Communications on Stochastic Analysis
Vol. 3, No. 1 (2009) 69-84

Serials Publications
www.serialspublications.com

ON THE DISTRIBUTIONS OF THE SUP AND INF OF THE CLASSICAL RISK PROCESS WITH EXPONENTIAL CLAIM*

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Abstract. The purpose of this article is to use the double Laplace transform of the occupation measure of the classical risk process X with exponential claim to deduce the distributions of the random variables $\sup\{X_s : s \leq t\}$ and $\inf\{X_s : s \leq t\}$, for every $t > 0$. As a consequence, we also get the distributions of the time to ruin in finite time and the first passage of a given level.

1. Introduction

In this paper, we deal with the classical risk process with exponential claim defined on a complete probability space $(\Omega, \mathcal{F}, P)$. More precisely, let

$$X_t = x_0 + ct - \sum_{k=1}^{N_t} R_k, \quad t \geq 0. \tag{1.1}$$

Here $x_0 \geq 0$ is the initial capital, $c > 0$ is the premium income per unit of time, $N = \{N_t, t \geq 0\}$ is an homogeneous Poisson process with rate λ and $\{R_k, k = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables independent of N. Henceforth we suppose that R_1 has exponential distribution with mean $1/r$.

Our goal is to calculate explicit expressions for the distributions of the random variables $\sup\{X_s : s \leq t\}$ and $\inf\{X_s : s \leq t\}$, $t > 0$. Toward this end, we apply the complex inversion theorem of the Laplace transform (or Lerch's theorem) to the double Laplace transforms of some occupation measures of X (see Section 2 below). As a consequence, we are also able to give the distribution of the first passage of certain level $x \in \mathbb{R}$ of the process X. It means, the distributions of

$$S_x = \inf\{t > 0 : X_t = x\} \quad \text{and} \quad T_x = \inf\{t > 0 : X_t < x\},$$

because the right-continuity of the process X yields

$$\{S_x < t\} = \left\{\sup_{s \leq t} X_s > x\right\}, \quad x > x_0$$

and

$$\{T_x \leq t\} = \left\{\inf_{s \leq t} X_s < x\right\}, \quad x < x_0.$$

2000 *Mathematics Subject Classification.* Primary 60K30; Secondary 60K99.
Key words and phrases. Classical risk process, Laplace transform, ruin probability, time to ruin.
* Partially supported by a sabbatical year grant of CONACyT and PIM 08-2 of UAA.

The study of the distribution of the time to ruin T_0 in finite time (i.e., for the case $x = 0$) is extensive in the literature on risk theory due to its applications in business activities. For instance, numerical procedures have been utilized by several authors in the analysis of T_0 (see, for example, Dickson and Waters [6] or Seal [12]). This numerical approximations have been improved in several works by deriving expressions for the mentioned distribution (see Asmussen [1, 2], Dickson [4], Dickson et al. [5, 7], Drekic and Willmot [9], and Ignatov and Kaishev [11], among others). In particular, the method used in [5], [7] and [9] is based on the complex inversion theorem, as we does here.

For $x \in \mathbb{R}$, the process S_x have been analyzed by doss Reis [8] and Gerber [10] via a martingale method.

The paper is organized as follows. In Section 2 we relate the Laplace transforms of the functions

$$P\left\{\sup_{s\leq\bullet} X_s \leq x\right\} \quad \text{and} \quad P\left\{\inf_{s\leq\bullet} X_s \leq x\right\} \tag{1.2}$$

to the double Laplace transform of some occupation measures of X. Then, in Section 3 we use the complex inversion theorem to calculate the two probabilities in (1.2).

2. Occupation Measures

Now we are interested in the Laplace transforms of the occupation measures

$$Y_x(t) = \int_0^t 1_{(x,+\infty)}(X_s)ds \quad \text{and} \quad Y^x(t) = \int_0^t 1_{(-\infty,x)}(X_s)ds,$$

with $x \in \mathbb{R}$ and $t > 0$. So, we assume that the reader is familiar with the elementary properties of the Laplace transform as they are presented, for example, in Spiegel [13].

Throughout, the Laplace transform of a measurable function $h : [0, \infty) \to \mathbb{R}$ is denoted by $\mathcal{L}(h)$. That is,

$$\mathcal{L}(h)(s) = \int_0^\infty e^{-st}h(t)dt,$$

for $s \in \mathbb{R}$ such that this integral is convergent.

The relation between the occupation measures Y_x and Y^x, and the probabilities in (1.2) is given by the following result.

Proposition 2.1. *Let X be the classical risk process defined in (1.1) and $x \in \mathbb{R}$. Then for each $t > 0$,*

$$\{Y_x(t) = 0\} = \left\{\sup_{s\leq t} X_s \leq x\right\} \quad \text{and} \quad \{Y^x(t) = 0\} = \left\{\inf_{s\leq t} X_s \geq x\right\}.$$

Proof. We first observe that $\{Y_x(t) = 0\} \supset \left\{\sup_{s\leq t} X_s \leq x\right\}$ is trivial. Now we see the reverse inclusion. Let $\omega \in \Omega$ be such that

$$\int_0^t 1_{(x,+\infty)}(X_s(\omega))ds = 0. \tag{2.1}$$

If there is some $s_0 \in (0,t)$ such that $X_{s_0}(\omega) > x$, then, by the right-continuity of X, there exists a non-empty open interval $I_{s_0} \subset (0,t)$ such that $s_0 \in I_{s_0}$ and $X_s(\omega) > x$, for all $s \in I_{s_0}$. Consequently,

$$\int_0^t 1_{(x,+\infty)}(X_s(\omega))ds \geq \int_{I_{s_0}} 1_{(x,+\infty)}(X_s(\omega))ds = |I_{s_0}| > 0,$$

where $|I_{s_0}|$ is the length of I_{s_0}. But this is a contradiction to (2.1). Therefore $X_s(\omega) \leq x$ for all $s \leq t$, which implies that ω also belongs to $\{\sup_{s\leq t} X_s \leq x\}$.

