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MARKOVIAN SYSTEMS OF TRANSITION EXPECTATIONS

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Abstract. We propose a definition of markovian systems of transition expectation as a generalization of Liebscher's continuous time version of Accardi's quantum Markov chains and we show in a reconstruction theorem that the transition expectations may be recovered with the help product systems of Hilbert modules and units for them in the sense of Bhat and Skeide.

1. Introduction

In [6], Liebscher proposed a continuous time version of *quantum Markov chains* in the sense of Accardi [1, 2]. For $0 \leq t \leq \infty$ let us consider the Hilbert spaces $H_t = G \bar{\otimes} \Gamma(L^2([0,t), \mathfrak{H})$ (with initial space G and another Hilbert space $\mathfrak{H}$). Denoting $\mathcal{B} = \mathscr{B}(G)$, $\mathcal{A}_t = \mathscr{B}(H_t)$ and $\mathcal{A}^c_t = \mathscr{B}(\Gamma(L^2([0,t), \mathfrak{H}))$, we have $\mathcal{A}_t = \mathcal{B} \bar{\otimes}^s \mathcal{A}^c_t$ and $\mathcal{A}^c_{s+t} = \mathcal{A}^c_s \bar{\otimes}^s \mathcal{A}^c_t$, hence, also $\mathcal{A}_{s+t} = \mathcal{A}_s \bar{\otimes}^s \mathcal{A}^c_t$. ($\overline{\otimes}^s$ denotes the tensor product of von Neumann algebras.) Liebscher defines a *system of transition expectations* as a family $\left(T^L_t\right)_{t \in \mathbb{R}_+}$ of unital (normal) completely postitive mappings $T^L_t \colon \mathcal{A}_t \to \mathcal{B}$ fulfilling $T^L_{s+t} = T^L_t \circ (T_s \otimes \mathsf{id})$. One can show that the typical system of transition expectations arises via $T^L_t = u^*_t \bullet u_t$ from a family $u = \left(u_t\right)_{t \in \mathbb{R}_+}$ of isometries $u_t \colon G \to H_t$ fulfilling $u_{s+t} = (\mathfrak{s}_t u_s \otimes \mathsf{id}) u_t$ (where $\mathfrak{s}_t$ denotes the time shift on the Fock space). Since this property reminds us of a cocycle property, Liebscher called u a *cocycle of type (H)*.

Example 1.1. By restriction to $\mathfrak{H} = \{0\}$, i.e. $\mathcal{A}^c_t = \mathbb{C}$, we are concerned with the case of unital CP-semigroups on $\mathcal{B}$.

Example 1.2. The discrete version in the case where $\mathcal{A}^c_n = \mathcal{A}^{c \otimes n}_1$ ($\mathcal{A}^c_\infty = \ldots \otimes \mathcal{A}^c_1 \otimes \mathcal{A}^c_1$) and $\mathcal{A}_n = \mathcal{B} \otimes \mathcal{A}^c_n$ gives us back Accardi's quantum Markov chains. Here the transition expectations T^A_n are determined uniquely by $T^A_1 \colon \mathcal{B} \otimes \mathcal{A}^c_1 \to \mathcal{B}$ and the composition property.

In Skeide [8] it was pointed out (see also [5, 10] for a more systematic treatment) that $E_t = \mathscr{B}(G, H_t)$ (with $\mathscr{B}(G)$–valued inner product $\langle x, y \rangle = x^* y$) is just the (strong closure of the) *symmetric Fock module* over $\mathscr{B}(G) \otimes L^2([0,t), \mathfrak{H})$. (An element $b \otimes f$ gives rise to a mapping $g \mapsto bg \otimes f$ in E_t and we may recover H_t as the *internal tensor product* $E_t \bar{\odot} G$.) We have $E_s \bar{\odot}^s E_t$ where $x \odot y = (\mathfrak{s}_t x \otimes \mathsf{id}) y$

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gives the identification so that $u_s \odot u_t = u_{s+t}$; see Bhat and Skeide [5] for details. In other words, the E_t form a product system $E^{\bar{\odot}^s} = \big(E_t\big)_{t\in\mathbb{R}_+}$ (to be precise a product system of von Neumann modules) and the u_t form a unit $u^\odot = \big(u_t\big)_{t\in\mathbb{R}_+}$ for $E^{\bar{\odot}^s}$ such that $T_t(a) = \langle u_t, au_t\rangle$.

In general, a family $E^\odot = \big(E_t\big)_{t\in\mathbb{T}}$ ($\mathbb{T} = \mathbb{R}_+$ or $\mathbb{T} = \mathbb{N}_0$) of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules E_t (for some unital C^*–algebra $\mathcal{B}$) is a *product system*, if $E_0 = \mathcal{B}$ and

$$E_s \odot E_t \;=\; E_{s+t} \tag{1.1}$$

such that $(E_r \odot E_s) \odot E_t = E_r \odot (E_s \odot E_t)$. Completion and strong closure are indicated by $^-$ and $^{-s}$, respectively. A family $\xi^\odot = \big(\xi_t\big)_{t\in\mathbb{T}}$ of elements $\xi_t \in E_t$ is a *unit* for $E^\odot$, if $\xi_0 = \mathbf{1}$ and $\xi_s \odot \xi_t = \xi_{s+t}$. If $\xi^\odot$ is unital (i.e. $\langle \xi_t, \xi_t\rangle = \mathbf{1}$), then $T_t(a_t) = \langle \xi_t, a_t\xi_t\rangle$ defines a family of unital completely positive mappings $\mathcal{B}^a(E_t) \to \mathcal{B}$ fulfilling $T_{s+t}(a_s \odot a_t^c) = T_t(T_s(a_s)a_t^c)$ for all $a_s \in \mathcal{B}^a(E_s)$ and $a_t^c \in \mathcal{B}^{a,bil}(E_t)$.

