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USING WEIGHTS FOR THE DESCRIPTION OF STATES OF BOSON SYSTEMS

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Abstract. For a locally normal state of a boson system we construct, based on the Skorokhod integral and Malliavin derivative respectively their compound versions, weights showing similar properties as the reduced and the compound Campbell measures of classical point process theory.

As application we provide several characterisations of coherent states and states with property (γ_h), generalising a respective notion from point process theory and appearing in coherence considerations [37].

1. Introduction

Consider a (unit intensity) Poisson process Π on the real. In the following, Π will be described as random counting measure on $\mathbb{R}$ with distribution Q. Then the so-called Campbell formula states for suitable functions f:

$$\int Q(\mathrm{d}\Pi) \int \Pi(\mathrm{d}t) f(t) = \int \ell(\mathrm{d}t) f(t), \tag{1.1}$$

where ℓ is Lebesgue measure, see e.g. section VI.3 in [10].

In [27] the authors provided a generalisation of (1.1) both to arbitrary point processes (random point configurations) and to more general spaces G instead of $\mathbb{R}$. On such a space, point configurations are described by locally finite counting measures φ, forming the set M. For any point process Q (which is a probability measure on M) the reduced Campbell measure $C^!_Q$ on $G \times M$ and the compound Campbell measure C^∞_Q on $M \times M$ are defined by

$$C^!_Q(Z) = \int Q(\mathrm{d}\varphi) \int \varphi(\mathrm{d}x) \mathbf{1}_Z(x, \varphi - \delta_x), \tag{1.2}$$

$$C^\infty_Q(Z) = \int Q(\mathrm{d}\varphi) \sum_{\hat\varphi \le \varphi, \hat\varphi(G) < \infty} \binom{\varphi}{\hat\varphi} \mathbf{1}_Z(\hat\varphi, \varphi - \hat\varphi), \tag{1.3}$$

for measurable sets Z. There, $\varphi, \hat\varphi \in M$ are point configurations, $x \in G$, and the combinatorial factor $\binom{\varphi}{\hat\varphi}$ is explained below. Thus, in (1.2), one considers all selections of one point x from the (possibly infinite) configuration φ and in (1.3) even all selections of finite sub-configurations $\hat\varphi$. This way, $C^!_Q$ and C^∞_Q can characterise dependencies between the random points within a realisation of Q.

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It is well-known [26, 38] that $C^!_Q$ and C^∞_Q are σ-finite measures but usually not probability measures. Nevertheless, these measures are crucial for different forms of conditioning in point process theory, e.g. Palm measures and Gibbs measures, see [26, 27, 30, 32, 18, 19] among others. E.g., the generalisation of formula (1.1) obtains the simple form

$$C^!_Q = \ell \otimes Q \tag{1.4}$$

and characterises the distribution of Π. Interestingly, this equation makes quite explicit the strong independence properties of Poisson processes. Campbell measures play also a rôle in forming independent and dependent (stochastic) evolutions of particle systems [2, 3].

In this work we form analogues of $C^!_Q$ and C^∞_Q for locally normal states of bosons system. Those are the distributions of a possibly infinite but locally finite system of bosons, quantum particles. In the papers [11, 17] there was introduced a functional $(Y, A) \mapsto \omega(\mathrm{S}(Y, A))$, where Y is a set of point configurations, A is a (bounded) operator on a Fock space and ω is a locally normal state. Formally,

$$\mathrm{S}(Y, A) = \mathcal{S}^{\mathsf{c}} \mathrm{O}_Y \otimes A \mathcal{D}^{\mathsf{c}},$$

where $\mathcal{S}^{\mathsf{c}}$ is the compound Skorokhod integral, $\mathcal{D}^{\mathsf{c}}$ is the compound Malliavin derivative and O_Y is the operator of multiplication by the indicator function of Y. The operators $\mathcal{S}^{\mathsf{c}}, \mathcal{D}^{\mathsf{c}}$ act of suitable functions u via

$$\begin{aligned} \mathcal{S}^{\mathsf{c}} u(\varphi) &= \sum_{\hat\varphi \le \varphi, \hat\varphi(G) < \infty} \binom{\varphi}{\hat\varphi} u(\hat\varphi, \varphi - \hat\varphi) \\ \mathcal{D}^{\mathsf{c}} u(\varphi_1, \varphi_2) &= u(\varphi_1 + \varphi_2). \end{aligned}$$

From these formulae the close connection to (1.3) becomes obvious. $\mathcal{S}^{\mathsf{c}}, \mathcal{D}^{\mathsf{c}}$ were introduced in [12, 16], in section 3 we present more details.

Now one may replace in the term $\mathcal{S}^{\mathsf{c}} \mathrm{O}_Y \otimes A \mathcal{D}^{\mathsf{c}}$ the set Y by some bounded measurable function to get more general operators. Another variant is to consider kernel operators of the kind $\mathcal{S}^{\mathsf{c}} A_k \mathcal{D}^{\mathsf{c}}$ where A_k is some compound of an integral kernel operator and a multiplication operator [28, 21]. But, we do not want to consider these functionals in their relation to some form of multiplication or kernel operators. Instead, we take a more conceptual point of view. The quantum analogues of measures are weights. So we will look in section 4 at the weights given formally for operators A by

$$W^\infty_\omega(A) = \omega(\mathcal{S}^{\mathsf{c}} A \mathcal{D}^{\mathsf{c}}) \qquad \text{and} \qquad W^!_\omega(A) = \omega(\mathcal{S} A \mathcal{D}).$$

Unfortunately, $\mathcal{S}^{\mathsf{c}}, \mathcal{D}^{\mathsf{c}}$ are unbounded. Thus, there are normal states ω for which these formulae do not extend to semi-finite normal weights $W^!_\omega, W^\infty_\omega$ on the whole cone of positive operators on Fock space. The reason for this is that the analogue of Caratheodory's theorem might fail for general weights on pre-von Neumann algebras or that an associated quadratic form might be non-closable. Thus we will first look at a sufficiently large algebra of operators on which these functionals are well-defined. This suggests a more flexible approach to weights, comparable to the use of cylindrical measures in infinite dimensional analysis. Afterwards, we single out a collection of normal states for which these functionals extend to semifinite normal weights on the whole von Neumann algebra.

Similar procedures are available for locally normal states, provided by section 5.

Having these semifinite normal weights one could look for conditioning procedures. But we will leave this aside, it is already dealt with by [11, 17, 22]. Instead, we will look in section 6 at the generalisation of the characterisation of classical microcanonical ideal gases due to [29, 26] by analogues of (1.4). The corresponding class of states has a property we call (γ_h) for some $h \in L^2_{\mathsf{loc}}(G, \nu)$. This class of states coincides with the class of states with first order coherence introduced by [37], leaving aside the existence of some moments required by [37].

2. Basic Definitions and Notations

For the natural numbers, positive natural numbers, integers, real, positive real and complex numbers we use the symbols $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^* = \{1, 2, \ldots\}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{C}$ respectively.

Any function space we consider is complex. Thereby we will identify functions and the almost everywhere classes they refer to.

If $\mathcal{H}_0 \subset \mathcal{H}$ are two Hilbert spaces, $\mathcal{H}_0^\perp$ will denote the orthogonal complement. $\mathrm{Pr}_{\mathcal{H}_0}$ will denote the orthogonal projection onto $\mathcal{H}_0$, in the case $\mathcal{H}_0 = \mathbb{C}u$, $u \in \mathcal{H}$ we write simply Pr_u. We denote $\mathfrak{L}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$.

Especially, we consider Hilbert spaces of the form $\mathcal{H} = L^2(G, \mathfrak{G}, \nu)$ where G is a complete separable metric space G with Borel sets $\mathfrak{G}$ and the ring $\mathfrak{B}$ of bounded Borel sets. Further, the measure ν should be locally finite, i.e. $\nu(K) < \infty$ for all $K \in \mathfrak{B}$. For applications, $G = \mathbb{R}$ with Lebesgue measure or $G = \mathbb{N}$ with counting measure are sufficiently general. In the sequel, we suppress the σ-field and write $L^2(G, \nu)$.

The complement of some $K \in \mathfrak{G}$ we denote by K^{C}, the indicator function of K is $\mathbf{1}_K$. If $f : G \longmapsto \mathbb{C}$ is some function and $K \in \mathfrak{G}$ denote $f \cdot \mathbf{1}_K$ by f_K. If f is a bounded measurable function let denote $\mathrm{O}_f \in \mathfrak{L}(H)$ the operator of multiplication by f, i.e. for all $u \in L^2(G, \nu)$

$$(\mathrm{O}_f u)(x) = f(x)u(x) \qquad \nu\text{–a.e.} \tag{2.1}$$

If Y is a measurable set, set $\mathrm{O}_Y = \mathrm{O}_{\mathbf{1}_Y}$.

We will only consider C^*-algebras $\mathcal{A}$ having a unit $\mathbf{1}_{\mathcal{A}}$. All positive elements of $\mathcal{A}$, i.e. elements of the form A^*A form the set $\mathcal{A}_+$. A *state on* $\mathcal{A}$ is a linear functional $\omega : \mathcal{A} \longmapsto \mathbb{C}$ which is positive, i.e. $\omega(A) \geq 0$ for $A \in \mathcal{A}_+$, and normalised by $\omega(\mathbf{1}) = 1$.

Definition 2.1. The *symmetric Fock space over a separable Hilbert space* $\mathcal{H}$ is the Hilbert space

$$\Gamma(\mathcal{H}) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}_{\mathsf{sym}}, \tag{2.2}$$

where $\mathcal{H}^{\otimes n}_{\mathsf{sym}}$ is the n fold symmetric tensor product of $\mathcal{H}$, i.e. the Hilbert subspace of $\mathcal{H}^{\otimes n}$ generated by the vectors $u^{\otimes n}$, $u \in \mathcal{H}$ arbitrary.