We proceed similarly for the remainder of the proof. □

In order to express the double Laplace transform of Y^x and Y_x, we need to introduce the following notation. Let s be a positive real number. The positive and negative roots of the quadratic equation

$$cv^2 + (rc - \lambda - s)v - sr = 0$$

are denoted by v_s^+ and v_s^-, respectively.

Proposition 2.2. *Let s and α be two positive real numbers. Then*

$$\int_0^\infty e^{-st} E\left[e^{-\alpha Y_x(t)}\right] dt = \begin{cases} \frac{1}{s+\alpha}\left\{1 + \frac{\alpha(\frac{1}{c} - \frac{v_s^+}{s})}{v_{s+\alpha}^- - v_s^+ - \frac{\alpha}{c}} e^{(x_0-x)v_{s+\alpha}^-}\right\}, & x_0 \geq x, \\ \frac{1}{s} + \frac{\frac{\alpha sc}{s+\alpha}(\frac{1}{c} - \frac{v_s^+}{s})v_{s+\alpha}^- - \alpha\left(v_{s+\alpha}^- - v_s^+ - \frac{\alpha}{c}\right)}{sc\left(v_s^+ + \frac{\alpha}{c}\right)\left(v_{s+\alpha}^- - v_s^+ - \frac{\alpha}{c}\right)e^{(x-x_0)v_s^+}}, & x_0 < x, \end{cases}$$

and

$$\int_0^\infty e^{-st} E\left[e^{-\alpha Y^x(t)}\right] dt = \begin{cases} \frac{1}{s+\alpha}\left\{1 + \frac{\alpha(\frac{1}{c} - \frac{v_s^-}{s})}{v_{s+\alpha}^+ - v_s^- - \frac{\alpha}{c}} e^{(x_0-x)v_{s+\alpha}^+}\right\}, & x_0 \leq x, \\ \frac{1}{s} + \frac{\frac{\alpha sc}{s+\alpha}(\frac{1}{c} - \frac{v_s^-}{s})v_{s+\alpha}^+ - \alpha\left(v_{s+\alpha}^+ - v_s^- - \frac{\alpha}{c}\right)}{sc\left(v_s^- + \frac{\alpha}{c}\right)\left(v_{s+\alpha}^+ - v_s^- - \frac{\alpha}{c}\right)e^{(x-x_0)v_s^-}}, & x_0 \geq x. \end{cases}$$

Proof. For $a < b$ and $t > 0$ define the double Laplace transform of $T_{[a,b]}(t) = \int_0^t 1_{[a,b]}(X_r)dr$ as

$$f(x_0) = \int_0^\infty e^{-st} E_{x_0}\left[e^{-\alpha T_{[a,b]}(t)}\right] dt.$$

This is a Feynman-Kac representation of the solution of equation

$$\mathcal{A}f(x_0) + 1 = \begin{cases} (s+\alpha)f(x_0), & a < x_0 < b, \\ sf(x_0), & x_0 < a \text{ or } x_0 > b, \end{cases}$$

where $\mathcal{A}$ is the infinitesimal generator associated to the semigroup of process X. Solving this equation and letting $a \downarrow -\infty$ and $b \uparrow \infty$, respectively, we are done. For details see the paper of Chiu and Yin [3] (Corollary 4.1). □

The following result is a consequence of Proposition 2.2 and it will be used in Section 3.

Proposition 2.3. *Let X be the classical risk process given by (1.1). Then, for every $s > 0$, we have*

$$\mathcal{L}(P(Y_x(\cdot) = 0))(s) = \begin{cases} 0, & x_0 \geq x, \\ \frac{1}{s} - \frac{e^{-(x-x_0)v_s^+}}{s}, & x_0 < x, \end{cases} \tag{2.2}$$

and

$$\mathcal{L}(P\left(Y^x(\cdot)=0\right))(s)=\begin{cases} 0, & x_0\le x,\\ \frac{1}{s}-\frac{\lambda e^{-(x-x_0)v_s^-}}{cs\left(v_s^+ +r\right)}, & x_0>x. \end{cases} \tag{2.3}$$

Proof. We first deal with equality (2.2). By the dominated convergence theorem we can write

$$\begin{aligned}
&\lim_{\alpha\to\infty}\int_0^\infty e^{-st}E\left[e^{-\alpha Y_x(t)}\right]dt\\
&=\lim_{\alpha\to\infty}\int_0^\infty e^{-st}\left(\int_{\{Y_x(t)=0\}}+\int_{\{Y_x(t)>0\}}\right)e^{-\alpha Y_x(t)}dPdt\\
&=\int_0^\infty e^{-st}P\left(Y_x(t)=0\right)dt+\int_0^\infty e^{-st}\int_{\{Y_x(t)>0\}}\left(\lim_{\alpha\to\infty}e^{-\alpha Y_x(t)}\right)dPdt\\
&=\int_0^\infty e^{-st}P\left(Y_x(t)=0\right)dt.
\end{aligned}$$

Therefore, from Proposition 2.2 we get

$$\begin{aligned}
&\int_0^\infty e^{-st}P\left(Y_x(t)=0\right)dt \\
&\quad=\lim_{\alpha\to\infty}\begin{cases} \frac{1}{s+\alpha}\left\{1+\frac{\alpha(\frac{1}{c}-\frac{v_s^+}{s})}{v_{s+\alpha}^- -v_s^+-\frac{\alpha}{c}}e^{(x_0-x)v_{s+\alpha}^-}\right\}, & x_0\ge x,\\ \frac{1}{s}+\frac{\frac{\alpha sc}{s+\alpha}(\frac{1}{c}-\frac{v_s^+}{s})v_{s+\alpha}^- -\alpha(v_{s+\alpha}^- -v_s^+-\frac{\alpha}{c})}{sc(v_s^+ +\frac{\alpha}{c})(v_{s+\alpha}^- -v_s^+-\frac{\alpha}{c})e^{(x-x_0)v_s^+}}, & x_0<x. \end{cases}
\end{aligned} \tag{2.4}$$

Note that the definition of $v_{s+\alpha}^-$ implies that $\lim_{\alpha\to\infty}\left(v_{s+\alpha}^-/\alpha\right)=0$, which leads us to

$$\lim_{\alpha\to\infty}\left(\alpha\left(v_{s+\alpha}^- -v_s^+-\frac{\alpha}{c}\right)^{-1}\right)=-c.$$

Hence, the fact that $v_{s+\alpha}^-<0$, for all $\alpha>0$, together with (2.4), yields that equality (2.2) is true for $x_0\ge x$.