($\mathcal{B}^a(E)$ denotes the algebra of bounded adjointable (and, therefore, right linear) mappings on a pre-Hilbert $\mathcal{B}$–module E whereas $\mathcal{B}^{a,bil}(E)$ denotes the subet of $\mathcal{B}$–$\mathcal{B}$–linear elements in $\mathcal{B}^a(E)$ for a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module E. If E is a Hilbert module then any adjointable mapping is closed and, therefore, bounded. If E is a *von Neumann module* (i.e. if E is a strongly closed subset of some $\mathcal{B}(G, H)$), then any bounded right linear mapping is adjointable. See Skeide [10] for details.)

Contrary to the case $\mathcal{B} = \mathcal{B}(G)$, here we do not know, whether $\mathcal{B} \subset \mathcal{B}^a(E_t)$ and $\mathcal{B}^{a,bil}(E_t) \subset \mathcal{B}^a(E_t)$ are in *tensor position* (i.e., whether the subalgebra $\mathsf{span}(\mathcal{B}\mathcal{B}^{a,bil}(E_t))$ of $\mathcal{B}^a(E_t)$ is isomorphic to $\mathcal{B} \otimes \mathcal{B}^{a,bil}(E_t)$ (this is certainly wrong if $\mathcal{B} \cap \mathcal{B}^{a,bil}(E_t) \neq \mathbb{C}\mathbf{1}$), nor do we know, whether this subalgebra is strongly dense in $\mathcal{B}^a(E_t)$. The same questions are open for the (mutually commuting) subalgebras $\mathsf{id}_{E_s} \odot \mathcal{B}^{a,bil}(E_t)$ and $\mathcal{B}^{a,bil}(E_s) \odot \mathsf{id}_{E_t}$ of $\mathcal{B}^{a,bil}(E_{s+t})$.

In Section 2 we give our definition of transition expectations and related structures. We pay particular attention in order that the definitions work for any type of closure under natural compatibility conditions. In Section 3 we extend the construction from [5] (starting from a completely positive semigroup on $\mathcal{B}$) to transition expectations. We find for any such system of expectations a product system $E^\odot$ and a unit $\xi^\odot$ such that $T_t = \langle \xi_t, \bullet\xi_t\rangle$. In Section 4 we construct a natural time shift endomorphism semigroup also based on a construction from [5].

We remark that the above product system of symmetric Fock modules is isomorphic to a product system of time ordered Fock modules as introduced by Bhat and Skeide [5] (roughly speaking, the functions in the n–particle sector are not symmetric under permutation of the n time arguments, but the times have to be ordered decreasingly). The norm continuous units were found in Skeide and Liebscher [7] and have a comparably simple form. Thus, we have lots of examples for transition expectations.

We remark further that in the case $\mathcal{B} = \mathcal{B}(G)$ and *normal* transition expectations the members of the products sytem (of von Neumann modules) have the simple form $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_t)$ where the $\mathfrak{H}_t$ form a product sytem of Hilbert spaces (under a mild measurability requirement on T_t, indeed, a product system in the sense of Arveson [4]); see [5, 10] for details.

2. Definition and Basic Properties

We present a set of axioms on families $\big(\mathcal{A}_t\big)_{t\in\mathbb{T}}$ and $\big(\mathcal{A}^c_t\big)_{t\in\mathbb{T}}$ of pre–C^*–algebras and transition expectations $\big(T_t\big)_{t\in\mathbb{T}}$ which allows us to show a reconstruction theorem. More precisely, we want to find a product system $E^\odot$ such that on each E_t we have a representation of $\mathcal{A}_t$, and a unit $\xi^\odot$ such that $T_t(a_t) = \langle \xi_t, a_t\xi_t\rangle$. The obstacles mentioned in the introduction show that we may not hope to conclude backwards, i.e. to find such families from a given pair $(E^\odot, \xi^\odot)$. The reconstruction will follow very much the lines of the construction for CP-semigroups discovered in [5]. As this construction is purely algebraical, we start also here on an algebraical level, pointing out the places where to put topological conditions like contractivity or normality. We only recall that $\mathcal{B}$ is always a unital C^*–algebra, sometimes a von Neumann algebra on a Hilbert space G. For details about von Neumann modules over von Neumann algebras we refer the reader to [9, 5]. A complete introduction to everything we need here can be found in [10].

