxdsdt\Bigg\} \\
&:= C_{H,T}(J_1 + J_2).
\end{aligned}$$

We treat J_1 first. Using (6.13), we get:

$$\int_{\mathbb{R}^d} |T_{t-s}[g_x^k(s,\cdot)](x)|^2 dx = \sum_{i=1}^{m_k} |F_i^k|^2 1_{(t_{i-1}^k, t_i^k]}(s) \int_{\mathbb{R}^d} |T_{t-s} g_{ix}^k(x)|^2 dx$$
$$= \sum_{i=1}^{m_k} |F_i^k|^2 1_{(t_{i-1}^k, t_i^k]}(s) \int_{\mathbb{R}^d} |\xi|^2 e^{-(t-s)|\xi|^2} |\mathcal{F} g_i^k(\xi)|^2 d\xi,$$

where the second equality is due to the fact that for $l = 1, \ldots, d$,

$$\mathcal{F}(T_{t-s} g_{ix_l}^k)(\xi) = i\xi_l \mathcal{F}(T_{t-s} g_i^k)(\xi) = i\xi_l e^{-(t-s)|\xi|^2} \mathcal{F} g_i^k(\xi), \tag{6.14}$$

which can be proved using integration by parts and (6.9).

Using Fubini's theorem and the fact that

$$\int_s^T e^{-(t-s)|\xi|^2} dt \leq \frac{1 - e^{-T|\xi|^2}}{|\xi|^2} \leq \frac{1}{|\xi|^2}, \quad \forall \xi \in \mathbb{R}^d, \tag{6.15}$$

we obtain:

$$J_1 = \sum_{k=1}^{\infty} E \int_0^T \int_0^t \sum_{i=1}^{m_k} |F_i^k|^2 1_{(t_{i-1}^k, t_i^k]}(s) \int_{\mathbb{R}^d} |\xi|^2 e^{-(t-s)|\xi|^2} |\mathcal{F} g_i^k(\xi)|^2 d\xi ds dt$$
$$= \sum_{k=1}^{\infty} E \int_0^T \sum_{i=1}^{m_k} |F_i^k|^2 1_{(t_{i-1}^k, t_i^k]}(s) \int_{\mathbb{R}^d} \left(\int_s^T e^{-(t-s)|\xi|^2} dt \right) |\xi|^2 |\mathcal{F} g_i^k(\xi)|^2 d\xi ds$$
$$\leq \sum_{k=1}^{\infty} E \int_0^T \sum_{i=1}^{m_k} |F_i^k|^2 1_{(t_{i-1}^k, t_i^k]}(s) \int_{\mathbb{R}^d} |\mathcal{F} g_i^k(\xi)|^2 d\xi ds$$
$$= \sum_{k=1}^{\infty} E \int_0^T \int_{\mathbb{R}^d} |g^k(s,x)|^2 dx ds = \|g\|_{\mathbb{H}_2^0(l_2)}^2. \tag{6.16}$$

We treat J_2 next. Using Minkowski's inequality, we get:

$$J_2 = \sum_{k=1}^{\infty} E \int_0^T \int_0^T \int_{\mathbb{R}^d} \left(\int_0^T 1_{\{s \leq t\}} |D_\theta^{\beta^k} \{T_{t-s}[g_x^k(s,\cdot)](x)\}|^{1/H} d\theta \right)^{2H} dx dt ds$$
$$\leq \sum_{k=1}^{\infty} E \int_0^T \left[\int_0^T \left(\int_s^T \int_{\mathbb{R}^d} |D_\theta^{\beta^k} \{T_{t-s}[g_x^k(s,\cdot)](x)\}|^2 dx dt \right)^{1/(2H)} d\theta \right]^{2H} ds.$$

By (6.13), we have

$$|D_\theta^{\beta^k} \{T_{t-s}[g_x^k(s,\cdot)](x)\}|^2 = \sum_{i=1}^{m_k} |D_\theta^{\beta^k} F_i^k|^2 1_{(t_{i-1}^k, t_i^k]}(s) |T_{t-s} g_{ix}^k(x)|^2,$$