An interesting class of vectors on $\Gamma(\mathcal{H})$ is built up by the exponential vectors:

Definition 2.2. Let be $h \in \mathcal{H}$. Then the *exponential vector* $\psi_h \in \Gamma(\mathcal{H})$ *associated with* h is defined through

$$\psi_h = 1 \oplus \bigoplus_{n=1}^{\infty} \frac{1}{\sqrt{n!}} h^{\otimes n}. \tag{2.3}$$

For any $K \in \mathfrak{G}$ we get the Hilbert space $\mathcal{M}_K \cong \Gamma(L^2(K,\nu))$ and abbreviate $\mathcal{M} = \mathcal{M}_G$. From [31, Proposition 19.6] we obtain that for disjoint $K, K' \in \mathfrak{G}$ there is a unitary $\mathcal{I}_{K,K'} : \mathcal{M}_K \otimes \mathcal{M}_{K'} \longmapsto \mathcal{M}_{K\cup K'}$ characterised by

$$\mathcal{I}_{K,K'}\left(\psi_{h_1} \otimes \psi_{h_2}\right) = \psi_{h_1+h_2}$$

for all $h_1 \in L^2(K)$, $h_2 \in L^2(K')$. Now we set for $K \in \mathfrak{G}$

$$\mathcal{A}_K = \mathcal{I}_{K,K^{\mathsf{c}}}(\mathfrak{L}\left(\mathcal{M}_K\right) \otimes \mathbf{1}_{\mathcal{M}_{K^{\mathsf{c}}}})\mathcal{I}^{-1}_{K,K^{\mathsf{c}}} \cong \mathfrak{L}\left(\mathcal{M}_K\right) \otimes \mathbf{1}_{\mathcal{M}_{K^{\mathsf{c}}}},$$

i.e. $\mathcal{A}_K \subset \mathfrak{L}(\mathcal{M})$. For $K \in \mathfrak{B}$ the algebra $\mathcal{A}_K$ is called *local algebra (over* K*)*. The C^*-algebra of quasilocal observables is given by $\mathfrak{A} = \overline{\bigcup_{K\in\mathfrak{B}} \mathcal{A}_K}$, where the bar denotes the closure in the uniform topology of $\mathfrak{L}(\mathcal{M})$.

Remark 2.3. For disjoint $K, K' \in \mathfrak{B}$ the algebras $\mathcal{A}_K$ and $\mathcal{A}_{K'}$ commute. This means that we are dealing with models for Boson systems.

A *normal state* ω on $\mathfrak{L}(\mathcal{M})$ or some $\mathcal{A}_K$ is given by $A \mapsto \omega(A) = \mathrm{Tr}\varrho A$ for some positive trace-class operator ϱ on $\mathcal{M}$. A state ω on $\mathfrak{A}$ is called *locally normal state* if for all $K \in \mathfrak{B}$ the state $\omega_K = \omega\mid_{\mathcal{A}_K}$ is normal.

Non zero vectors in a Hilbert space $\mathcal{H}'$ define in a natural manner a normal state on $\mathfrak{L}(\mathcal{H}')$.

Definition 2.4. For $h \in \mathcal{H}$ define the *coherent state associated with* h by

$$\phi^h(A) = \mathrm{e}^{-\|h\|^2}\left\langle \psi_h, A\psi_h \right\rangle$$

for all $A \in \mathfrak{L}(\Gamma(\mathcal{H}))$.

The density matrix (i.e. the operator ϱ above) of ϕ^h is given by $\mathrm{e}^{-\|h\|^2}\mathrm{B}_{h,h}$ where for $f, g \in \mathcal{H}$ the operator $\mathrm{B}_{f,g}$ operates as

$$\mathrm{B}_{f,g} u = \left\langle \psi_f, u \right\rangle \psi_g, \qquad (u \in \Gamma(\mathcal{H})). \tag{2.4}$$

On $\mathfrak{L}(\mathcal{M})$, we have thus (normal) coherent states. Now we introduce locally normal states with similar properties. Define (recall $g_K = g \cdot \mathbf{1}_K$)

$$L^2_{\mathsf{loc}}(G,\nu) = \left\{ g : G \longmapsto \mathbb{C} : g_K \in L^2(G,\nu) \text{ for all } K \in \mathfrak{B} \right\},$$

i.e. $L^2_{\mathsf{loc}}(G,\nu)$ is the space of locally square integrable functions.

For any pair of disjoint sets $K, K' \in \mathfrak{B}$ and $g \in L^2(K \cup K')$ one gets

$$\mathcal{I}^{-1}_{K,K'}\psi_g = \psi_{g_K} \otimes \psi_{g_{K'}}.$$

Clearly, this implies for any $A \in \mathcal{A}_K$ and $h \in L^2(G,\nu)$

$$\phi^h(A) = \phi^{h_K}(A).$$

Consequently, to define a locally normal coherent state from h, we need to have $h_K \in L^2(G,\nu)$ only for all $K \in \mathfrak{B}$, i.e. $h \in L^2_{\mathsf{loc}}(G,\nu)$. Then we call the unique locally normal state ϕ^h such that

$$\phi^h(A) = \phi^{h_K}(A) \tag{2.5}$$

for all $K \in \mathfrak{B}$ and all local observables $A \in \mathcal{A}_K$ again *coherent state.*

Further, we abbreviate the Dirac measure concentrated in $x \in G$ to δ_x, the zero measure is denoted by $\mathfrak{o}$ ($\mathfrak{o}(K) = 0\ \forall K \in \mathfrak{G}$). Now let M be the set of all locally finite counting measures on G, i.e.

$$M = \{\varphi : \varphi \text{ is measure on } G,\ \varphi(K) \in \mathbb{N} \text{ for all } K \in \mathfrak{B}\}\,.$$

Remark 2.5. Each $\varphi \in M$ is given by $\varphi = \sum_{j\in J} \delta_{x_j}$ with $x_j \in G$ and an at most countable index set J (cf. [26]). Therefore we can interpret φ as locally finite point configuration in G, $\mathfrak{o}$ represents the void configuration.

$\operatorname{supp}\varphi$ stands for the support of $\varphi \in M$, i.e. $\operatorname{supp}\varphi = \{x \in G : \varphi(\{x\}) > 0\}$. For $\varphi, \hat\varphi \in M$ we say $\hat\varphi \le \varphi$ if $\varphi - \hat\varphi \in M$. Then we define a number $\binom{\varphi}{\hat\varphi} \in \mathbb{N}$ by

$$\binom{\varphi}{\hat\varphi} = \prod_{x\in\operatorname{supp}\varphi} \binom{\varphi(\{x\})}{\hat\varphi(\{x\})}. \tag{2.6}$$

For $K \in \mathfrak{G}$ we denote by M_K the set of all counting measures φ with $\varphi(K^{\mathsf{C}}) = 0$, i.e. the set of configurations which have only points inside K.

On M we introduce the σ-field $\mathfrak{M}$ which is the smallest σ-field making the map $\varphi \mapsto \varphi(K')$ measurable for all $K' \in \mathfrak{B}$. In the case that K' is only varying over the bounded Borel sets inside $K \in \mathfrak{G}$, the corresponding σ-field is denoted by ${}_K\mathfrak{M}$. A probability measure on $(M,\mathfrak{M})$ is called *point process on G*.

$\Gamma(L^2(G,\nu))$ has itself the form of an L^2 space. We define the following σ-finite measure F on $(M,\mathfrak{M})$ by

$$F(Y) = \mathbf{1}_Y(\mathfrak{o}) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathbf{1}_Y\left(\textstyle\sum_{i=1}^n \delta_{x_i}\right) \nu^{\otimes n}(\mathrm{d}x_1,\dots,\mathrm{d}x_n) \tag{2.7}$$

for arbitrary $Y \in \mathfrak{M}$.

Remark 2.6. Identifying for diffuse ν the counting measure $\sum_{i=1}^n \delta_{x_i}$ with the set $\{x_1, x_2, \dots, x_n\}$, we get an isomorphism to the symmetric measure spaces used in [20, 24, 25, 23] and others. Observe that the above description works for atomic ν too, see [9, 8, 22].

The following lemma is easy to prove.

Lemma 2.7. *$L^2(M,F)$ is isomorphic to $\Gamma(L^2(G,\nu))$ under the isomorphism I determined by*

$$Iu^{\otimes n}(\varphi) = \mathbf{1}_{\{\tilde\varphi : |\tilde\varphi| = n\}}(\varphi) \prod_{x\in\operatorname{supp}\varphi} u(x)^{\varphi(\{x\})}. \tag{2.8}$$

In the sequel we will identify the spaces $\mathcal{M}$ and $L^2(M,\mathfrak{M},F)$ and denote both as symmetric Fock spaces.

Exponential vectors from $\mathcal{M}$ correspond to the following functions $\exp_h$. For a measurable function $h : G \longmapsto \mathbb{C}$ introduce $\exp_h : M \longmapsto \mathbb{C}$ by

$$\exp_h(\varphi) = \begin{cases} \prod_{x \in \operatorname{supp}\varphi} h(x)^{\varphi(\{x\})}, & \text{if } 0 < |\varphi| < \infty, \\ 1, & \text{if } \varphi = \mathfrak{o}, \\ 0, & \text{otherwise.} \end{cases}$$

We want to mention the following connection to point process theory. Observe that for $K \in \mathfrak{B}$ and ${}_K\mathfrak{M}$ measurable f we have $\mathrm{O}_f \in \mathcal{A}_K$.