$\mathcal{B} \otimes \mathcal{A}^c_t$ is a particularly simple example of a *$\mathcal{B}$–algebra*, i.e. a $*$–algebra with unit $\mathbf{1}$ containing a $*$–subalgebra $\mathcal{B} \ni \mathbf{1}$. Of course, a $\mathcal{B}$–algebra is a $\mathcal{B}$–$\mathcal{B}$–module, and as such it can be centered or not. Recall from [8] that a $\mathcal{B}$–$\mathcal{B}$–module E is *centered*, if it is generated (maybe, topologiaclly) by it *$\mathcal{B}$–center* $C_\mathcal{B}(E)$, i.e. the set of all element in E commuting with all elements in $\mathcal{B}$. We are interested in $\mathcal{B}$–algebras with a distinguished subalgebra $\mathcal{A}^c \subset C_\mathcal{B}(\mathcal{A})$ of the $\mathcal{B}$–center of $\mathcal{A}$ such that $\mathcal{A}$ is (topologically) spanned by $\mathcal{B}\mathcal{A}^c$. From the introduction we know that $\mathcal{A}_c$ may be, but need not be all of $C_\mathcal{B}(\mathcal{A})$.

Definition 2.1. Let $\mathcal{A}^\otimes = \big(\mathcal{A}_t\big)_{t\in\mathbb{T}}$ be a family of $\mathcal{B}$–algebras with a family $\mathcal{A}^{c\otimes} = \big(\mathcal{A}^c_t\big)_{t\in\mathbb{T}}$ of $*$–subalgebras $\mathcal{A}^c_t \subset C_\mathcal{B}(\mathcal{A}_t)$ of the $\mathcal{B}$–center of $\mathcal{A}_t$ such that $\mathsf{span}(\mathcal{B}\mathcal{A}^c_t) = \mathcal{A}_t$. We require $\mathcal{A}_0 = \mathcal{B}$ and $\mathcal{A}^c_0 = \mathbb{C}$.

Let $\alpha = \big(\alpha_{s,t}\big)_{s,t\in\mathbb{T}}$ be a family of unital homomorphisms $\mathcal{A}_s \otimes \mathcal{A}^c_t \to \mathcal{A}_{s+t}$ such that $\mathsf{span}\,\alpha_{s,t}(\mathcal{A}^c_s \otimes \mathcal{A}^c_t) = \mathcal{A}^c_{s+t}$. (This implies, in particular, that $\mathsf{span}\,\alpha_{s,t}(\mathcal{A}_s \otimes \mathcal{A}^c_t) = \mathcal{A}_{s+t}$.) We require that $\alpha_{0,t}$ is the canonical mapping $b \otimes a_t \mapsto ba_t$ and that $\alpha_{t,0} = \mathsf{id}_{\mathcal{A}_t}$, We define $\alpha^c_{s,t} = \alpha_{s,t} \upharpoonright (\mathcal{A}^c_s \otimes \mathcal{A}^c_t)$. We say $(\mathcal{A}^\otimes, \mathcal{A}^{c\otimes}, \alpha)$ is a *left tensor product system of $\mathcal{B}$–algebras* with *central tensor product system* $\mathcal{A}^{c\otimes}$, if α fulfill the associativity condition

$$\alpha_{r,s+t} \circ (\mathsf{id} \otimes \alpha^c_{s,t}) \;=\; \alpha_{r+s,t} \circ (\alpha_{r,s} \otimes \mathsf{id}).$$

If we speak about pre–C^*–algebras, we require that the $\alpha_{s,t}$ are contractive. If we speak about von Neumann algebras $\alpha_{s,t}$ should be normal. For C^*–algebras or von Neumann algebras, instead of the linear span we take the closure (in the respective topology) of the linear span.

Definition 2.2. Let $T = \big(T_t\big)_{t\in\mathbb{T}}$ be a family of unital completely positive mappings $T_t \colon \mathcal{A}_t \to \mathcal{B}$ with $T_0 = \mathsf{id}_\mathcal{B}$ and

$$T_{s+t} \circ \alpha_{s,t} \;=\; T_t \circ \alpha_{0,t} \circ (T_s \otimes \mathsf{id}).$$

We say T is a *system of transition expectations*, if there exists a family $\big(\mathcal{T}_{s+t,t}\big)_{s,t\in\mathbb{T}}$ of mappings $\mathcal{T}_{s+t,t} \colon \mathcal{A}_{s+t} \to \mathcal{A}_t$ such that

$$\mathcal{T}_{s+t,t} \circ \alpha_{s,t} \;=\; \alpha_{0,t} \circ (T_s \otimes \mathsf{id}). \tag{2.1}$$

We use the conventions as in Definition 2.1 for topological variants.

If $\mathcal{T}$ exists, then it is uniquely determined by (2.1). Basically, (2.1) tells us that the unital completely positive mapping $ba_s \otimes a_t \mapsto T_s(ba_s)a_t$ ($b \in \mathcal{B}, a_s \in \mathcal{A}_s^c, a_t \in \mathcal{A}_t^c$) factors through $\alpha_{s,t}$. Therefore, also $\mathcal{T}_{s+t,t}$ is unital and completely positive. If all $\alpha_{s,t}$ are injective (like in [6]), then we may forget about (2.1), at least, from the algebraical point of view. Later on, (2.1) shows to be responsible for the possibility to define a representation of $\mathcal{A}_t^c$ on the member E_t of the *GNS-system* $E^\odot$ of T.

Another aspect of (2.1), even if the $\alpha_{s,t}$ are injective, is the topological one. (The fact that $\mathcal{A}_s \otimes \mathcal{A}_t^c$ is algebraically isomorphic to (a dense subset of) $\mathcal{A}_{s+t}$, does not mean that some natural topology on the tensor product $\mathcal{A}_s \otimes \mathcal{A}_t^c$ gives us back the correct topology on $\mathcal{A}_{s+t}$.) The topological requirements on $\mathcal{T}$ provide us with all necessary information in order that the construction of the GNS-system is compatible with existing topological structures.