and therefore,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} 1_{\{s\le t\}} |D_\theta^{\beta^k}\{T_{t-s}[g_x^k(s,\cdot)](x)\}|^2 dxdt \\
&= \sum_{i=1}^{m_k} |D_\theta^{\beta^k} F_i^k|^2 1_{(t_{i-1}^k, t_i^k]}(s) \int_s^T \int_{\mathbb{R}^d} |\mathcal{F}(T_{t-s} g_{ix}^k)(\xi)|^2 d\xi dt \\
&= \sum_{i=1}^{mk} |D_\theta^{\beta^k} F_i^k|^2 1_{(t_{i-1}^k, t_i^k]}(s) \int_{\mathbb{R}^d} |\xi|^2 \mathcal{F} g_i^k(\xi) \left(\int_s^T e^{-(t-s)|\xi|^2} dt \right) d\xi \\
&\le \sum_{i=1}^{m_k} |D_\theta^{\beta^k} F_i^k|^2 1_{(t_{i-1}^k, t_i^k]}(s) \int_{\mathbb{R}^d} |g_i^k(x)|^2 dx \\
&= \int_{\mathbb{R}^d} \left| \sum_{i=1}^{m_k} (D_\theta^{\beta^k} F_i^k) 1_{(t_{i-1}^k, t_i^k]}(s) g_i^k(x) \right|^2 dx = \|D_\theta^{\beta^k} g^k(s,\cdot)\|_{L_2}^2,
\end{aligned}$$

where we used (6.14) for the second equality above, and (6.15) for the inequality. From here, we obtain that

$$J_2 \le \sum_{k=1}^{\infty} E \int_0^T \left(\int_0^T \|D_\theta^{\beta^k} g^k(s,\cdot)\|_{L_2}^{1/H} d\theta \right)^{2H} ds = \|Dg\|^2_{\mathbb{H}^0_{2,H}(l_2)}. \tag{6.17}$$

Relation (6.5) follows by taking the sum of (6.16) and (6.17), and using (4.5). Finally, (6.6) follows from (6.5) since $\|u_{xx} - u_{1xx}\|_{\mathbb{H}_2^{-1}} \le N \|u_x - u_{1x}\|_{\mathbb{H}_2^0}$. □

We are now ready to give the proof of the main result.

Proof. (of Theorem 6.1) By Proposition 5.4, it suffices to take $n = -1$.

Case 1. Suppose that $g^k \in \mathcal{S}_{\beta^k}(\mathcal{E}_{C_0^\infty})$ for $k \le K$ and $g^k = 0$ for $k > K$. Let $u(t,x) = v(t,x) + z(t,x)$, where v is as in Proposition 6.3, and z satisfies $z(t,x) = \int_0^t (\Delta z + \Delta v + f)(s,x) ds = \int_0^t (\Delta u + f)(s,x) ds$. Clearly, u is a solution of (6.1).

We now check that $u \in \mathcal{H}^1_{2,H}$, i.e. u satisfies (i)-(iii) of Definition 5.1. Since $u(0,\cdot) = 0$, (i) holds. Also, (iii) holds with $\mathbf{D}u = \Delta u + f$ and $\mathbf{S}u = g$. It remains to check (ii). Let $u_1(t,x) = \int_0^t T_{t-s}[f(s,\cdot)](x) ds$. From the PDE theory,

$$\begin{aligned}
\|u_1\|_{L_2([0,T],L_2)} &\le N \|f\|_{L_2([0,T],H_2^{-1})} && (6.18)\\
\|u_{1xx}\|_{L_2([0,T],H_2^{-1})} &\le N \|f\|_{L_2([0,T],H_2^{-1})}, && (6.19)
\end{aligned}$$

where N is a constant depending on d and T. Using (6.18) and (6.4), we get:

$$\begin{aligned}
\|u\|_{\mathbb{H}_2^0} &\le \|u_1\|_{\mathbb{H}_2^0} + \|u - u_1\|_{\mathbb{H}_2^0} \\
&\le N(\|f\|_{\mathbb{H}_2^{-1}} + \|g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)}). && (6.20)
\end{aligned}$$

Using (6.19) and (6.6), we get:

$$\begin{aligned}
\|u_{xx}\|_{\mathbb{H}_2^{-1}} &\le \|u_{1xx}\|_{\mathbb{H}_2^{-1}} + \|u_{xx} - u_{1xx}\|_{\mathbb{H}_2^{-1}} \\
&\le N(\|f\|_{\mathbb{H}_2^{-1}} + \|g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)}). && (6.21)
\end{aligned}$$

Using the fact that $\|\phi\|_{H_2^1} \leq \|\phi\|_{L_2} + \|\phi_{xx}\|_{H_2^{-1}}$, (6.20) and (6.21), we get:

$$\begin{aligned}\|u\|_{\mathbb{H}_2^1} &\leq \|u\|_{\mathbb{H}_2^0} + \|u_{xx}\|_{\mathbb{H}_2^{-1}} \\ &\leq 2N(\|f\|_{\mathbb{H}_2^{-1}} + \|g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)}).\end{aligned}$$