Proposition 2.8 (cf. [11, Theorem 2.15])**.** *For any locally normal state ω on $\mathfrak{A}$ there is exactly one point process Q_ω on G which fulfils for all $K \in \mathfrak{B}$ and all $f \in L^\infty(M, {}_K\mathfrak{M}, F)$.*

$$\int \mathrm{Q}_\omega(\mathrm{d}\varphi) f(\varphi) = \omega(\mathrm{O}_f).$$

The position distribution of the coherent state ϕ^g is easily found to be the *Poisson process* $\Pi_{|g|^2\nu}$ with intensity measure $|g|^2\nu$. I.e., for all disjoint bounded sets $K_1, K_2, \ldots, K_m \in \mathfrak{B}$ and all m–tuple $(n_1, n_2, \ldots n_m) \in \mathbb{N}^m$

$$\mathrm{Q}_{\phi^g}\left(\{\varphi : \forall i, 1 \le i \le m \quad \varphi(K_i) = n_i\}\right) = \prod_{i=1}^m \exp(-\int_{K_i} |g|^2 \,\mathrm{d}\nu) \frac{(\int_{K_i} |g|^2 \,\mathrm{d}\nu)^{n_i}}{n_i!}.$$

3. Malliavin Derivative and Skorokhod Integral on Fock space

In this section we will introduce the Malliavin derivative $\mathcal{D}$ and Skorokhod integral $\mathcal{S}$ on Fock space as well as their compound versions $\mathcal{S}^{\mathsf{c}}$ and $\mathcal{D}^{\mathsf{c}}$. These operators are basic for many operations on Fock space. We will state only the facts, for proofs and related topics we refer to [12, 16].

Let N denote the number operator on $\mathcal{M}$. I.e. N is defined on

$$\operatorname{dom}(N) = \left\{ u \in \mathcal{M} : \int F(\mathrm{d}\varphi)\varphi(G)^2 \left|u(\varphi)\right|^2 < \infty \right\} \tag{3.1}$$

via

$$Nu(\varphi) = \varphi(G)u(\varphi).$$

Now it is clear what we have to understand under $\operatorname{dom}(\sqrt{N})$. On this domain we define the *Malliavin derivative* $\mathcal{D} : \operatorname{dom}(\sqrt{N}) \longmapsto L^2(G, \nu) \otimes \mathcal{M}$ by

$$\mathcal{D}u(x, \varphi) = u(\varphi + \delta_x). \tag{3.2}$$

$\mathcal{D}$ is a closed operator and its adjoint, the *Skorokhod integral* $\mathcal{S} : L^2(G, \nu) \otimes \mathcal{M} \longmapsto \mathcal{M}$, fulfils

$$\mathcal{S}u(\varphi) = \int \varphi(\mathrm{d}x) u(x, \varphi - \delta_x). \tag{3.3}$$

$\operatorname{dom}(\mathcal{S})$ is thereby the maximal domain on which this formula defines a square integrable function.

Similarly define the *compound Malliavin derivative* $\mathcal{D}^{\mathsf{c}} : \operatorname{dom}(\sqrt{2}^N) \longmapsto \mathcal{M} \otimes \mathcal{M}$ and the *compound Skorokhod integral* $\mathcal{S}^{\mathsf{c}} : \mathcal{M} \otimes \mathcal{M} \longmapsto \mathcal{M}$ by

$$\mathcal{D}^{\mathsf{c}} u(\varphi_1, \varphi_2) = u(\varphi_1 + \varphi_2) \tag{3.4}$$

and

$$\mathcal{S}^{c}u(\varphi)=\sum_{\hat{\varphi}\leq\varphi,\hat{\varphi}(G)<\infty}\binom{\varphi}{\hat{\varphi}}u(\hat{\varphi},\varphi-\hat{\varphi}). \tag{3.5}$$

For later use, note the following formulae for $g\in L^2(G,\nu)$:

$$\mathcal{D}\psi_g = g\otimes\psi_g, \tag{3.6}$$

$$\mathcal{D}^{c}\psi_g = \psi_g\otimes\psi_g. \tag{3.7}$$

4. Campbell Weights for Normal States

Definition 4.1 ([5, Definition 2.7.8])**.** A *weight* on a C^*-algebra $\mathcal{A}$ is a functional $\eta:\mathcal{A}_+\longmapsto[0,\infty]$ such that

$$\eta(\lambda_1A+\lambda_2B)=\lambda_1\eta(A)+\lambda_2\eta(B),\qquad(\lambda_1,\lambda_2\geq 0,\quad A,B\in\mathcal{A}_+)$$

under the convention $0\cdot\infty=0$.

If $\mathcal{A}$ is a von Neumann algebra η is called *normal* if

$$\eta(\sup_{\alpha\in\mathbb{A}}A_\alpha)=\sup_{\alpha\in\mathbb{A}}\eta(A_\alpha)$$

for each monotonously increasing bounded net $(A_\alpha)_{\alpha\in\mathbb{A}}\subset\mathcal{A}_+$. η is called *semi-finite weight* if

$$L^1(\eta):=\operatorname{span}\{A\in\mathcal{A}_+:\eta(A)<\infty\} \tag{4.1}$$

is σ-weakly dense in $\mathcal{A}$.

For a moment, let ω be a normal state and ϱ the density matrix of ω, i.e. $\omega(A)=\mathrm{Tr}\varrho A$. Then formally

$$\omega(\mathcal{S}^{c}A\mathcal{D}^{c})=\mathrm{Tr}(\varrho\mathcal{S}^{c}A\mathcal{D}^{c})=\mathrm{Tr}((\mathcal{D}^{c}\varrho\mathcal{S}^{c})A). \tag{4.2}$$

It is well-known that Tr is a faithful semifinite normal weight [33]. The only problem is that $(\mathcal{D}^{c}\varrho\mathcal{S}^{c})A$ is in general not a positive operator. But formally, under the convention $\mathcal{D}^{c}\varrho\mathcal{S}^{c}=Q$

$$\mathrm{Tr}QA=\mathrm{Tr}\sqrt{Q}A\sqrt{Q} \tag{4.3}$$

and $\sqrt{Q}A\sqrt{Q}$ is positive (may be, it is only a non-closable quadratic form). To avoid problems with non-closability, we restrict to a smaller class of operators. Define

$$\mathcal{B}=\left\{A\in\mathfrak{L}(\mathcal{M}):\exists m\in\mathbb{N}:\mathbf{1}_{[0,m]}(N)A\mathbf{1}_{[0,m]}(N)=A\right\}.$$

and similarly $\mathcal{B}^!\subset\mathfrak{L}(\mathcal{H}\otimes\mathcal{M})$ and $\mathcal{B}^\infty\subset\mathfrak{L}(\mathcal{M}\otimes\mathcal{M})$:

$$\mathcal{B}^! = \left\{A\in\mathfrak{L}(\mathcal{H}\otimes\mathcal{M}):\exists m\in\mathbb{N}:\mathbf{1}\otimes\mathbf{1}_{[0,m]}(N)A\mathbf{1}\otimes\mathbf{1}_{[0,m]}(N)=A\right\},$$

$$\mathcal{B}^\infty = \left\{A\in\mathfrak{L}(\mathcal{M}\otimes\mathcal{M}):\exists m\in\mathbb{N}:\mathbf{1}\otimes\mathbf{1}_{[0,m]}(N)A\mathbf{1}\otimes\mathbf{1}_{[0,m]}(N)=A\right\}.$$

Definition 4.2. Let ω be a normal state on $\mathfrak{L}(\mathcal{M})$. We define its *reduced weight* $W^!_\omega$ and its *compound weight* W^∞_ω on $\mathcal{B}^!$ respectively $\mathcal{B}^\infty$ by

$$W^!_\omega(A)=\mathrm{Tr}\varrho\mathcal{S}A\mathcal{D},\qquad(A\in\mathcal{B}^!),$$

$$W^\infty_\omega(A)=\mathrm{Tr}\varrho\mathcal{S}^{c}A\mathcal{D}^{c},\qquad(A\in\mathcal{B}^\infty).$$

Since neither $\mathcal{B}^!$ nor $\mathcal{B}^\infty$ are C^*-algebras, this definition does not follow the above definition of weights. We make another generalisation.

Definition 4.3. Suppose $\mathfrak{C} = \bigcup_{\alpha\in\mathbb{A}} \mathfrak{C}_\alpha$ is an algebra where all $\mathfrak{C}_\alpha$ are von Neumann algebras. We say that a functional $\eta : \mathfrak{C}_+ \longmapsto [0,\infty]$ is a *semifinite locally normal weight* if for all $\alpha \in \mathbb{A}$ $\eta\,|_{\mathfrak{C}_\alpha}$ is a semifinite normal weight.

Remark 4.4. This way to define a semifinite locally normal weight is comparable to the definition of cylindrical measures on an infinite dimensional vector space [35]. Such cylindrical measures are defined on a field rather than a σ field. Moreover, they are additive on this field but σ additive on certain sub σ fields.

In our case, $\mathcal{B}^! = \bigcup_{m\in\mathbb{N}} \mathcal{B}^!_m$ with

$$\mathcal{B}^!_m = \left\{ A \in \mathfrak{L}(\mathcal{H}\otimes\mathcal{M}) : \mathbf{1}\otimes\mathbf{1}_{[0,m]}(N) A \mathbf{1}\otimes\mathbf{1}_{[0,m]}(N) = A \right\}$$

and $\mathcal{B}^\infty = \bigcup_{m\in\mathbb{N}} \mathcal{B}^\infty_m$ with

$$\mathcal{B}^\infty_m = \left\{ A \in \mathfrak{L}(\mathcal{M}\otimes\mathcal{M}) : \mathbf{1}_{[0,m]}(N\otimes\mathbf{1}+\mathbf{1}\otimes N) A \mathbf{1}_{[0,m]}(N\otimes\mathbf{1}+\mathbf{1}\otimes N) = A \right\}.$$

Lemma 4.5. *$W^!_\omega$ and W^∞_ω are well-defined semifinite locally normal weights.*

Proof. For $A \in \mathcal{B}^!_m$ the operator

$$\mathcal{S}A\mathcal{D} = \mathcal{S}\mathbf{1}\otimes\mathbf{1}_{[0,m]}(N)A\mathbf{1}\otimes\mathbf{1}_{[0,m]}(N)\mathcal{D} = \mathbf{1}_{[0,m-1]}(N)\mathcal{S}A\mathcal{D}\mathbf{1}_{[0,m-1]}(N)\mathcal{D}$$

is well-defined and bounded on $\mathrm{dom}(\sqrt{N})$, see [12]. Therefore, it has a unique bounded extension to $\mathfrak{L}(\mathcal{M})$. If $(A_\alpha)_{\alpha\in\mathbb{A}}$ is a bounded increasing net of positive operators in $\mathcal{B}^!_m$, it follows that

$$\sup_{\alpha\in\mathbb{A}} \mathcal{S}A_\alpha\mathcal{D} = \mathcal{S}\sup_{\alpha\in\mathbb{A}} A_\alpha\mathcal{D}.$$

This shows that $W^!_\omega$ is a semifinite locally normal weight if ω is normal.