On the other hand, for $x_0<x$,

$$\begin{aligned}
&\lim_{\alpha\to\infty}\left(\frac{1}{s}+\frac{\frac{\alpha sc}{s+\alpha}\left(\frac{1}{c}-\frac{v_s^+}{s}\right)v_{s+\alpha}^- -\alpha\left(v_{s+\alpha}^- -v_s^+-\frac{\alpha}{c}\right)}{sc\left(v_s^+ +\frac{\alpha}{c}\right)\left(v_{s+\alpha}^- -v_s^+-\frac{\alpha}{c}\right)e^{(x-x_0)v_s^+}}\right)\\
&=\lim_{\alpha\to\infty}\left(\frac{1}{s}+\frac{sc\left(\frac{1}{c}-\frac{v_s^+}{s}\right)\frac{v_{s+\alpha}^-}{s+\alpha}-\left(v_{s+\alpha}^- -v_s^+-\frac{\alpha}{c}\right)}{sc\left(\frac{v_s^+}{\alpha}+\frac{1}{c}\right)\left(v_{s+\alpha}^- -v_s^+-\frac{\alpha}{c}\right)}e^{-(x-x_0)v_s^+}\right)\\
&=\frac{1}{s}-\frac{e^{-(x-x_0)v_s^+}}{s}.
\end{aligned}$$

Thus (2.2) holds.

Finally, in order to see that (2.3) is satisfied, we only need to observe that

$$\lim_{\alpha\to\infty}\left(v_{s+\alpha}^+/\alpha\right)=\frac{1}{c}\quad\text{and}\quad\lim_{\alpha\to\infty}\left(v_{s+\alpha}^+ -v_s^- -\frac{\alpha}{c}\right)=v_s^+ +r,$$

and proceed as in the beginning of this proof. □

3. The Distributions of the Sup and Inf of X Within Finite Time

The purpose of this section is to apply Lerch's theorem to the Laplace transforms obtained in Proposition 2.3 to calculate the distributions of the sup and inf of X (see Proposition 2.1).

In order to state the main result of this paper, we need to introduce the following notation.

Note first that

$$\begin{aligned}(s+\lambda-rc)^2+4crs &= (s+\lambda+rc)^2-4\lambda rc \\ &= (s+\lambda+rc-2\sqrt{\lambda rc})(s+\lambda+rc+2\sqrt{\lambda rc}) \\ &= (s-r_1)(s-r_2),\end{aligned}$$

with $r_1 = 2\sqrt{\lambda rc}-\lambda-rc$ and $r_2=-2\sqrt{\lambda rc}-\lambda-rc$. Hence $r_2<r_1<0$,

$$v_s^+ = \frac{s+\lambda-rc+\sqrt{(s-r_1)(s-r_2)}}{2c}, \tag{3.1}$$

and

$$v_s^- = \frac{s+\lambda-rc-\sqrt{(s-r_1)(s-r_2)}}{2c}. \tag{3.2}$$

For sake of simplicity, let us utilize the conventions $a=\frac{x-x_0}{2c}$ and $b=\lambda+rc$.

Theorem 3.1. *Let X be the classical risk process (1.1) and $t>0$. Then we have*

$$\begin{aligned}&P\left(\sup_{s\le t} X_s > x\right) \qquad (3.3)\\ &= e^{-a(\lambda-rc)}\left(e^{-a\sqrt{r_1r_2}}+\frac{1}{\pi}\int_{r_2}^{r_1}\frac{e^{(t-a)u}\sin(a\left|u-r_1\right|^{1/2}\left|u-r_2\right|^{1/2})}{u}du\right),\end{aligned}$$

for every $x\in(x_0,x_0+ct)$, and

$$\begin{aligned}P\left(\inf_{s\le t} X_s < x\right) &= 2\lambda e^{-a(\lambda-rc)}\left(\frac{e^{a\sqrt{r_1r_2}}}{b+\sqrt{r_1r_2}}\right. \qquad (3.4)\\ &\left.+\frac{1}{\pi}Im\left(\int_{r_2}^{r_1}\frac{e^{(t-a)u-ai|u-r_1|^{1/2}|u-r_2|^{1/2}}}{u\left(u+b-i|u-r_1|^{1/2}|u-r_2|^{1/2}\right)}du\right)\right),\end{aligned}$$

for every $x<x_0$.

Remark 3.2. We make two remarks.

i) Note that

$$P\left(\sup_{s\le t} X_s > x\right)=0 \text{ for } x\ge x_0+ct, \quad \text{and} \quad P\left(\sup_{s\le t} X_s > x\right)=1 \text{ for } x<x_0.$$

ii) Similarly we have

$$P\left(\inf_{s\le t} X_s < x\right)=1 \quad \text{for } x>x_0.$$

In the following two subsections, we separate the proofs of (3.3) and (3.4) for the convenience of the reader because both of them are long and tedious.