From here, we conclude that $u \in \mathbb{H}_2^1$ and $u_{xx} \in \mathbb{H}_2^{-1}$, i.e. u verifies condition (ii) of Definition 5.1. Since $\mathbf{D}u = \Delta u + f$, we also infer that

$$\begin{aligned}\|u\|_{\mathcal{H}_{2,H}^1} &= \|u_{xx}\|_{\mathbb{H}_2^{-1}} + \|\Delta u + f\|_{\mathbb{H}_2^{-1}} + \|g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)} \\ &\leq N(\|f\|_{\mathbb{H}_2^{-1}} + \|g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)})).\end{aligned}$$

The uniqueness of the solution of (6.1) in $\mathcal{H}_{2,H}^1$ follows from the uniqueness of the solution of the classical heat equation.

Case 2. Let $g = (g^k)_k \in \widetilde{\mathbb{L}}_H^{1,2}(L_2, l_2)$ be arbitrary. By Theorem 4.4, there exists a sequence $(g_j)_j \subset \mathbb{L}_H^{1,2}(L_2, l_2)$ such that $\|g_j - g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)} \to 0$, $g_j^k \in \mathcal{S}_{\beta^k}(\mathcal{E}_{C_0^\infty})$ for $k \leq K_j$ and $g_j^k = 0$ for $k > K_j$.

Using the result proved in *Case 1*, we know that there exists a unique solution $u_j \in \mathcal{H}_{2,H}^1$ of the equation

$$du_j(t,x) = (\Delta u_j + f)(t,x)dt + \sum_{k=1}^{\infty} g_j^k(t,x)\delta\beta_t^k,\ t \in [0,T], \tag{6.22}$$

with zero initial condition. This solution satisfies:

$$\|u_{jxx}\|_{\mathbb{H}_2^{-1}} \leq N(\|f\|_{\mathbb{H}_2^{-1}} + \|g_j\|_{\mathbb{L}_H^{1,2}(L_2,l_2)}), \tag{6.23}$$

$$\|u_j\|_{\mathcal{H}_{2,H}^1} \leq N(\|f\|_{\mathbb{H}_2^{-1}} + \|g_j\|_{\mathbb{L}_H^{1,2}(L_2,l_2)}). \tag{6.24}$$

From here, it follows that $(u_j)_j$ is a Cauchy sequence in $\mathcal{H}_{2,H}^1$. By Theorem 5.8.(b), there exists $u \in \mathcal{H}_{2,H}^1$ such that $\|u_j - u\|_{\mathcal{H}_{2,H}^1} \to 0$.

We now prove that u is a solution of (6.1). Since $\|u_j - u\|_{\mathcal{H}_{2,H}^1} \to 0$ and the operators $\mathbf{D} : \mathcal{H}_{2,H}^1 \to \mathbb{H}_2^{-1}$ and $\mathbf{S} : \mathcal{H}_{2,H}^1 \to \mathbb{L}_H^{1,2}(L_2, l_2)$ are continuous, it follows that

$$\|\mathbf{D}u_j - \mathbf{D}u\|_{\mathbb{H}_2^{-1}} \to 0 \quad \text{and} \quad \|\mathbf{S}u_j - \mathbf{S}u\|_{\mathbb{L}_H^{1,2}(L_2,l_2)} \to 0. \tag{6.25}$$

On the other hand, from (6.22) it follows that $\mathbf{D}u_j = \Delta u_j + f$ and $\mathbf{S}u_j = g_j$. Since $\|u_{jxx} - u_{xx}\|_{\mathbb{H}_2^{-1}} \leq \|u_j - u\|_{\mathcal{H}_{2,H}^1} \to 0$, we get that $\mathbf{D}u_j \to \Delta u + f$ in $\mathbb{H}_2^{-1}$. Hence

$$\|\mathbf{D}u_j - (\Delta u + f)\|_{\mathbb{H}_2^{-1}} \to 0 \quad \text{and} \quad \|\mathbf{S}u_j - g\|_{\mathbb{L}_H^{1,2}(L_2,l_2)} \to 0. \tag{6.26}$$

From (6.25) and (6.26), we infer that $\mathbf{D}u = \Delta u + f$ and $\mathbf{S}u = g$, i.e. u satisfies (6.1). Finally, (6.2) and (6.3) are obtained by passing to the limit in (6.23) and (6.24). □

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