Before we come to the construction of the GNS-system, we draw some general consequences from Definition 2.2.

Corollary 2.3. *The embeddings $\alpha_{s,t} \restriction (\mathbf{1} \otimes \mathcal{A}_t^c)$ of $\mathcal{A}_t^c$ into $\mathcal{A}_{s+t}$ are injective. In other words, we may consider $\mathcal{A}_t^c$ as a subalgebra of $\mathcal{A}_{s+t}$ and, in particular, (putting $s = 0$) as a subalgebra of $\mathcal{A}_t$.*

Proof. We apply (2.1) to $\mathbf{1} \otimes a_t$ and obtain $\mathcal{T}_{s+t,t} \circ \alpha_{s,t}(\mathbf{1} \otimes a_t) = \alpha_{0,t} \circ (T_s \otimes \mathsf{id})(\mathbf{1} \otimes a_t) = a_t$. Therefore, $\mathcal{T}_{s+t,t} \circ \alpha_{s,t} \restriction (\mathbf{1} \otimes \mathcal{A}_t^c)$ and *a fortiori* $\alpha_{s,t} \restriction (\mathbf{1} \otimes \mathcal{A}_t^c)$ is injective. □

The embedding, in general, does not extend to $\mathcal{A}_t$. Saying that the copy of $\mathcal{B}$ in $\mathcal{A}_t$ is attached to time t, it is, roughly speaking, acting at the wrong time to be imbedded into $\mathcal{A}_{s+t}$ where it should act at time $s + t$. We will see later on very clearly that the different actions of $\mathcal{B}$ at different times correspond to a weak Markov flow. Of course, there is an embedding $\alpha_{s,t} \restriction (\mathcal{A}_s \otimes \mathbf{1})$ of $\mathcal{A}_s$ into $\mathcal{A}_{s+t}$, but (except for $s = 0$) it need not be injective.

Corollary 2.4. *The $\mathcal{A}_t^c$ with the embeddings form an inductive system with inductive limit $\mathcal{A}^c$. On $\mathcal{A}^c$ we define an E_0–semigroup Θ by setting $\Theta_t(a_s) = \alpha_{s,t}^c(a_s \otimes \mathbf{1})$ where we identify $a_s \in \mathcal{A}_s^c$ and $\alpha_{s,t}(a_s \otimes \mathbf{1}) \in \mathcal{A}_{s+t}^c$ with the corresponding elements in $\mathcal{A}^c$. In this identification we define a completely positive unital mapping $\tau \colon \mathcal{A}^c \to \mathcal{B}$ by setting $\tau(a_t) = T_t \circ \alpha_{0,t}(\mathbf{1} \otimes a_t)$.*

Proposition 2.5. $\mathcal{T}_{s+t,t} \circ \mathcal{T}_{r+s+t,s+t} = \mathcal{T}_{r+s+t,t}$.

Proof. We have

$$\begin{aligned} &\mathcal{T}_{r+s+t,t} \circ \alpha_{r+s,t} \circ (\alpha_{r,s} \otimes \mathsf{id}) \;=\; \alpha_{0,t} \circ (T_{r+s} \otimes \mathsf{id}) \circ (\alpha_{r,s} \otimes \mathsf{id}) \\ &= \alpha_{0,t} \circ (T_s \otimes \mathsf{id}) \circ (\alpha_{0,s} \otimes \mathsf{id}) \circ (T_r \otimes \mathsf{id} \otimes \mathsf{id}) \;=\; \mathcal{T}_{s+t,t} \circ \alpha_{s,t} \circ (\alpha_{0,s} \otimes \mathsf{id}) \circ (T_r \otimes \mathsf{id} \otimes \mathsf{id}) \\ &= \mathcal{T}_{s+t,t} \circ \alpha_{0,s+t} \circ (\mathsf{id} \otimes \alpha_{s,t}^c) \circ (T_r \otimes \mathsf{id} \otimes \mathsf{id}) \;=\; \mathcal{T}_{s+t,t} \circ \alpha_{0,s+t} \circ (T_r \otimes \mathsf{id} \otimes \mathsf{id}) \circ (\mathsf{id} \otimes \alpha_{s,t}^c) \\ &= \mathcal{T}_{s+t,t} \circ \mathcal{T}_{r+s+t,s+t} \circ \alpha_{r,s+t} \circ (\mathsf{id} \otimes \alpha_{s,t}^c) \;=\; \mathcal{T}_{s+t,t} \circ \mathcal{T}_{r+s+t,s+t} \circ \alpha_{r+s,t} \circ (\alpha_{r,s} \otimes \mathsf{id}). \end{aligned}$$

Since, the range of $\alpha_{r+s,t} \circ (\alpha_{r,s} \otimes \mathsf{id})$ is total in $\mathcal{A}_{r+s+t}$ this shows the statement. □

The mappings $\mathcal{T}_{s+t,t}$ have some aspects from a *markovian system of conditional expectations*; see Accardi [3]. Of course, neither $\mathcal{T}_{s+t,t}$ nor $\mathcal{T}_{s+t,t} \upharpoonright (\mathcal{A}^c_{s+t})$ are conditional expectations. The former are not, because (cf. the discussion before Proposition 2.5) $\mathcal{A}_t$ cannot be identified with a subalgebra of $\mathcal{A}_{s+t}$, and the latter are not, because they map into $\mathcal{A}_t$, not $\mathcal{A}^c_t$.