Similar arguments work for W^∞_ω. □

These preliminary results are not so exiting, as it is preferable to have a weight on the full $\mathfrak{L}(\mathcal{H}\otimes\mathcal{M})$ or $\mathfrak{L}(\mathcal{M}\otimes\mathcal{M})$. The extension of $W^!_\omega$ and W^∞_ω, if possible, will come from the following reformulation of (4.3).

Let Q be any positive densely defined closed operator on a Hilbert space H. We define the semifinite normal weight w_Q on $\mathfrak{L}(H)$ by

$$w_Q(A) = \sum_{i\in\mathbb{N}} \left\langle \sqrt{Q}e_i, A\sqrt{Q}e_i \right\rangle \in [0,\infty], \qquad (A\in\mathfrak{L}(\mathcal{H})_+),$$

for some complete orthonormal system $(e_i)_{i\in\mathbb{N}} \subset \mathrm{dom}(\sqrt{Q})$.

Remark 4.6. It is a standard argument comparable to [33] to show that $w_Q(A)$ does not depend on the choice of the complete orthonormal system $(e_i)_{i\in\mathbb{N}}$. In the appendix, Proposition 7.1, we prove that w_Q is really a semifinite normal weight. Moreover, several other properties of w_Q are collected there.

For using the construct w_Q on $W^!_\omega$ there remains the problem whether $\mathcal{D}\varrho\mathcal{S}$ is densely defined. We can reduce this to the question whether $\sqrt{\varrho}\mathcal{S}$ is closable as then $\mathcal{D}\varrho\mathcal{S} = (\sqrt{\varrho}\mathcal{S})^*\overline{\sqrt{\varrho}\mathcal{S}}$.

Definition 4.7. Let ω be a normal state on $\mathfrak{L}(\mathcal{M})$ represented by the density matrix ϱ such that $\sqrt{\varrho}\mathcal{S}$ or $\sqrt{\varrho}\mathcal{S}^{\mathsf{c}}$ is closable. We define the *reduced weight* $W_\omega^!$ or the *compound weight* W_ω^∞ by

$$W_\omega^!(A) = w_{\mathcal{D}\varrho\mathcal{S}}(A), \qquad (A \in \mathfrak{L}(\mathcal{H} \otimes \mathcal{M})) \tag{4.4}$$

or

$$W_\omega^\infty(A) = w_{\mathcal{D}^{\mathsf{c}}\varrho\mathcal{S}^{\mathsf{c}}}(A), \qquad (A \in \mathfrak{L}(\mathcal{M} \otimes \mathcal{M}) \tag{4.5}$$

respectively.

Proposition 4.8. *If they exist on the whole von Neumann algebra, $W_\omega^!$ and W_ω^∞ are semifinite normal weights. Moreover, if $R \in \mathfrak{L}(\mathcal{H} \otimes \mathcal{M})$ or $R \in \mathfrak{L}(\mathcal{M} \otimes \mathcal{M})$ is such that $R\mathcal{D}$ or $R\mathcal{D}^{\mathsf{c}}$ extends to a bounded operator then*

$$W_\omega^!(R^*AR) \;=\; \omega((R\mathcal{D})^*A(R\mathcal{D}), \qquad (A \in \mathfrak{L}(\mathcal{H} \otimes \mathcal{M})),$$

or

$$W_\omega^\infty(R^*AR) \;=\; \omega((R\mathcal{D}^{\mathsf{c}})^*A(R\mathcal{D}^{\mathsf{c}})), \qquad (A \in \mathfrak{L}(\mathcal{M} \otimes \mathcal{M})),$$

respectively. Especially, $W_\omega^!$ and W_ω^∞ coincide with their versions on $\mathcal{B}^!$ and $\mathcal{B}^\infty$ respectively.

Proof. The first part is just a special case of Proposition 7.1. For the second part we only prove the result for the reduced weight. Then

$$\omega((R\mathcal{D})^*A(R\mathcal{D})) = \mathrm{Tr}(\varrho(R\mathcal{D})^*A(\overline{R\mathcal{D}})) = \mathrm{Tr}((\overline{R\mathcal{D}})\varrho(R\mathcal{D})^*A).$$

As $(\overline{R\mathcal{D}})\varrho(R\mathcal{D})^* = \overline{R\mathcal{D}}\varrho\overline{\mathcal{S}R^*}$ is a trace class operator we can use again Proposition 7.1 and get

$$\omega((R\mathcal{D})^*A(\overline{R\mathcal{D}})) = w_{\mathcal{D}\varrho\mathcal{S}}(R^*AR) = W_\omega^!(R^*AR).$$

Using this result with $R = \mathbf{1} \otimes \mathbf{1}_{[0,m]}(N)$ shows that $W_\omega^!$ extends the previous definition from $\mathcal{B}^!$ to $\mathfrak{L}(\mathcal{H} \otimes \mathcal{M})$. □

The definitions of $\mathcal{D}$ and $\mathcal{D}^{\mathsf{c}}$ show that both operators work by decomposing a given configuration into two sub-configurations. So we may interpret a functional of the kind $\omega(\mathcal{D}^{\mathsf{c}}A\mathcal{S}^{\mathsf{c}})$ as "state" of the decomposed quantum system. We can make this more precise by using beam splittings to divide the quantum system into two [14, 13].

For $\zeta, \xi \in \mathbb{C}$ with $|\zeta|^2 + |\xi|^2 = 1$ the beam splitting isometry $\mathcal{V}_{\zeta,\xi}$ fulfils

$$\mathcal{V}_{\zeta,\xi}\psi_h = \psi_{\zeta h} \otimes \psi_{\xi h}. \tag{4.6}$$

For any operator $T \in \mathfrak{L}(\mathcal{H})$, $\|T\| \le 1$ the operator of second quantisation of T, $\Gamma(T) \in \mathfrak{L}(\mathcal{M})$ operates as

$$\Gamma(T)\psi_h = \psi_{Th}.$$

For $z \in \mathbb{C}, |z| \le 1$ we write briefly $\Gamma(z) = \Gamma(z\mathbf{1})$.

Corollary 4.9. *It holds*

$$\omega(\mathcal{V}_{\zeta,\xi}^*A\mathcal{V}_{\zeta,\xi}) = W_\omega^\infty(\Gamma(\overline{\zeta}) \otimes \Gamma(\overline{\xi})A\Gamma(\zeta) \otimes \Gamma(\xi))$$

for all $A \in \mathcal{B}^\infty$ and, if W_ω^∞ exists on $\mathfrak{L}(\mathcal{M} \otimes \mathcal{M})$, also for all $A \in \mathfrak{L}(\mathcal{M} \otimes \mathcal{M})$.

Proof. It is immediate, that $\mathcal{V}_{\zeta,\xi}$ is a bounded extension of $\Gamma(\zeta) \otimes \Gamma(\xi)\mathcal{D}^{\mathsf{c}}$. Proposition 4.8 completes the proof. □

We remark the following two results concerning the existence of the reduced respectively the compound weight. A state ω is called *gauge invariant* if

$$\omega(\Gamma(\mathrm{e}^{-\mathrm{i}t})\cdot\Gamma(\mathrm{e}^{\mathrm{i}t}))=\omega,\qquad(t\in\mathbb{R}).$$

Lemma 4.10. *For every gauge invariant normal state* $W^{!}_{\omega}$ *and* W^{∞}_{ω} *exist.*

Proof. The hypothesis implies that ϱ commutes with N, which is the generator of $t\mapsto\Gamma(\mathrm{e}^{\mathrm{i}t})$. Therefore, $\sqrt{\varrho}$ maps $\mathrm{dom}(\sqrt{N})$ and $\mathrm{dom}(\sqrt{2}^N)$ into themselves and $\mathcal{D}\sqrt{\varrho}$ and $\mathcal{D}^{\mathsf{c}}\sqrt{\varrho}$ are densely defined. Hence $\sqrt{\varrho}\mathcal{S}$ and $\sqrt{\varrho}\mathcal{S}^{\mathsf{c}}$ possess a densely defined adjoint. Consequently, they are closable. □

Lemma 4.11. *If* W^{∞}_{ω} *exists on* $\mathfrak{L}(\mathcal{M}\otimes\mathcal{M})$ *then* $W^{!}_{\omega}$ *exists on* $\mathfrak{L}(\mathcal{H}\otimes\mathcal{M})$.

Proof. This follows from the fact that we can think of $\mathcal{S}$ as restriction of $\mathcal{S}^{\mathsf{c}}$ by identifying $\mathcal{H}$ with $\mathcal{H}^{\otimes 1}_{\mathsf{sym}}\subset\Gamma(\mathcal{H})=\mathcal{M}$. □

As example, we evaluate $W^{!}_{\omega}$ and W^{∞}_{ω} for coherent states.