3.1. Proof of equality (3.3). Let $h_M(s) = \frac{e^{-as-a\sqrt{(s-r_1)(s-r_2)}}}{s}$, $s \in \mathbb{C}\backslash[r_2, r_1]$, with $\sqrt{(s-r_1)(s-r_2)} = |(s-r_1)(s-r_2)|^{1/2}\exp(i\chi(s))$, where

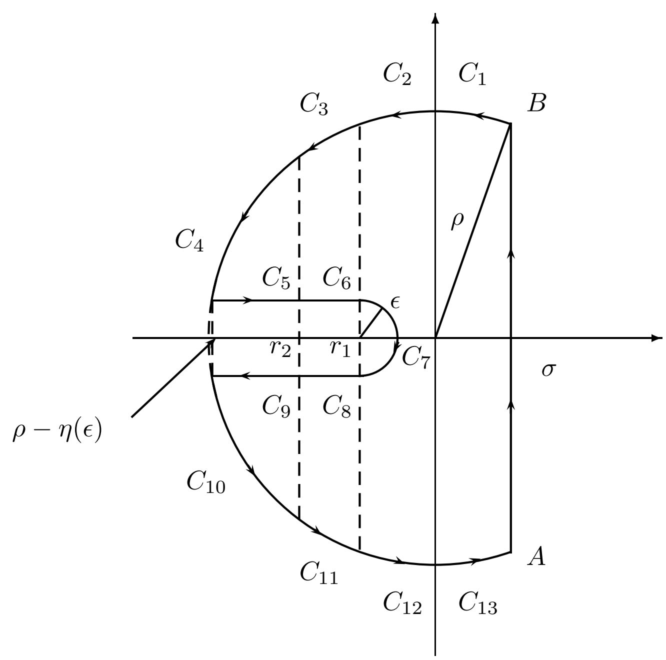
$$\chi(s) = \frac{\arg(s-r_1)+\arg(s-r_2)}{2}.$$

Then, by (2.2), (3.1), the inverse theorem of the Laplace transform (see for example [13]) and the fact that h_M is an analytic function on $\mathbb{C}\backslash[r_2, r_1]$, we have, for σ large enough,

$$\begin{aligned} P\left(Y_x(t)=0\right) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}\left(\frac{1}{s}-\frac{e^{-(x-x_0)v_s^+}}{s}\right)ds \\ &= 1-\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}\frac{e^{a(rc-\lambda-s-\sqrt{(s-r_1)(s-r_2)})}}{s}ds \\ &= 1-\frac{e^{a(rc-\lambda)}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}h_M(s)ds. \end{aligned}$$

Notice that we can use the inverse theorem because it is not difficult to see that $t \mapsto P(X_t = x) = 0$ is continuous, which follows from the fact that $P(X_t = x) = 0$.

Let $C(\rho, \varepsilon) = C_1 \cup \cdots \cup C_{13}$ be the following contour of integration:



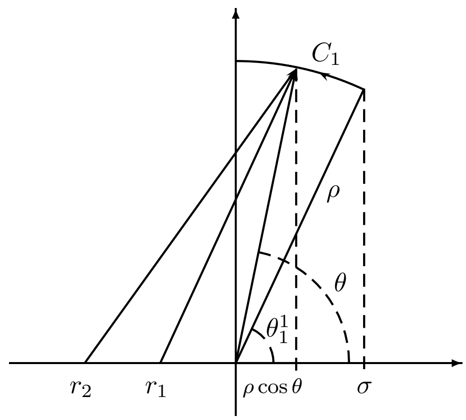
Now observe that 0 is a pole of order one and r_1, r_2 are branch points of h_M. Therefore, by the residue theorem (see [13]),

$$\int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts}h_M(s)ds = 2\pi i e^{-a\sqrt{r_1r_2}} - \lim_{\rho\to\infty}\lim_{\varepsilon\to 0}\int_{C(\rho,\varepsilon)} e^{ts}h_M(s)ds. \tag{3.5}$$

Note that we only need to analyze the integral in the right-hand side of (3.5) on each arc $C_j,\ j \in \{1,\dots,13\}$, in order to finish the proof. To do so, now we divide the proof in several steps.

Step 1. We begin our study on the arcs $C_1(\rho)$ and $C_{13}(\rho)$.

For $C_1(\rho)$ we take the parametrization $s = \rho e^{i\theta},\ \theta_1^1 \le \theta \le \pi/2$, where ρ and θ_1^1 are indicated in the following figure:



In this case, it is easy to see that

$$0 \le \arg(\rho e^{i\theta} - r_1) < \frac{\pi}{2} \quad \text{and} \quad 0 \le \arg(\rho e^{i\theta} - r_2) < \frac{\pi}{2},$$

which give $0 \le \chi(\rho e^{i\theta}) < \pi/2$. Since $\cos\theta \ge 0$, when $\theta \in [0, \pi/2]$, we have

$$\mathrm{Re}\left(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}\right) = |\rho e^{i\theta} - r_1|^{1/2} |\rho e^{i\theta} - r_2|^{1/2} \cos(\chi(\rho e^{i\theta})) > 0.$$

Using this and the fact that $a > 0$, we can conclude

$$e^{-a\mathrm{Re}\left(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}\right)} \le 1. \tag{3.6}$$

Now note that $t > a$, due to $x_0 + ct > x$, and that $\rho\cos\theta < \sigma$. Thus,

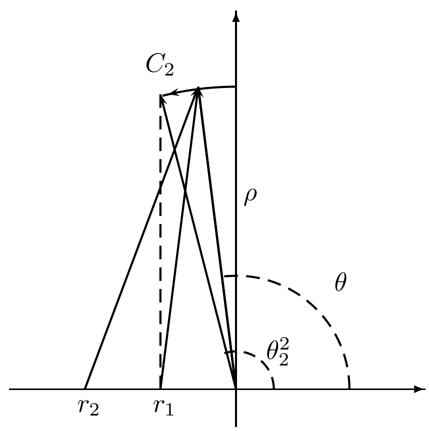
$$\begin{aligned}
\left|\int_{C_1(\rho)} e^{ts} h_M(s) ds\right| &= \left|\int_{\theta_1^1}^{\frac{\pi}{2}} \frac{e^{(t-a)\rho e^{i\theta} - a\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}}}{\rho e^{i\theta}} \rho i e^{i\theta} d\theta\right| \\
&\le \int_{\theta_1^1}^{\frac{\pi}{2}} e^{(t-a)\rho\cos\theta - a\mathrm{Re}\left(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}\right)} d\theta \\
&\le \int_{\theta_1^1}^{\frac{\pi}{2}} e^{(t-a)\sigma} d\theta \\
&= e^{(t-a)\sigma} \sin^{-1}\left(\frac{\sigma}{\rho}\right) \le e^{(t-a)\sigma}\left(\frac{\sigma}{\rho}\right)\frac{\pi}{2}.
\end{aligned}$$

Hence $\lim_{\rho\to\infty} \int_{C_1(\rho)} e^{ts} h_M(s) ds = 0$.