3. The Reconstruction Theorem

We come to the construction of the GNS-system. Any completely positive mapping $T: \mathcal{A} \to \mathcal{B}$ gives rise to a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E, the *GNS-module*, with a *cyclic vector* ξ such that $T(a) = \langle \xi, a\xi \rangle$ and $E = \operatorname{span} \mathcal{A}\xi\mathcal{B}$. This *GNS-construction* preserves all desirable topological properties. See [5, 10] for details.

Denote by $\breve{E}_{s+t,t}$ the GNS-module of $\mathcal{T}_{s+t,t}$ with cyclic vector $\breve{\xi}_{s+t,t}$ and denote by $\breve{E}_t$ the GNS-module of T_t with cyclic vector $\breve{\xi}_t$. We may consider the $\mathcal{A}_{s+t}$–$\mathcal{A}_t$–module $\breve{E}_{s+t,t}$ also as $\mathcal{A}_s$–$\mathcal{B}$–module (of course not, pre-Hilbert module, because the inner product takes values in $\mathcal{A}_t$, not in $\mathcal{B}$) via the embeddings $\mathcal{A}_s \to \alpha_{s,t}(\mathcal{A}_s \otimes \mathbf{1}) \subset \mathcal{A}_{s+t}$ and $\mathcal{B} \to \alpha_{0,t}(\mathcal{B} \otimes \mathbf{1}) \subset \mathcal{A}_t$.

Proposition 3.1. *The $\mathcal{A}_s$–$\mathcal{B}$–submodule of $\breve{E}_{s+t,t}$ generated by $\breve{\xi}_{s+t,t}$ is isomorphic to the pre-Hilbert $\mathcal{A}_s$–$\mathcal{B}$–module $\breve{E}_s$ and the $\mathcal{A}_s$–$\mathcal{B}$–linear extension of the mapping*

$$\breve{\xi}_{s+t,t} \longmapsto \breve{\xi}_s$$

is the isomorphism.

Proof. It is sufficient to show that

$$\langle \breve{\xi}_{s+t,t}, \alpha_{s,t}(a_s \otimes \mathbf{1})\breve{\xi}_{s+t,t} \rangle \;=\; \mathcal{T}_{s+t,t} \circ \alpha_{s,t}(a_s \otimes \mathbf{1}) \;=\; \alpha_{0,t}(T_s(a_s) \otimes \mathbf{1})$$

for all $a_s \in \mathcal{A}_s$. Then also the $\mathcal{A}_s$–$\mathcal{B}$–linear extension is isometric (of course, it is surjective) and, therefore, well-defined. *A fortiori* the inner product of the submodule of $\breve{E}_{s+t,t}$ takes values in $\alpha_{0,t}(\mathcal{B} \otimes \mathbf{1}) \cong \mathcal{B} \subset \mathcal{A}_t$. □

We observe that the elements of $\mathcal{A}^c_t$ ($\subset \mathcal{A}_{s+t}$) commute with all elements in the the $\mathcal{A}_s$–$\mathcal{B}$–submodule $\breve{E}_s \subset \breve{E}_{s+t,t}$, and that $\breve{E}_{s+t,t}$ is generated by $\mathcal{A}^c_t$ and $\breve{E}_s$.

Recall that the tensor product of a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E and a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module F is the (unique) pre-Hilbert $\mathcal{A}$–$\mathcal{C}$–module $E \odot F$ which is spanned by *elementary tensors* $x \odot y$ ($x \in E, y \in F$) and whose inner product is determined by $\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$.

Proposition 3.2. *Let F be a pre-Hilbert $\mathcal{A}_t$–$\mathcal{C}$–module (which we may also consider as a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module). Then*

$$\breve{E}_s \odot F \;=\; \breve{E}_{s+t,t} \odot F.$$

In particular, $\breve{E}_s \odot F$ is a pre-Hilbert $\mathcal{A}_{s+t}$–$\mathcal{C}$–module.

Proof. $\breve{E}_{s+t,t} \odot F$ is spanned by elements of the form $x_s a_t \odot y = x_s \odot a_t y \in \breve{E}_s \odot F$ ($x_s \in \breve{E}_s, a_t \in \mathcal{A}^c_t, y \in F$). □

For $t > 0$ we set $\mathbb{I}_t = \{\mathfrak{t} = (t_n, \ldots, t_1) : n \in \mathbb{N}; t = t_n > t_{n-1} > \ldots > t_1 > t_0 = 0\}$. Clearly, $\mathbb{I}_t$ with the partial order defined by "inclusion" is a lattice. The bijective mapping $\mathfrak{o}\colon \mathfrak{t} \mapsto \mathfrak{s}$ with $s_i = \sum_{j=1}^{i} t_j$ induces a lattice structure also on the set $\mathbb{J}_t = \{\mathfrak{t} = (t_n, \ldots, t_1)\colon n \in \mathbb{N}; t_i > 0, |\mathfrak{t}| = t\}$ where $|\mathfrak{t}| = \sum_{i=1}^{n} t_i$.