Lemma 4.12. *For* $\omega=\phi^h$, $h\in\mathcal{H}$ *both the reduced and the compound weight exist on the full von Neumann algebras and*

$$W^{!}_{\omega}(A\otimes B)=\langle h,Ah\rangle\,\phi^h(B),\qquad(A\in\mathfrak{L}(\mathcal{H}),B\in\mathfrak{L}(\mathcal{M})),\tag{4.7}$$

$$W^{\infty}_{\omega}(A\otimes B)=\langle\psi_h,A\psi_h\rangle\,\phi^h(B),\qquad(A,B\in\mathfrak{L}(\mathcal{M})).\tag{4.8}$$

Proof. We prove only the formula for the reduced weight. The density matrix of ϕ^h is $\mathrm{e}^{-\|h\|^2}\mathrm{B}_{h,h}$, cf. (2.4). Now (3.6) shows for $u\in\mathrm{dom}(\mathcal{S})$

$$\mathcal{D}\mathrm{B}_{h,h}\mathcal{S}u=\langle\psi_h,\mathcal{S}u\rangle\,\mathcal{D}\psi_h=\langle\mathcal{D}\psi_h,u\rangle\,\mathcal{D}\psi_h=\langle h\otimes\psi_h,u\rangle\,h\otimes\psi_h.$$

Thus $\mathcal{D}\mathrm{B}_{h,h}\mathcal{S}$ extends to the trace class operator $\|h\|^2\,\mathrm{e}^{\|h\|^2}\mathrm{Pr}_{h\otimes\psi_h}$. Proposition 7.1 finishes the proof. □

5. Campbell Weights for Locally Normal States

To obtain the above defined weights also for locally normal states, we have to be careful. The first attempt is to insert into (4.4) and (4.5) some A from $\mathfrak{L}(\mathcal{H})\otimes\mathfrak{A}$ or $\mathfrak{A}\otimes\mathfrak{A}$. But we have in the decomposition $\Gamma(\mathcal{H})=\Gamma(L^2(K))\otimes\Gamma(L^2(K^{\mathsf{C}}))$ formally

$$\mathcal{S}^{\mathsf{c}}_K\otimes\mathcal{S}^{\mathsf{c}}_{K^{\mathsf{c}}}A'\otimes\mathbf{1}_{K^{\mathsf{c}},K^{\mathsf{c}}}\mathcal{D}^{\mathsf{c}}_K\otimes\mathcal{D}^{\mathsf{c}}_{K^{\mathsf{c}}}=\mathcal{S}^{\mathsf{c}}_K A'\mathcal{D}^{\mathsf{c}}_K\otimes 2^{N_{K^{\mathsf{c}}}}.$$

In some sense the distribution of $2^{N_{K^{\mathsf{c}}}}$ under ω should have finite values for W^{∞}_{ω} taking other values than 0 and ∞. But one can show [11, 22] that this implies that ω is a normal state.

Remark 5.1. An analogous statement is also true for point processes. This is the reason why in formula (1.3) summation is restricted to finite sub-configurations of φ only: one considers $\hat{\varphi}\leq\varphi$, $\hat{\varphi}(G)<\infty$. Then $C^{!}_{Q}$ and C^{∞}_{Q} are σ-finite measures. We have to give these considerations a more algebraic form.

For any $K\in\mathfrak{B}$, $m\in\mathbb{N}$ we have an algebra $\mathcal{B}^{!}_{K,m}$ of operators $A\in\mathfrak{L}(\mathcal{H}\otimes\mathcal{M})$ such that there is an operator $A_K\in\mathfrak{L}(\mathcal{H}\otimes\mathcal{M}_K)$ with

$$A(h\otimes u_K\otimes u_{K^{\mathsf{c}}})=(A_K(h\otimes u_K))\otimes u_{K^{\mathsf{c}}},\quad(h\in\mathcal{H},u_K\in\mathcal{M}_K,u_{K^{\mathsf{c}}}\in\mathcal{M}_{K^{\mathsf{c}}}),$$

and

$$\mathbf{1}_K \otimes \mathbf{1}_{[0,m]}(N_K) A_K \mathbf{1}_K \otimes \mathbf{1}_{[0,m]}(N_K) = A_K,$$

where N_K is the number operator on K. Similarly, we define $\mathcal{B}^{\infty}_{K,m}$ and introduce the algebras

$$\hat{\mathcal{B}}^! = \bigcup_{K\in\mathfrak{B}, m\in\mathbb{N}} \mathcal{B}^!_{K,m},$$

$$\hat{\mathcal{B}}^\infty = \bigcup_{K\in\mathfrak{B}, m\in\mathbb{N}} \mathcal{B}^\infty_{K,m}.$$

We define $W^!_\omega$ as functional on $\hat{\mathcal{B}}^!$ through

$$W^!_\omega(A) = W^!_{\omega_K}((\mathbf{1}\otimes \mathrm{O}_{M_K})A(\mathbf{1}\otimes \mathrm{O}_{M_K})), \qquad (A \in \mathcal{B}^!_K).$$

In a similar fashion we define

$$W^\infty_\omega(A) = W^\infty_{\omega_K}((\mathbf{1}\otimes \mathrm{O}_{M_K})A(\mathbf{1}\otimes \mathrm{O}_{M_K})), \qquad (A \in \mathcal{B}^\infty_K).$$

Proposition 5.2. *For any locally normal state ω, $W^!_\omega$ and W^∞_ω are well-defined semifinite locally normal weights.*

Proof. We prove this only for $W^!_\omega$, the proof for W^∞_ω follows the same lines. Further, we need only to prove that $W^!_\omega$ is well-defined. Take $K' \supseteq K \in \mathfrak{B}$ and let $\varrho_{K'}$ be the density matrix of $\omega_{K'}$. Then by Proposition 4.8 and the fact that $\big(\mathrm{O}_K \otimes \mathrm{O}_{M_{K'}} \mathbf{1}_{[0,m]}(N_K)\big)\mathcal{D}$ is bounded we derive for $A \in \mathcal{B}^!_K$

$$W^!_{\omega_{K'}}(A) = \mathrm{Tr}\varrho_{K'}\mathcal{S}\mathbf{1}\otimes \mathrm{O}_{M_{K'}\cap M_K^{\leq n}} A\mathbf{1}\otimes \mathrm{O}_{M_{K'}}\mathbf{1}_{[0,m]}(N_K)\mathcal{D}.$$

The locality property of A implies

$$\mathcal{S}\mathbf{1}\otimes \mathrm{O}_{M_{K'}}\mathbf{1}_{[0,m]}(N_K)M_{K'}\cap M_K^{\leq n} A\mathbf{1}\otimes \mathrm{O}_{M_{K'}}\mathbf{1}_{[0,m]}(N_K)\mathcal{D} = \\ \mathcal{S}\mathbf{1}\otimes \mathbf{1}_{[0,m]}(N_K)A'\mathbf{1}\otimes \mathbf{1}_{[0,m]}(N_K)\mathcal{D}\otimes \mathrm{O}_{M_{K'\setminus K}}$$

for some A'. This completes the proof. □

In analogy to normal states, we would like to extend these weights for certain locally normal states to a larger algebra, related to the quasilocal algebra the state is defined on. E.g., the reduced weight $W^!_\omega$ should be defined on $\overline{\bigcup_{K\in\mathfrak{B}} (\mathfrak{L}(L^2(K))\otimes \mathcal{A}_K)}$. But (4.7) and (4.8) should remain true if h is chosen from $L^2_{\mathsf{loc}}(G,\nu)$. Then we have to know what is $\langle h, Ah\rangle$ if $A \in \overline{\bigcup_{K\in\mathfrak{B}} \mathfrak{L}(L^2(K))}$. E.g. all operators Pr_f are from this space. Then formally $\langle h, \mathrm{Pr}_f h\rangle = |\langle f, h\rangle|^2$ which is the square of an integral which may converge, diverge or oscillate without having any limit depending on the choice of f and h. We conclude that we should restrict the weight to purely local algebras. We set

$$\mathfrak{A}^! = \bigcup_{K\in\mathfrak{B}} \left(\mathfrak{L}(L^2(K))\otimes \mathcal{A}_K\right),$$

$$\mathfrak{A}^\infty = \bigcup_{K\in\mathfrak{B}} \left(\mathfrak{L}(\mathcal{M}_K)\otimes \mathcal{A}_K\right).$$

Suppose ω is a locally normal state such that for any $K \in \mathfrak{B}$ the normal state ω_K given through

$$\omega_K(A) = \omega(\mathrm{O}_{M_K} A \mathrm{O}_{M_K} \otimes \mathbf{1}_{\mathcal{M}_{K^{\mathsf{c}}}})$$

possesses the reduced weight. Define $W^!_\omega$ as functional on $\bigcup_{K\in\mathfrak{B}}(\mathfrak{L}(L^2(K))\otimes\mathcal{A}_K)_+$ through

$$W^!_\omega(A) = W^!_{\omega_K}((\mathbf{1}\otimes \mathrm{O}_{M_K})A(\mathbf{1}\otimes \mathrm{O}_{M_K})), \qquad (A \in (\mathfrak{L}(L^2(K))\otimes\mathcal{A}_K)_+).$$

In a similar fashion we define

$$W^\infty_\omega(A) = W^\infty_{\omega_K}((\mathbf{1}\otimes \mathrm{O}_{M_K})A(\mathbf{1}\otimes \mathrm{O}_{M_K})), \qquad (A \in (\mathfrak{L}(\mathcal{M}_K)\otimes\mathcal{A}_K)_+).$$

Again we find

Proposition 5.3. *If they exist $W^!_\omega$ and W^∞_ω are well-defined semifinite locally normal weights.*

Now we want to know to which extent the reduced and the compound weight determine the original state.

Theorem 5.4. *A locally normal state ω is uniquely determined by W^∞_ω.*

Proof. We know that $\mathrm{O}_{M_K}\otimes \mathrm{O}_{M_{K'}}\mathcal{D}^{\mathsf{c}}$ is a bounded operator if $K\cap K' = 0$ and extends to $\mathrm{O}_{M_{K\cup K'}}$. Thus for $A\in\mathcal{A}_K\cap\mathcal{B}$, $B\in\mathcal{A}_{K'}\cap\mathcal{B}$ we get

$$\begin{aligned} &W^\infty_\omega(A\otimes(B\otimes\mathbf{1}_{(K\cup K')^{\mathsf{c}}}) \\ &= W^\infty_{\omega_{K\cup K'}}(A\otimes B) \\ &= \omega_{K\cup K'}(((\mathrm{O}_{M_K}\otimes \mathrm{O}_{M_{K'}})\mathcal{D}^{\mathsf{c}})^* A\otimes B(\mathrm{O}_{M_K}\otimes \mathrm{O}_{M_{K'}})\mathcal{D}^{\mathsf{c}}) \\ &= \omega_{K\cup K'}(A\otimes B). \end{aligned} \tag{5.1}$$

Thus W^∞_ω determines all ω_K and therefore ω. □

For $W^!_\omega$ the question of determination becomes more complicated and we need more notation. Any normal state on some $L^2(G,\nu)$ has a *kernel* $k\in L^2(G^2,\nu^2)$ such that

$$\omega(\mathrm{Pr}_f) = \int \nu^2(\mathrm{d}x,\mathrm{d}y)\overline{f(x)}f(y)k(x,y).$$

We may introduce the same concept also for semifinite normal weights.