We can proceed in the same way to see that $\lim_{\rho\to\infty} \int_{C_{13}(\rho)} e^{ts} h_M(s) ds$ is also equal to zero.

Note that an important point in this analysis is the fact that $t-a>0$ and $-a<0$. This will be also important in the remaining of this proof.

Step 2. Now we consider the integral over $C_2(\rho)$ and $C_{12}(\rho)$.
Over $C_2(\rho)$, we consider the parametrization $s=\rho e^{i\theta}$, $\pi/2\le\theta\le\theta_2^2<\pi$, as it is ilustrated in the next figure:



As in the previous case, the inequality (3.6) is still true. So, taking into account that $\sin\theta\ge 2\theta/\pi$, for $\theta\in[0,\pi/2]$, we conclude

$$\begin{aligned}\left|\int_{C_2(\rho)} e^{ts}h_M(s)ds\right| &\le \int_{\frac{\pi}{2}}^{\theta_2^2} e^{(t-a)\rho\cos\theta - a\mathrm{Re}\left(\sqrt{(\rho e^{i\theta}-r_1)(\rho e^{i\theta}-r_2)}\right)}d\theta\\ &\le \int_{\frac{\pi}{2}}^{\theta_2^2} e^{(t-a)\rho\cos\theta}d\theta\end{aligned}$$

$$\begin{aligned}&= \int_0^{\theta_2^2-\frac{\pi}{2}} e^{(t-a)\rho\cos\left(\theta+\frac{\pi}{2}\right)}d\theta\\ &= \int_0^{\theta_2^2-\frac{\pi}{2}} e^{-(t-a)\rho\sin\theta}d\theta\\ &\le \int_0^{\theta_2^2-\frac{\pi}{2}} e^{-(t-a)\rho(2\theta/\pi)}d\theta \le \frac{\pi}{2\rho(t-a)}.\end{aligned}$$

From which we get

$$\lim_{\rho\to\infty}\int_{C_2(\rho)} e^{ts}h_M(s)ds=0.$$

Similarly,

$$\lim_{\rho\to\infty}\int_{C_{12}(\rho)} e^{ts}h_M(s)ds=0.$$

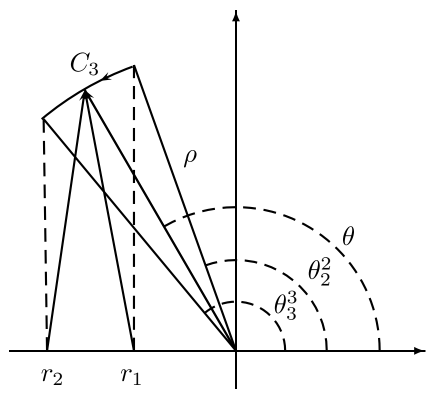
Step 3. Here we will show that

$$\lim_{\rho\to\infty}\int_{C_3(\rho)} e^{ts}h_M(s)ds = 0$$

and

$$\lim_{\rho\to\infty}\int_{C_{11}(\rho)} e^{ts}h_M(s)ds = 0.$$

On $C_3(\rho)$, we still use the parametrization $s=\rho e^{i\theta}$, with $\frac{\pi}{2}<\theta_2^2\le\theta\le\theta_3^3<\pi$:



Since $x<x_0+ct$, then $t>2a$. Therefore we can take $\eta>0$ such that

$$t>\left(1+(1+\eta)^{1/4}\right)a. \tag{3.7}$$

Moreover take $\rho>0$ such that

$$\eta\rho^2>|r_2-r_1|^2. \tag{3.8}$$

Notice that

$$\frac{\pi}{2}\le\arg(\rho e^{i\theta}-r_1)<\theta \quad\text{and}\quad 0\le\arg(\rho e^{i\theta}-r_2)\le\frac{\pi}{2}.$$

Hence

$$\frac{\pi}{4}\le\chi(\rho e^{i\theta})<\frac{\theta+\frac{\pi}{2}}{2}<\theta.$$

Since $\cos\theta$ is decreasing on $[\pi/4,\pi]$, we have

$$\mathrm{Re}\left(\sqrt{(\rho e^{i\theta}-r_1)(\rho e^{i\theta}-r_2)}\right)\ge\left|\rho e^{i\theta}-r_1\right|^{1/2}\left|\rho e^{i\theta}-r_2\right|^{1/2}\cos\theta. \tag{3.9}$$

On the other hand, the fact that $r_1\ge\rho\cos\theta$ implies

$$\left|\rho e^{i\theta}-r_1\right|\le\rho. \tag{3.10}$$

And using (3.8), we get

$$\begin{aligned}\left|\rho e^{i\theta}-r_2\right| &\le \sqrt{\left|r_1-r_2\right|^2+\rho^2\sin^2(\theta)}\\ &\le \sqrt{\left|r_1-r_2\right|^2+\rho^2}\\ &\le (\eta+1)^{1/2}\rho. \end{aligned}\tag{3.11}$$

Therefore (3.9), (3.10) and (3.11) yields

$$\begin{aligned}&\mathrm{Re}\left(\sqrt{(\rho e^{i\theta}-r_1)(\rho e^{i\theta}-r_2)}\right)\\ &\ge \rho^{1/2}(\eta+1)^{1/4}\rho^{1/2}\cos\theta \\ &= (\eta+1)^{1/4}\rho\cos\theta.\end{aligned}\tag{3.12}$$