Corollary 3.3. *Let $\mathfrak{t} \in \mathbb{J}_t$ and $\mathfrak{s} = \mathfrak{o}(\mathfrak{t}) \in \mathbb{I}_t$. Then*

$$\breve{E}_{\mathfrak{t}} \ := \ \breve{E}_{t_n} \odot \ldots \odot \breve{E}_{t_1} \ = \ \breve{E}_{s_n, s_{n-1}} \odot \breve{E}_{s_{n-1}, s_{n-2}} \odot \ldots \odot \breve{E}_{s_1, 0}$$

is a pre-Hilbert $\mathcal{A}_t$–$\mathcal{C}$–module.

Remark 3.4. The crucial point here is that, although we construct $\breve{E}_{\mathfrak{t}}$ as multiple tensor product of $\mathcal{B}$–$\mathcal{B}$–modules, it carries a well-defined left action of $\mathcal{A}_t$. The reason why this works can be traced back to the condition in (2.1). The message is that an element $a = a_{t_n} \ldots a_{t_1} \in \mathcal{A}_t^c$ which is thought of, roughly speaking, as a product of elements $a_{t_i} \in \mathcal{A}_{t_i}^c$ suitably shifted to the interval $[s_{i-1}, s_i]$ acts as $a(x_{t_n} \odot \ldots \odot x_{t_1}) = a_{t_n} x_{t_n} \odot \ldots \odot a_{t_1} x_{t_1}$. We do not formulate this in a more precise manner. We only want to give an intuitive idea.

Now we are reduced precisely to the situation in [5] for CP-semigroups. For $\mathfrak{s} \leq \mathfrak{t} \in \mathbb{J}_t$ (i.e. $\mathfrak{t} = (s_m^{k_m}, \ldots, s_m^1, \ \ldots, \ s_1^{k_1}, \ldots, s_1^1)$ where $\mathfrak{s}_\ell = (s_\ell^{k_\ell}, \ldots, s_\ell^1) \in \mathbb{J}_{s_\ell}$) we define two-sided isometric mappings $\beta_{\mathfrak{t}\mathfrak{s}}\colon \breve{E}_{\mathfrak{s}} \to \breve{E}_{\mathfrak{t}}$ by multiple $\mathcal{B}$–linear extension of

$$\xi_{s_m} \odot \ldots \odot \xi_{s_1} \ \longmapsto \ \xi_{t_n} \odot \ldots \odot \xi_{t_1}$$

and construct an inductive limit E_t. The only difference as compared with [5] is that we are concerned with an inductive system of pre-Hilbert $\mathcal{A}_t$–$\mathcal{B}$–modules. Consequently, also the inductive limit E_t is a pre-Hilbert $\mathcal{A}_t$–$\mathcal{B}$–module. Nevertheless, considering E_t as a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module, the E_t form a product system $E^\odot$. Also here the $\breve{\xi}_t$ give rise to elements $\xi_t \in E_t$ which form a unital unit $\xi^\odot$ for $E^\odot$. We collect these and some more results which are fairly obvious from [5].

Theorem 3.5. *Let T be a system of transition expectations for $(\mathcal{A}^\otimes, \mathcal{A}^{c\otimes}, \alpha)$. Then there exists a pair $(E^\odot, \xi^\odot)$ consisting of a product system pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules $E^\odot$ and a unital unit $\xi^\odot$ for $E^\odot$, fulfilling the following properties.*

E_t is also a pre-Hilbert $\mathcal{A}_t$–$\mathcal{B}$–module, and generated as such by ξ_t. The restriction of the left multiplication of $\mathcal{A}_t$ to the subset $\mathcal{B}$ gives back the correct left multiplication of $\mathcal{B}$. In particular, $\mathcal{A}_t^c$ is represented as a subset of $\mathcal{B}^{a,bil}(E_t)$. Finally, $T_t(a) = \langle \xi_t, a\xi_t \rangle$ for $a \in \mathcal{A}_t$.

The pair $(E^\odot, \xi^\odot)$ is determined by these properties up to isomorphism. We call $E^\odot$ the GNS-system *of T.*

4. The Time Shift

The inductive limit leading to the product system in Theorem 3.5 is a limit of two-sided modules. We recall quickly a second inductive limit (also from [5]) which embeds all E_t into a single module E equiped with an E_0*–semigroup* (i.e. a semigroup of unital endomorphisms of $\mathcal{B}^a(E)$). This is, however, only one-sided and, consequently, the resulting pre-Hilbert module is only a right module.

The mapping $\xi_s \odot \mathsf{id}_{E_t}\colon x \mapsto \xi_s \odot x$ defines an isometric embedding of E_t into E_{s+t}, obviously, giving rise to an inductive limit E. The factorization (1.1) turns over (roughly speaking, by sending formally s to ∞) and gives $E \odot E_t = E$. The associativity turns into $(E \odot E_s) \odot E_t = E \odot (E_s \odot E_t)$. Consequently, the mappings $\vartheta_t : a \mapsto a \odot \mathsf{id}_{E_t}$ on $\mathcal{B}^a(E)$ form an E_0–semigroup ϑ.