Definition 5.5. Suppose η is a semifinite normal weight. We say that $k : G^2 \longmapsto \mathbb{C}$ is *a kernel of* η if it holds for any multiplication operator O_f with $\eta(\mathrm{O}_{|f|^2}) < \infty$ that $k^f(x,y) = \overline{f(x)}k(x,y)f(y)$ is a kernel of the normal functional $A\mapsto \eta(\mathrm{O}_{\overline{f}} A\mathrm{O}_f)$ and for any $Y\in\mathfrak{G}$ there is such an f that $f\mathbf{1}_Y \neq 0$ on a set of ν positive measure.

Lemma 5.6. *Suppose ω is a normal state on $\mathfrak{L}(\mathcal{M})$ with kernel k. Then $W^!_\omega$ and W^∞_ω possess the kernels $k^!$ and k^∞ given by*

$$k^!(x,\varphi_1,y,\varphi_2) = k(\varphi_1+\delta_x,\varphi_2+\delta_y) \tag{5.2}$$

and

$$k^\infty(\varphi_1,\varphi_1',\varphi_2,\varphi_2') = k(\varphi_1+\varphi_1',\varphi_2+\varphi_2'). \tag{5.3}$$

Proof. We have just to look at (3.2), (3.3), (3.4), (3.5) and Proposition 7.1 completes the proof. □

Lemma 5.7. *Suppose that ω is a normal state the density matrix ϱ of which commutes with $\mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}}$. Then $W^!_\omega$ determines ω uniquely.*

Proof. From (5.2) we know that $W^!_\omega$ determines the kernel k of ω on the set $\{(\varphi_1,\varphi_2) : \varphi_1(G), \varphi_2(G) > 0\}$. As the commutator of ϱ with $\mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}}$ vanishes, $k(\mathfrak{o},\varphi)$ and $k(\varphi,\mathfrak{o})$ vanishes for F–a.a. $\varphi \neq \mathfrak{o}$. Finally, $k(\mathfrak{o},\mathfrak{o})$ is uniquely determined as we know $\mathrm{Tr}\varrho = 1$. Consequently, $k^!$ determines k. □

Corollary 5.8. *$W^!_\omega$ determines $(\mathbf{1} - \mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}})\varrho(\mathbf{1} - \mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}})$.*

Corollary 5.9. *Suppose $\omega(\mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}}) = 0$ for a normal state ω. Then $W^!_\omega$ determines ω uniquely.*

Remark 5.10. Please note, that the previous results differ from the point process case, where both $C^!_Q$ and C^∞_Q determine Q uniquely.

We can also draw some conclusions for locally normal states.

Corollary 5.11. *Suppose ω is a gauge-invariant locally normal state. Then $W^!_\omega$ determines ω uniquely.*

Corollary 5.12. *Suppose ω is a locally normal state with $\mathrm{Q}_\omega(\{\mathfrak{o}\}) = 0$. Then for any locally normal state ω' with $W^!_\omega = W^!_{\omega'}$ it holds $\omega = \omega'$.*

Proof. Suppose two normal states ω', ω'' fulfil $W^!_{\omega'} = W^!_{\omega''}$. Then we derive from above that ω', ω'' coincide on $\mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}} A \mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}} + (\mathbf{1} - \mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}})A(\mathbf{1} - \mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}})$ for any $A \in \mathfrak{L}(\mathcal{M})$. Consequently, we obtain for $A = A^*$

$$|\omega'(A) - \omega''(A)| \leq 2\left|\omega'(\mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}} A(\mathbf{1} - \mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}})) - \omega''(\mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}} A(\mathbf{1} - \mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}}))\right|.$$

Using [5, Lemma 2.3.10] we have for $\omega''' = \omega', \omega''$

$$\left|\omega'''(\mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}} A(\mathbf{1} - \mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}}))\right|^2 \leq \|A\|^2 \omega'''(\mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}})\omega'''(\mathbf{1} - \mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}}).$$

Take a monotonously increasing sequence $(K_n)_{n\in\mathbb{N}} \subset \mathfrak{B}$ with $\bigcup_{n\in\mathbb{N}} K_n = G$. Since Q_ω is a measure we obtain from the hypothesis

$$\lim_{n\to\infty} \omega_{K_n}(\mathrm{Pr}_{\mathbf{1}_{\{\mathfrak{o}\}}}) = \lim_{n\to\infty} \mathrm{Q}_\omega(\{\varphi : \varphi(K_n) = 0\}) = \mathrm{Q}_\omega(\{\mathfrak{o}\}) = 0.$$

The above estimate shows for the states ω, ω' from the assumption that

$$\lim_{n\to\infty} \left\|\omega_{K_n} - \omega'_{K_n}\right\| = 0.$$

Thus $\omega = \omega'$. □

There is also a simple connection to the Campbell measures mentioned in the introduction.

Lemma 5.13. *Let ω be a locally normal state and the set Z be such that $\mathrm{O}_Z \in \hat{\mathcal{B}}^!$ or $\mathrm{O}_Z \in \hat{\mathcal{B}}^\infty$. Then*

$$W^!_\omega(\mathrm{O}_Z) = C^!_{\mathrm{Q}_\omega}(Z) \qquad \textit{respectively} \qquad W^\infty_\omega(\mathrm{O}_Z) = C^\infty_{\mathrm{Q}_\omega}(Z).$$

This formula extends to all measurable sets Z with $\mathrm{O}_Z \in \mathfrak{A}^!$ or $\mathrm{O}_Z \in \mathfrak{A}^\infty$ if $W^!_\omega$ or W^∞_ω exist.

6. A Condition (γ_h)

In [26] (see also section 1.6 of the german edition of that work) there are introduced conditions ($g_{\nu,K}$) on a point process Q for $K \in \mathfrak{B}$ and a locally finite measure ν on G via

($g_{\nu,K}$)**:** For all $n \in \mathbb{N}$ and $Y \in \mathfrak{M}(G,K)$ with $Y \subseteq \{\varphi \in M : \varphi(K) = n\}$ it holds

$$Q(Y) = \begin{cases} \delta_{\mathfrak{o}}(Y) & \text{if } \nu(K) = 0 \\ Q(\{\varphi : \varphi(K) = n\})\nu(K)^{-n}\mathcal{F}^{\nu}(Y \cap M_K) \end{cases}.$$

Furthermore, the point process Q is said to obey condition (g_ν) if it obeys ($g_{\nu,K}$) for all $K \in \mathfrak{B}$.

We want to provide in the sequel a similar condition on locally normal states. Thereby we have to replace $N(G)$ by $L^2_{\mathsf{loc}}(G,\nu)$.

Definition 6.1. We say that a locally normal state ω on $\mathfrak{A}$ obeys the *condition* (γ_h) for some $h \in L^2_{\mathsf{loc}}(G,\nu)$ if it fulfils for all $K \in \mathfrak{B}$

($\gamma_{h,K}$)**:** It holds $\omega(\Gamma(\mathrm{Pr}_{\mathbb{C}h_K}) \otimes \mathbf{1}_{\mathcal{M}_{K^\complement}}) = 1$.

Remark 6.2. If we consider $\mathfrak{A}$ to give a model for a quantised electromagnetic field we can think of a state ω with property (γ_h) as a state where only one mode, namely that one related to h, is exited. In [37] such states were shown to be exactly the states showing first order coherence. If such a state had a Poisson process as position distribution it was called state with full coherence. Proposition 6.16 below characterises a subclass of these states.

Naturally, our condition (γ_h) is build in a way to we recover (g_ν) for the position distribution.

Proposition 6.3. *If ω has the property ($\gamma_{h,K}$) respectively (γ_h) then Q_ω has the property ($g_{|h|^2\nu,K}$)respectively ($g_{|h|^2\nu}$).*

We will prove this below but note immediately an easy implication.

Proposition 6.4. *Let ω be a locally normal state fulfilling (γ_h) for some $h \in L^2(G,\nu)$. Then ω is a normal state.*

Proof. We consider the point process Q_ω which must fulfil the condition ($g_{|h|^2\nu}$). As h is ν–square integrable $|h|^2\,\nu$ is a finite measure. Now [26] implies that Q_ω is a finite point process. We conclude from [17] that ω is really a normal state. □

Proposition 6.5. *Fix $h \in L^2_{\mathsf{loc}}(G,\nu)$. Then the locally normal states with property ($\gamma_{h,K}$) respectively (γ_h) form a weak-$*$-closed convex face in the set of all locally normal states.*

Proof. Closedness of ($\gamma_{h,K}$) is obvious. The face property follows from the fact that $\omega(\Gamma(\mathrm{Pr}_{\mathbb{C}h_K}) \otimes \mathbf{1}_{\mathcal{M}_{K^\complement}}) \in [0,1]$ for all states ω on $\mathfrak{A}$. Clearly, the statement for property (γ_h) follows from the fact that the decreasing intersection of closed convex faces is again a closed convex face. □

Proposition 6.6. *Suppose ω is a locally normal state.*

(1) *Assume* $K, K' \in \mathfrak{B}$, $K' \subseteq K$ *are such that* ω *has the property* $(\gamma_{h,K})$. *Then it has also property* $(\gamma_{h,K'})$.
(2) *Suppose* ω *has the properties* (γ_{h,K_n}) *where* $(K_n)_{n\in\mathbb{N}} \subset \mathfrak{B}$ *fulfils* $K_{n+1} \supseteq K_n$ *and* $\mathfrak{B} \ni K = \bigcup_{n\in\mathbb{N}} K_n$. *Then* ω *has also the property* $(\gamma_{h,K})$.