From this and (3.7) we obtain

$$\begin{aligned}\left|\int_{C_3(\rho)} e^{ts}h_M(s)ds\right| &\le \int_{\theta_2^2}^{\theta_3^3} e^{(t-a)\rho\cos\theta-a\mathrm{Re}\left(\sqrt{(\rho e^{i\theta}-r_1)(\rho e^{i\theta}-r_2)}\right)}d\theta\\ &\le \int_{\theta_2^2}^{\theta_3^3} e^{(t-a)\rho\cos\theta-a(\eta+1)^{1/4}\rho\cos\theta}d\theta\\ &= \int_{\theta_2^2}^{\theta_3^3} e^{(t-a(1+(\eta+1)^{1/4}))\rho\cos\theta}d\theta\\ &= \int_{\theta_2^2-\frac{\pi}{2}}^{\theta_3^3-\frac{\pi}{2}} e^{\left(t-a(1+(\eta+1)^{1/4})\right)\rho\cos\left(\theta+\frac{\pi}{2}\right)}d\theta\\ &= \int_{\theta_2^2-\frac{\pi}{2}}^{\theta_3^3-\frac{\pi}{2}} e^{-\left(t-a(1+(\eta+1)^{1/4})\right)\rho\sin\theta}d\theta\\ &\le \int_{\theta_2^2-\frac{\pi}{2}}^{\theta_3^3-\frac{\pi}{2}} e^{-\left(t-a(1+(\eta+1)^{1/4})\right)\rho\frac{2\theta}{\pi}}d\theta\\ &\le \frac{\pi}{2\rho(t-a(1+(\eta+1)^{1/4}))}.\end{aligned}$$

This implies that

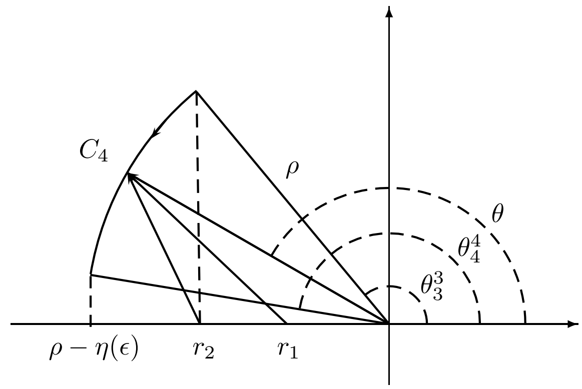
$$\lim_{\rho\to\infty}\int_{C_3(\rho)} e^{ts}h_M(s)ds=0.$$

Proceeding as the beginning of this step, we can conclude that we also have

$$\lim_{\rho\to\infty}\int_{C_{11}(\rho)} e^{ts}h_M(s)ds=0.$$

Step 4. Now we deal with the arcs $C_4(\rho,\varepsilon)$ and $C_{10}(\rho,\varepsilon)$.

Here we consider the same parametrization of previous steps. That is, $s=\rho e^{i\theta},\ \pi/2\le\theta_3^3\le\theta\le\theta_4^4\le\pi$:



Notice that

$$\frac{\pi}{2} \leq \arg(\rho e^{i\theta} - r_1) \leq \theta \quad \text{and} \quad \frac{\pi}{2} \leq \arg(\rho e^{i\theta} - r_2) \leq \theta.$$

Thus, $\pi/2 \leq \chi(\rho e^{i\theta}) \leq \theta \leq \pi$. Moreover since $r_1, r_2 \geq \rho \cos\theta$ then

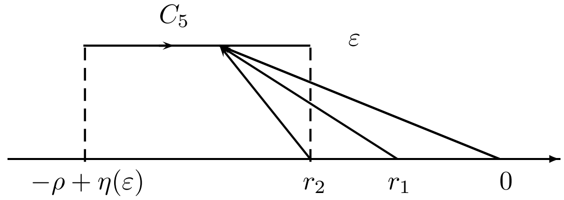
$$\left|\rho e^{i\theta} - r_1\right| \leq \rho \quad \text{and} \quad \left|\rho e^{i\theta} - r_2\right| \leq \rho,$$

and using the monotony of cos on $[\pi/2, \pi]$, we have

$$\begin{aligned} \operatorname{Re}\left(\sqrt{(\rho e^{i\theta} - r_1)(\rho e^{i\theta} - r_2)}\right) &= |\rho e^{i\theta} - r_1|^{1/2} |\rho e^{i\theta} - r_2|^{1/2} \cos(\chi(\rho e^{i\theta})) \\ &\geq |\rho e^{i\theta} - r_1|^{1/2} |\rho e^{i\theta} - r_2|^{1/2} \cos\theta \\ &\geq \rho \cos\theta. \end{aligned}$$

The above estimation is analogous to (3.12). Now the conclusion follows as in Step 3.

Step 5. Here we consider $C_5(\rho, \varepsilon)$ and $C_9(\rho, \varepsilon)$.
Over $C_5(\rho, \varepsilon)$ we take the parametrization $s = u + \varepsilon i$, $-\rho + \eta(\varepsilon) \leq u \leq r_2$:



Notice that

$$\frac{\pi}{2} \leq \arg(u + \varepsilon i - r_1) \leq \pi \quad \text{and} \quad \frac{\pi}{2} \leq \arg(u + \varepsilon i - r_2) \leq \pi,$$

then $\pi/2 \leq \chi(u+\varepsilon i) \leq \pi$. Since cosine is negative and decreasing on $[\pi/2, \pi]$ we have, for $\varepsilon < |r_1 - r_2|$,

$$\begin{aligned} & -a\mathrm{Re}\left(\sqrt{(u+\varepsilon i - r_1)(u+\varepsilon i - r_2)}\right) \\ & \qquad = a|u+\varepsilon i - r_1|^{1/2}|u+\varepsilon i - r_2|^{1/2}(-\cos(\chi(u+\varepsilon i))) \\ & \qquad \leq 2^{1/2} a\,|u - r_1|\,. \end{aligned}$$

Hence

$$\begin{aligned} \left|e^{ts} h_M(s)\right| & \leq \frac{e^{(t-a)u - a\mathrm{Re}\left(\sqrt{(u+\varepsilon i - r_1)(u+\varepsilon i - r_2)}\right)}}{|u+\varepsilon i|} \\ & \leq \frac{e^{(t-a)r_2 + 2^{1/2} a|\rho - r_1|}}{|r_2|}. \end{aligned} \tag{3.13}$$