Under the inductive limit all ξ_t in E_t are identified with the same unit vector $\xi \in E$. By $j_0(b) = \xi b \xi^*$ we define a (usually, non-unital) representation of $\mathcal{B}$ on E. Then one of the main results from [5] asserts that the mappings $j_t = \vartheta_t \circ j_0$ form a *weak Markov flow* j for the CP-semigroup $T_t^\xi(b) = \langle \xi_t, b\xi_t \rangle$ on $\mathcal{B}$, i.e. setting $p_t = j_t(\mathbf{1})$, we have $p_t j_{s+t}(b) p_t = j_t \circ T_s^\xi(b)$. The following theorem follows precisely as in [5].

Theorem 4.1. *On the one-sided inductive limit E for the unit $\xi^\odot$, besides the weak Markov flow j of $\mathcal{B}$, we have a family $j^c = \big(j_t^c\big)_{t \in \mathbb{T}}$ of unital representations $a_t \mapsto \mathsf{id}_E \odot a_t$ of $\mathcal{A}_t^c$. These representations are compatible with the inductive structure of the $\mathcal{A}_t^c$ (i.e. $j_{s+t}^c \circ \alpha_{s+t} \restriction (\mathbf{1} \otimes \mathcal{A}_t^c) = j_t^c$). Therefore, there is a unique unital representation j_∞^c of $\mathcal{A}^c$ on E. (As there is, in general, no natural left action of $\mathcal{B}$ on E, it does not make sense to speak about bilinear operators on E.) Moreover, $j_\infty^c \circ \Theta_t = \vartheta_t \circ j_\infty^c$.*

By $J_t \circ \alpha_{0,t} = m \circ (j_t \otimes j_t^c)$ (where m denotes multiplication in $\mathcal{B}^a(E)$) we define a family $J = \big(J_t\big)_{t \in \mathbb{T}}$ of representations J_t of $\mathcal{A}_t$. These representations fulfill the generalized Markov property

$$p_t J_{s+t}(a) p_t \;=\; J_t \circ \mathcal{T}_{s+t,t}(a).$$

In particular, $p_0 J_t(a) p_0 = j_0 \circ T_t(a)$, i.e. $\langle \xi, J_t(a)\xi \rangle = T_t(a)$.

Theorem 4.2. *An* adapted *unitary cocycle $\mathfrak{u}^c$ for Θ (i.e. $\mathfrak{u}_t^c \in \mathcal{A}_t^c$) gives rise to a* local *cocycle $\mathfrak{u}$ for ϑ (i.e. $\mathfrak{u}_t$ commutes with $\vartheta_t(a)$ for all $a \in \mathcal{B}^a(E)$) via $\mathfrak{u}_t = \mathsf{id} \odot \mathfrak{u}_t^c$.*

Needless, to say that all statements in these notes extend to completions or closures under the assumed compatibility conditions. We do not go into details, because it is fairly clear from the corresponding arguments in [5]. We only mention as typical example for the argument that the assumption of normality for $\mathcal{T}_{s+t,t}$ (when $\mathcal{B}$ and $\mathcal{A}_t$ are von Neumann algebras) guarantees that $\overline{\breve{E}}^s_{s+t,t}$ is a von Neumann $\mathcal{A}_t$–$\mathcal{B}$–module and that the tensor product and the inductive limit of von Neumann modules result in von Neumann modules.

5. The Discrete Case

We have a closer look at the discrete example as indicated in Example 1.2 (where all α are just identifications). Here no inductive limit has to be computed and E_n is just $E_1^{\odot n}$ and E_1 is just the GNS-module of $T_1 : \mathcal{B} \otimes \mathcal{A}_1^c \to \mathcal{B}$. In other words, E_1 is $(\mathcal{B} \otimes \mathcal{A}_1^c) \otimes \mathcal{B}$ with obvious $\mathcal{B} \otimes \mathcal{A}_1^c$–$\mathcal{B}$–module structure and inner product

$$\langle (b \otimes a) \otimes \bar{b}, (b' \otimes a') \otimes \bar{b}' \rangle \;=\; \bar{b}^* T_1(b^* b' \otimes a^* a') \bar{b}'$$

divided by the set of length-zero elements $\mathcal{N}$. One easily check that $[(b \otimes a) \otimes \bar{b}] \odot [(b' \otimes a') \otimes \bar{b}'] = [(b \otimes a) \otimes \mathbf{1}] \odot [(\bar{b} b' \otimes a') \otimes \bar{b}']$. Therefore, we may identify conveniently

$$E_1 \odot E_1 \;=\; (\mathcal{B} \otimes \mathcal{A}_1^c) \otimes (\mathcal{B} \otimes \mathcal{A}_1^c) \otimes \mathcal{B} \;/\; \mathcal{N}$$

and correspondingly for higher tensor powers. The unit ξ_n is the element having $\mathbf{1}$'s at every position. Consequently, the second inductive limit is

$$E \;=\; \ldots \otimes (\mathcal{B} \otimes \mathcal{A}_1^c) \otimes (\mathcal{B} \otimes \mathcal{A}_1^c) \otimes \mathcal{B} \;/\; \mathcal{N}$$

and also ξ has $\mathbf{1}$'s everywhere.