Proof. The first statement follows from $\Gamma(\mathbb{C}h_K) \supset \Gamma(\mathbb{C}h_{K'})$, considered as subspaces of $\mathcal{M}_{K'}$ for $K' \supseteq K$.

The second relation is based on

$$\sigma\text{-w-}\lim_{n\to\infty} \Gamma(\mathrm{Pr}_{\mathbb{C}h_{K_n}}) = \Gamma(\mathrm{Pr}_{\mathbb{C}h_K})$$

as well as σ-weak continuity of ω. □

In the same manner,

Corollary 6.7. *Suppose* ω *is a normal state with property* (γ_h) *where* $\|h\| = 1$. *Then* ω *is given by a trace class operator* ϱ *with* $\Gamma(\mathrm{Pr}_h)\varrho\Gamma(\mathrm{Pr}_h) = \varrho$. *Moreover,*

$$\varrho = \sum_{n,m\in\mathbb{N}} c_{n,m} \mathrm{Pr}_{\sqrt{n!}h^{\otimes n}, \sqrt{m!}h^{\otimes m}}$$

where $(c_{n,m})_{n,m\in\mathbb{N}}$ *is a positive definite matrix with* $\sum_{n\in\mathbb{N}} c_{n,n} = 1$.

Now we have an easy

Proof of Proposition 6.3. We have only to prove the assertion concerning $(\gamma_{h,K})$, fix $K \in \mathfrak{B}$ and $n \in \mathbb{N}$. We derive for any $Y \in \mathfrak{M}(G, K)$, $Y \subset \{\varphi : \varphi(K) = n\}$

$$\begin{aligned} \mathrm{Q}_\omega(Y) &= \omega(\mathrm{O}_Y) = \omega(\mathrm{O}_{Y\cap M_K} \otimes \mathbf{1}_{\mathcal{M}_{K^\complement}}) = c_{n,n} \left\langle (\hat{h}_K)^{\otimes n}, \mathrm{O}_{Y\cap M_K} (\hat{h}_K)^{\otimes n} \right\rangle \\ &= \frac{c_{n,n}}{\|h\|_K^{2n}} \int F^{|h|^2\nu}(\mathrm{d}\varphi) \mathbf{1}_{Y\cap M_K}(\varphi). \end{aligned}$$

This is nothing else than condition $(g_{|h|^2\nu,K})$. □

The structure of locally normal states with the property (γ_h) is more complicated. Consider for $K \in \mathfrak{B}$, $h \in L^2_{\mathsf{loc}}(G,\nu)$ with $h_K \neq 0$ the map $I^h_K : \mathbb{C} \longmapsto L^2(K)$ given by $z \mapsto z \frac{h_K}{\|h\|_K}$. Clearly, $I^h_K (I^h_K)^* = \mathrm{Pr}_{\mathbb{C}h_K}$. It is easy to see in the case $h_K = 0$ that we get the same property by setting $I^h_K z = 0$. From the properties of the operators of second quantisation we derive

Lemma 6.8. *For any* $K \in \mathfrak{B}$, $h \in L^2_{\mathsf{loc}}(G,\nu)$ *there is a one-to-one correspondence between the local states* ω_K *of locally normal states* ω *with property* $(\gamma_{h,K})$ *and normal states* ω^h_K *on* $\Gamma(\mathbb{C})$ *given by*

$$\omega^h_K(A) = \omega_K(\Gamma(I^h_K)A\Gamma((I^h_K)^*) \otimes \mathbf{1}_{K^\complement}).$$

Proof. As

$$\omega^h_K(\mathbf{1}) = \omega_K(\Gamma(I^h_K)\Gamma((I^h_K)^*)) = \omega_K(\Gamma(\mathrm{Pr}_{\mathbb{C}h_K}))$$

we see that ω^h_K is a normal state if and only if ω has the property $(\gamma_{h,K})$. Conversely, $h_K = 0$ if and only if $\omega^h_K = \phi$. In the case $h_K \neq 0$ the map I^h_K is an isometry and therefore ω^h_K characterises ω_K. □

For the next lemma, we need the quasifree maps $\mathcal{P}_\zeta : \mathfrak{L}(\mathcal{M}) \longmapsto \mathfrak{L}(\mathcal{M})$ for all $\zeta \in \mathbb{C}$, $|\zeta| \leq 1$. They are given by

$$\mathcal{P}_\zeta(A) = \mathcal{V}^*_{\zeta,\sqrt{1-|\zeta|^2}} A \otimes \mathbf{1} \mathcal{V}^*_{\zeta,\sqrt{1-|\zeta|^2}}, \qquad (A \in \mathfrak{L}(\mathcal{M})).$$

Lemma 6.9. *Fix $K, K' \in \mathfrak{B}$, $K \subseteq K'$ and $h \in L^2_{\mathsf{loc}}(G,\nu)$, $h_{K'} \neq 0$. For any locally normal states ω with property $(\gamma_{h,K'})$ it holds*

$$\omega^h_K = \omega^h_{K'} \circ \mathcal{P}_{\|h\|_K / \|h\|_{K'}}.$$

Proof. It is enough to prove this for the coherent states $\omega = \phi^{zh}$, $z \in \mathbb{C}$. Then $\omega^h_{K'} = \phi^{z\|h_{K'}\|}$ and $\omega^h_K = \phi^{z\|h_K\|}$. Since we know that

$$\phi^g \circ (\mathcal{V}^*_{\zeta,\xi} \cdot \mathcal{V}_{\zeta,\xi}) = \phi^{\zeta g} \otimes \phi^{\xi g},$$

the definition of the quasifree map $\mathcal{P}_{\|h\|_K / \|h\|_{K'}}$ completes the proof. □

As one consequence we derive

Proposition 6.10. *Assume $h \in L^2_{\mathsf{loc}}(G,\nu) \setminus L^2(G,\nu)$ and ω is a locally normal state on $\mathfrak{A}$ with property (γ_h). Then there is a unique measure σ_ω on $\mathbb{C}$ such that*

$$\omega = \int \sigma_\omega(\mathrm{d}z)\phi^{zh}.$$

Proof. We fix some $K \in \mathfrak{B}$ such that $h_K \neq 0$ and consider a sequence $(K_n)_{n\in\mathbb{N}} \subset \mathfrak{B}$, such that $K_n \supseteq K$ and $\|h\|_{K_n} \to_{n\to\infty} \infty$. From Lemma 6.9 and Theorem 4.5 of [15] it follows that ω^h_K is a mixture of coherent states over $\Gamma(\mathbb{C})$. Therefore, there is a measure σ_K on $\mathbb{C}$ such that

$$\omega^h_K = \int \sigma_K(\mathrm{d}z)\phi^z$$

or

$$\omega_K = \int \sigma_K(\mathrm{d}z)\phi^{\frac{z}{\|h\|_K} h_K}.$$

If we denote $\sigma^K = \sigma_K \circ (\|h\|_K)^{-1}$ we see

$$\omega_K = \int \sigma^K(\mathrm{d}z)\phi^{zh_K}.$$

Suppose $K' \supseteq K \in \mathfrak{B}$. In view of $\left(\phi^{zh_{K'}}\right)_K = \phi^{zh_K}$ we derive $\int \sigma^{K'}(\mathrm{d}z)\phi^{zh_K} = \int \sigma^K(\mathrm{d}z)\phi^{zh_K}$. Now uniqueness of the measure representing a mixture of coherent states, see again Theorem 4.5 of [15], implies that $\sigma = \sigma^K$ is independent of $K \in \mathfrak{B}$ and

$$\omega = \int \sigma(\mathrm{d}z)\phi^{zh}.$$

□

Example 6.11. The state of the condensate of an ideal Bose gas as described in [6], Theorem 5.2.32, or [15], Example 4.13., has the property $(\gamma_{\mathbf{1}_{\mathbb{R}^d}})$. Clearly, $\ell^d(\mathbb{R}^d) = \infty$ implies that it is not a normal state. The corresponding measure σ_ω is the rotation invariant Gaussian measure with variance $(\varrho - \varrho_c)c_\sigma$. The same way, the condensate state of [4] has the property $(\gamma_{\mathbf{1}_{\mathbb{R}^d}})$ with σ_ω being the rotation invariant measure concentrated on $\bar{\varrho}S^1$.