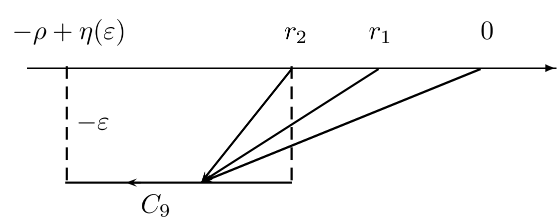
By (3.13) we can apply the dominated convergence theorem and since

$$\arg(u+\varepsilon i - r_1) \to \pi \quad \text{and} \quad \arg(u+\varepsilon i - r_2) \to \pi, \qquad \text{as } \varepsilon \to 0,$$

we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{C_5(\rho,\varepsilon)} e^{ts} h_M(s) ds & = \lim_{\varepsilon \to 0} \int_{-\rho+\eta(\varepsilon)}^{r_2} \frac{e^{(t-a)(u+\varepsilon i) - a\sqrt{(u+\varepsilon i - r_1)(u+\varepsilon i - r_2)}}}{u+\varepsilon i} du \\ & = \int_{-\rho}^{r_2} \frac{e^{(t-a)u - a|u-r_1|^{1/2}|u-r_2|^{1/2} e^{i\pi}}}{u} du \\ & = \int_{-\rho}^{r_2} \frac{e^{(t-a)u + a|u-r_1|^{1/2}|u-r_2|^{1/2}}}{u} du. \end{aligned}$$

On the other hand, on $C_9(\rho, \varepsilon)$ we use the parametrization $s = -u - \varepsilon i$, $-r_2 \leq u \leq \rho - \eta(\varepsilon)$:



Working as in previous case, and noting that

$$\arg(-u - \varepsilon i - r_1) \to \pi \quad \text{and} \quad \arg(-u-\varepsilon i - r_2) \to \pi, \qquad \text{as } \varepsilon \to 0,$$

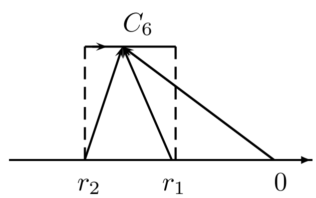
we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{C_9(\rho,\varepsilon)} e^{ts}h_M(s)ds\\
&=\lim_{\varepsilon\to 0}\int_{-r_2}^{\rho-\eta(\varepsilon)} \frac{e^{(t-a)(-u-\varepsilon i)-a\sqrt{(-u-\varepsilon i-r_1)(-u-\varepsilon i-r_2)}}}{-u-\varepsilon i}(-du)\\
&=-\int_{-\rho}^{r_2}\frac{e^{(t-a)u-a|u-r_1|^{1/2}|u-r_2|^{1/2}e^{i\pi}}}{u}du\\
&=-\int_{-\rho}^{r_2}\frac{e^{(t-a)u+a|u-r_1|^{1/2}|u-r_2|^{1/2}}}{u}du.
\end{aligned}$$

Therefore,

$$\lim_{\rho\to\infty}\left(\lim_{\varepsilon\to 0}\int_{C_5(\rho,\varepsilon)} e^{ts}h_M(s)ds+\lim_{\varepsilon\to 0}\int_{C_9(\rho,\varepsilon)} e^{ts}h_M(s)ds\right)=0.$$

Step 6. Now we deal with $C_6(\varepsilon)$. To do this, we take the parametrization $s=u+\varepsilon i$, $r_2\le u\le r_1$:



Notice that for $\varepsilon<1$,

$$\begin{aligned}
|e^{ts}h_M(s)| &= \left|\frac{e^{(t-a)(u+\varepsilon i)-a\sqrt{(u+\varepsilon i-r_1)(u+\varepsilon i-r_2)}}}{u+\varepsilon i}\right|\\
&= \frac{e^{(t-a)u-a|u+\varepsilon i-r_1|^{1/2}|u+\varepsilon i-r_2|^{1/2}\cos\chi(u+\varepsilon i)}}{|u+\varepsilon i|}\\
&\le \frac{e^{(t-a)r_1}e^{-a|u+\varepsilon i-r_1|^{1/2}|u+\varepsilon i-r_2|^{1/2}\cos\chi(u+\varepsilon i)}}{|r_1|}\\
&\le \frac{e^{(t-a)r_1}e^{a|u+\varepsilon i-r_1|^{1/2}|u+\varepsilon i-r_2|^{1/2}}}{|r_1|}\\
&\le \frac{e^{-(t-a)r_1+a(1+|r_1-r_2|^2)^{1/2}}}{|r_1|}. \qquad (3.14)
\end{aligned}$$

Since

$$\arg(u+\varepsilon i-r_1)\to\pi \quad\text{and}\quad \arg(u+\varepsilon i-r_2)\to 0,\quad \text{as } \varepsilon\to 0,$$

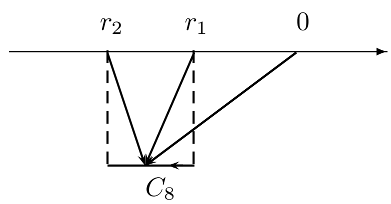
then

$$\begin{aligned}\sqrt{(u+\varepsilon i-r_1)(u+\varepsilon i-r_2)} &= |u+\varepsilon i-r_1|^{1/2}\,|u+\varepsilon i-r_2|^{1/2}\exp\left(i\chi(u+\varepsilon i)\right)\\ &\to |u-r_1|^{1/2}\,|u-r_2|^{1/2}\,i,\ \text{as } \varepsilon\to 0.\end{aligned}$$

Due to (3.14) we are able to apply the dominated convergence theorem:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{C_6(\varepsilon)} e^{ts}h_M(s)ds &= \lim_{\varepsilon\to 0}\int_{r_2}^{r_1}\frac{e^{(t-a)(u+\varepsilon i)-a\sqrt{(u+\varepsilon i-r_1)(u+\varepsilon i-r_2)}}}{u+\varepsilon i}du\\ &= \int_{r_2}^{r_1}\frac{e^{(t-a)u-a|u-r_1|^{1/2}|u-r_2|^{1/2}i}}{u}du.\end{aligned}$$