The representation of the inifinite tensor product $\mathcal{A}^c = \ldots \otimes \mathcal{A}_1^c \otimes \mathcal{A}_1^c$ acts in the obvious way. The n–th member j_n of the weak Markov flow is a representation of $\mathcal{B}$ which first projects down the infinite tensor product to the first n factors and then acts in the obvious way. The ranges of j_n and j_m do, in general, not commute and, in general, on E there does not exist a representation of $\mathcal{B}_\infty = \ldots \otimes \mathcal{B} \otimes \mathcal{B}$.

If $\mathcal{B} = \mathbb{C}$, then T_1 is a state on $\mathcal{A}_1^c$ and $a \mapsto \langle \xi, a\xi \rangle$ is just the product state on the infinite tensor product $\mathcal{A}^c$. More generally, if T_1 is a conditional expectation (in which case necessarily $T_1(\mathbf{1} \otimes a)$ is in the center of $\mathcal{B}$ for all $a \in \mathcal{A}$), then $(b \otimes a) \otimes \bar{b} = (\mathbf{1} \otimes a) \otimes b\bar{b}$ so that E_1 simplifies as $\mathcal{A} \otimes \mathcal{B} \;/\; \mathcal{N}$, $E_n = \mathcal{A}^{\otimes n} \otimes \mathcal{B} \;/\; \mathcal{N}$ and $E = \ldots \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{B} \;/\; \mathcal{N}$. Also here $\langle \xi, \ldots \otimes a_n \otimes \ldots \otimes a_1 \xi \rangle = \ldots T_1(\mathbf{1} \otimes a_n) \ldots T_1(\mathbf{1} \otimes a_1)$ is just the product of the center-valued mappings $a \mapsto T_1(\mathbf{1} \otimes a)$. As $\mathcal{A}$ and $\mathcal{B}$ need not have something in common, it is evident that E need not carry a representation of the infinite tensor product $\mathcal{B}_\infty$. The action of $j_n(b)$ of projecting out the tensor sites $> n$ by applying the relevant part of the product state, and to let act b on the factor $\mathcal{B}$. Although all j_t have unital extensions, it is clear that these unital extensions will not commute for different n, unless $\mathcal{B}$ is commutative.

The situation changes completely, if $\mathcal{B} = \mathcal{A}_1^c$. In this case, T_1 may be a conditional expectation onto the second factor of $\mathcal{B} \otimes \mathcal{A}_1^c$. Again $T_1(b \otimes \mathbf{1})$ must be in the center and, formally, E_n and E have look as before. However, the inner product on E_1 is $\langle b \otimes a, b' \otimes a' \rangle = T_1(b^*b')a^*a'$ so that

$$\begin{aligned} \langle \xi, a_n \otimes \ldots \otimes a_1 \otimes b\xi \rangle &= T_1(\ldots T_1(T_1(\mathbf{1} \otimes a_n) \otimes a_{n-1}) \ldots \otimes a_1)b \\ &= T_1(\ldots T_1(T_1(a_n \otimes \mathbf{1})a_{n-1} \otimes \mathbf{1}) \ldots \otimes \mathbf{1})a_1 b. \end{aligned}$$

Here again j_n projects out the factors $> n$ (producing some centered element) but then applies b to the n–th site $\mathcal{A}_1^c = \mathcal{B}$ in $\mathcal{A}^c$. Again all j_n allow for a unital extension and have mutually commuting range, giving rise to a representation of $\mathcal{B}_\infty$.

If now $\mathcal{B} = \mathcal{A}_1^c = \mathcal{C}(S)$ for some compact Hausdorff space S, if T_1 is defined as $[T_1(b \otimes a)](s) = \int P(ds', s)b(s')a(s)$ for some Markov kernel P on S, and if μ is some probability measure on S, then $\varphi\colon \ \ldots \otimes a_n \otimes \ldots \otimes a_1 \otimes \mathbf{1} \mapsto \varphi_\mu\big(\langle \xi, a_n \otimes \ldots \otimes a_1 \otimes \mathbf{1}\xi \rangle\big)$, where φ_μ is the state induced on $\mathcal{C}(S)$ by μ, defines a state on $\ldots \otimes \mathcal{C}(S) \otimes \mathcal{C}(S) = \mathcal{A}^c$. This state extends to the completion $\mathcal{C}(\ldots \times S \times S)$ of $\mathcal{A}^c$. By the Tychonov theorem the space $S_\infty = \ldots \times S \times S$ is compact so that there is a unique Borel measure μ_∞ on S_∞ inducing the state φ. This measure is the measure guaranteed by the Kolmogorov extension theorem for the stationary Markov process with transition kernel P and initial measure μ on S. The representation obtained from the unital extensions of the j_n is (equivalent to) the Markov process obtained by Daniel-Kolmogorov construction.

A continuous time version where $\mathcal{A}^c$ is defined as $\mathcal{C}\big(\prod_{t \in \mathbb{R}_+} S\big)$ and $\mathcal{A}_t^c$ are the appropriate subalgebras gives us the reconstruction theorem for continuous time Markov processes defined by transition kernels and an initial measure. As all methods explained in these notes have obvious generalizations to the non-stationary case (then we just have to start with a compatible family $\mathcal{T}_{s,t}$ instead of T_t) also non-stationary Markov processes are included. It should be possible to obtain the Kolmogorov theorem for arbitrary index sets as long as the measure spaces at each index are isomorphic to compact ones.

So far, the mentioned extensions were concerning only the commutative case. However, there are indications that also extensions to more general quantum Markov fields are possible.

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