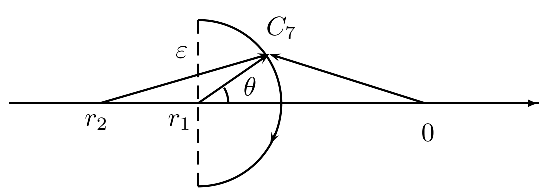
Step 7. Now we suppose that $C_8(\varepsilon)$ is defined by $s=-u-\varepsilon i$, $-r_1\le u\le -r_2$:



we can imitate Step 6 to we get

$$\lim_{\varepsilon\to 0}\int_{C_8(\varepsilon)} e^{ts}h_M(s)ds = -\int_{r_2}^{r_1}\frac{e^{(t-a)u+a|u-r_1|^{1/2}|u-r_2|^{1/2}i}}{u}du.$$

Step 8. Finally we deal with $C_7(\varepsilon)$. Here, we put $s=r_1+\varepsilon e^{-i\theta}$, $-\pi/2\le\theta\le\pi/2$:



Thus

$$\chi(\varepsilon e^{-i\theta}+r_1)\in\left[\frac{3}{2}\pi,2\pi\right]\cup\left[0,\frac{1}{2}\pi\right].$$

Consequently, using the fact that $\cos\theta\ge 0$, for $\theta\in[\frac{3}{2}\pi,2\pi]\cup[0,\frac{1}{2}\pi]$, we obtain

$$\operatorname{Re}\left(\sqrt{\varepsilon e^{-i\theta}(\varepsilon e^{-i\theta}+r_1-r_2)}\right)=\varepsilon^{1/2}|\varepsilon e^{-i\theta}+r_1-r_2|^{1/2}\cos\chi(\varepsilon e^{-i\theta}+r_1)\ge 0. \tag{3.15}$$

Also, for $0 < \varepsilon < -r_1/2$, we have

$$\begin{aligned} \left| r_1 + \varepsilon e^{-i\theta} \right| &= \sqrt{r_1^2 + 2r_1\varepsilon\cos(-\theta) + \varepsilon^2} \\ &\geq |r_1 + \varepsilon| = -r_1 - \varepsilon > -\frac{r_1}{2}. \end{aligned} \tag{3.16}$$

The estimations (3.15), (3.16) and $t > a$ yield

$$\begin{aligned} &\left| \int_{C_7(\varepsilon)} e^{ts} h_M(s) ds \right| \\ &\leq \varepsilon e^{(t-a)r_1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \frac{e^{(t-a)\varepsilon e^{-i\theta} - a\sqrt{\varepsilon e^{-i\theta}(\varepsilon e^{-i\theta} + r_1 - r_2)}}}{r_1 + \varepsilon e^{-i\theta}} \right| d\theta \\ &\leq \frac{2\varepsilon e^{(t-a)r_1}}{|r_1|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| e^{(t-a)\varepsilon e^{-i\theta} - a\sqrt{\varepsilon e^{-i\theta}(\varepsilon e^{-i\theta} + r_1 - r_2)}} \right| d\theta \\ &= \frac{2\varepsilon e^{(t-a)r_1}}{|r_1|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(t-a)\varepsilon\cos(-\theta) - a\mathrm{Re}\left(\sqrt{\varepsilon e^{-i\theta}(\varepsilon e^{-i\theta} + r_1 - r_2)}\right)} d\theta \\ &\leq \frac{2\varepsilon e^{(t-a)(r_1+\varepsilon)}}{|r_1|} \pi. \end{aligned}$$

Therefore $\lim_{\varepsilon\to 0} \int_{C_7(\varepsilon)} e^{ts} h_M(s) ds = 0$.

Step 9. To finish the proof, we only need to take into account Steps 1-8, together with (3.5), and Propositions 2.1 and 2.3. □

3.2. Proof of (3.4). Let us define

$$K(s) = s + b + \sqrt{(s - r_1)(s - r_2)}, \quad s \in \mathbb{C}\backslash[r_2, r_1].$$

Since

$$2b + r_1 + r_2 = 0 \neq -4\lambda rc = r_1 r_2 - b^2,$$

then

$$(s + b)^2 \neq (s - r_1)(s - r_2), \ \forall s \in \mathbb{C}.$$

This implies that $1/K$ is analytic over $\mathbb{C}\backslash[r_2, r_1]$. Moreover, it is not difficult to see that, for ρ large enough, we have

$$|K(s)| \geq \begin{cases} 2\sqrt{\lambda rc}, \quad s \in C_2 \cup C_4 \cup C_5 \cup C_7 \cup C_9 \cup C_{10} \cup C_{12}, \\ b, \quad s \in C_1 \cup C_{13}, \\ (\rho + r_1)^{1/2}(\rho + r_2)^{1/2} \sin\frac{\pi}{4}, \quad s \in C_3 \cup C_{11}, \\ \varepsilon + |\mathrm{Re}(s) - r_1|^{1/2} |\mathrm{Re}(s) - r_2|^{1/2} \sin\frac{\pi}{4}, \quad s \in C_6 \cup C_8. \end{cases} \tag{3.17}$$

As in the proof of (3.3) we have by (3.2)

$$\begin{aligned} P(Y^x(t) = 0) &= 1 - \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{ts} \frac{\lambda e^{-(x - x_0)v_s^-}}{cs\left(v_s^+ + r\right)} ds \\ &= 1 - \frac{\lambda e^{-a(\lambda - rc)}}{\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{ts} h_m(s) ds, \end{aligned}$$

where

$$h_m(s) = \frac{e^{-as+a\sqrt{(s-r_1)(s-r_2)}}}{sK(s)}.$$

Finally, the result follows from Proposition 2.1, (3.17) and the proof of (3.3). Observe that $a < 0$ and $t - 2a > 0$, together with (3.17), allow us to copy, line by line, the proof of (3.3) to show that (3.4) is also true. □

Acknowledgment. The first author would like to thank Universidad Autónoma de Aguascalientes and the second author appreciates the hospitality of Cinvestav-IPN and Universidad Juárez Autónoma de Tabasco during the realization of this work.

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