Now we want to use the reduced and the compound weight for characterising the property (γ_h) and coherent states. We follow the lines of [26]. To any $h \in L^2_{\mathsf{loc}}(G,\nu)$ we may associate the weights $\tau^!_h$ on $\bigcup_{K\in\mathfrak{B}} \mathfrak{L}(L^2(K))$ respectively τ^∞_h on $\bigcup_{K\in\mathfrak{B}} \mathcal{A}_K$, where

$$\tau^!_h(A) = \langle h_K, Ah_K\rangle, \qquad (A \in \mathfrak{L}(L^2(K))),$$
$$\tau^\infty_h(A) = \langle \psi_{h_K}, A\psi_{h_K}\rangle, \qquad (A \in \mathcal{A}_K).$$

Proposition 6.12. *A locally normal state ω has property (γ_h) for some $h \in L^2_{\mathsf{loc}}(G,\nu)$ if and only if $W^!_\omega$ exists and there is a semifinite locally normal weight $\widetilde{\omega}$ on $\mathfrak{A}$ such that*

$$W^!_\omega = \tau^!_h \otimes \widetilde{\omega}.$$

Proof. Clearly, it is only necessary to prove this in the case of a normal state and $h \in \mathcal{H}$. First we consider the *only if* part. According to section 3 we may restrict to the case $\mathcal{H} = L^2(\mathbb{N}, \#)$ and $h = c\mathbf{1}_{\{0\}}$. From Lemma 5.6 we know that $W^!_\omega$ has the kernel

$$k^!(x,\varphi_1,y,\varphi_2) = k(\varphi_1 + \delta_x, \varphi_2 + \delta_y).$$

We can conclude from Lemma 5.13 and the σ-finiteness of $C^!_{\mathrm{Q}_{\widetilde{\omega}}}$ that $\widetilde{\omega}$ has a kernel $\tilde{k}$. The assumption implies

$$k(\varphi_1 + \delta_x, \varphi_2 + \delta_y) = |c|^2\, \mathbf{1}_{\{0\}}(x)\mathbf{1}_{\{0\}}(y)\tilde{k}(\varphi_1,\varphi_2). \tag{6.1}$$

We conclude for $\varphi_1(\mathbb{N}) > 0$ and $\varphi_2 \neq 0$ that $k(\varphi_1,\varphi_2) = 0$. Especially $k(\varphi_1,\varphi_1) = 0$ which implies by the Cauchy Schwarz inequality $k(\varphi_1,\varphi_3) = 0$ for all $\varphi_3 \in M$, $\varphi_3(\mathbb{N}) < \infty$. In the same manner we derive $k(\varphi_3,\varphi_1) = 0$. This implies $\varrho = \Gamma(\mathrm{O}_{\{0\}})\varrho\Gamma(\mathrm{O}_{\{0\}})$ for the density matrix of ω, thus we get property (γ_h).

The *if* part follows easily from this lemma and the description of the kernel of $W^!_\omega$ given by (6.1). □

Proposition 6.13. *A locally normal state ω is the coherent state ϕ^h for some $h \in L^2_{\mathsf{loc}}(G,\nu)$ if and only if W^∞_ω exists and*

$$W^\infty_\omega = \tau^\infty_h \otimes \omega.$$

Proof. By the above proposition, we may assume that ω is a normal state on $\mathfrak{L}(\mathcal{M}(\{0\}))$ and $h = c\mathbf{1}_{\{0\}}$. We get again by using kernels for all $m, m, n, n' \in \mathbb{N}$

$$k((m+m')\delta_0, (n+n')\delta_0) = \overline{c}^m c^n k(m'\delta_0, n'\delta_0).$$

Immediately

$$k(m\delta_0, n\delta_0) = \overline{c}^m c^n k(\mathfrak{o},\mathfrak{o}).$$

Clearly, this implies $\omega = \phi^{c\mathbf{1}_{\{0\}}} = \phi^h$. □

This technique can be considerably strengthened.

Corollary 6.14. *A locally normal state ω is the coherent state ϕ^h for a fixed $h \in L^2_{\mathsf{loc}}(G,\nu)$ if and only if there is some semifinite locally normal weight $\widetilde{\omega}$ on $\mathfrak{A}$ such that*

$$W^\infty_\omega = \tau^\infty_h \otimes \widetilde{\omega}.$$

Proposition 6.15. *A locally normal state ω is a coherent state if and only if W_ω^∞ exists and there are some semifinite locally normal weights $\widetilde{\omega}$ on $\mathfrak{A}$ and ω' on $\bigcup_{K\in\mathfrak{B}}\mathcal{A}_K$ such that*

$$W_\omega^\infty = \omega' \otimes \widetilde{\omega}.$$

Proof. We may restrict to the case of a normal state. Now the assertion follows easily from Corollary 4.9 and Theorem 3.4 of [15]. □

In the case of the reduced weight $W_\omega^!$ we observe the same deficiency as for the characterisation of locally normal states: We cannot distinguish states related by a gauge transformation.

Proposition 6.16. *A locally normal state ω possesses the reduced weight $W_\omega^!$ and fulfils for some $h \in L^2_{\mathsf{loc}}(G,\nu)$ the equation*

$$W_\omega^! = \tau_h^! \otimes \omega$$

if and only if there exists a Borel probability measure μ on $[0,2\pi)$ such that

$$\omega = \int\limits_{[0,2\pi)} \mu(\mathrm{d}t)\phi^{\mathrm{e}^{\mathrm{i}t}h}. \tag{6.2}$$

Proof. Again, we look for the kernel k of ω in the case $G = \{0\}$. We see

$$k((m+1)\delta_0,(n+1)\delta_0) = |c|^2\, k(m\delta_0, n\delta_0).$$

We define the function $f : \mathbb{Z} \longmapsto \mathbb{C}$ by

$$f(l) = \begin{cases} k(l\delta_0, \mathfrak{o}) & \text{if } l \geq 0, \\ k(\mathfrak{o}, |l|\,\delta_0) & \text{if } l < 0. \end{cases}$$

Then by induction

$$k(m\delta_0, n\delta_0) = |c|^{2|m-n|}\, f(m-n).$$

We know that k is a positive definite kernel on M. Therefore, $(m,n) \mapsto f(m-n)$ is a positive definite too. Thus f is a positive definite function on $\mathbb{Z}$. By the classical Bochner-Khintchine theorem [34] f is the Fourier transform of a finite measure on $[0,2\pi)$, choose this measure to be $k(\mathfrak{o},\mathfrak{o})\mu$. (Observe that $k(\mathfrak{o},\mathfrak{o}) = 0$ implies $k(\varphi,\varphi) = 0$ for all $\varphi \in M$ or by positive definiteness $k = 0$.) Clearly, μ is a probability measure and (6.2) is valid. □

Remark 6.17. By equation (6.2) we can construct a lot of states having full coherence in the sense of [37]. We want to stress the fact that there are much more states with this property which are not of the form (6.2).

7. Building Weights from Positive Operators

Proposition 7.1. *η_Q is a semifinite normal weight. Moreover, if R is some bounded operator such that RQR^* extends to the trace class operator $\overline{RQR^*}$ then for all $A \in \mathfrak{L}(\mathcal{H})_+$ it holds $\eta_Q(R^*AR) < \infty$ and*

$$\dot{\eta}_Q(R^*AR) = \mathrm{Tr}(\overline{RQR^*}A), \qquad (A \in \mathfrak{L}(\mathcal{H})).$$

Proof. Suppose that RQR^* extends to a trace-class operator. Thus $R\sqrt{Q}$ extends to a Hilbert-Schmidt operator, let denote $\|R\sqrt{Q}\|_{HS}$ its Hilbert-Schmidt norm. Then we obtain for any $A \in \mathfrak{L}(\mathcal{H})_+$ and some complete orthonormal system $(e_i)_{i\in\mathbb{N}}$, $e_i \in \mathrm{dom}(\sqrt{Q})$

$$\begin{aligned}\eta_Q(R^*AR) &= \sum_{i\in\mathbb{N}} \langle \sqrt{Q}e_i, R^*AR\sqrt{Q}e_i\rangle = \sum_{i\in\mathbb{N}} \langle R\sqrt{Q}e_i, AR\sqrt{Q}e_i\rangle \\ &\leq \|A\| \sum_{i\in\mathbb{N}} \langle R\sqrt{Q}e_i, R\sqrt{Q}e_i\rangle = \|A\| \, \|R\sqrt{Q}\|_{HS} \,.\end{aligned}$$

This means $\eta_Q(R^*AR) < \infty$. Without restricting generality we may suppose that $R\sqrt{Q}$ is positive and $(e_i)_{i\in\mathbb{N}}$ is a system of eigenvectors for the eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$. Then we get

$$\begin{aligned}\eta_Q(R^*AR) &= \sum_{i\in\mathbb{N}} \lambda_i^2 \langle e_i, Ae_i\rangle = \sum_{i\in\mathbb{N}} \left\langle |R\sqrt{Q}|^2 \, e_i, Ae_i \right\rangle \\ &= \sum_{i\in\mathbb{N}} \left\langle |\sqrt{Q}R|^2 \, e_i, Ae_i \right\rangle = \sum_{i\in\mathbb{N}} \langle R^*QRe_i, Ae_i\rangle = \mathrm{Tr}(RQR^*A)\end{aligned}$$

and the formula for η_Q is proven. It remains to show that η_Q is semifinite. If P is some eigenprojection of Q related to a compact interval $[0, a]$, surely PQ is a bounded operator. Then for a finite rank projection P' the choice $R = P'P$ gives an operator such that RQR^* extends to a finite rank, i.e. trace class operator. Such a way all operators of the the kind $PP'AP'P$ with arbitrary $A \in \mathfrak{L}(\mathcal{H})$ are from the set $\{A \in \mathcal{A}_+ : \eta_Q(A) < \infty\}$. Clearly, the linear hull of this set is σ-weakly dense in $\mathfrak{L}(\mathcal{H})$. □

Lemma 7.2. *Let η be a semifinite normal weight on $\mathfrak{L}(\mathcal{H})$.*

Then the quadratic form q_η,

$$q_\eta(f) = \eta(\mathrm{Pr}_{f,f}),$$

on $\mathrm{dom}(q_\eta) = \{f \in \mathcal{H} : \eta(\mathrm{Pr}_{f,f}) < \infty\}$*, is densely defined.*

Moreover, if $\mathrm{dom}(q_\eta) = \mathcal{H}$*, it holds $\eta = \eta_Q$ for some $Q \in \mathfrak{L}(\mathcal{H})_+$. Further, $\eta(\mathbf{1}) < \infty$ if and only if Q is of trace-class.*

Proof. Let be $H = \mathrm{span}\{f : \eta(\mathrm{Pr}_{f,f}) < \infty\}^\perp$ and take $g \in H$, $\|g\| = 1$. For any $A \in \mathfrak{L}(\mathcal{H})_+$ with $\eta(A) < \infty$ we get $\eta(\sqrt{A}\mathrm{Pr}_g\sqrt{A}) < \infty$. But

$$\sqrt{A}\mathrm{Pr}_g\sqrt{A} = \mathrm{Pr}_g\sqrt{A}\mathrm{Pr}_g\sqrt{A}\mathrm{Pr}_g = \|\sqrt[4]{A}g\|^4 \, \mathrm{Pr}_g$$

and $\eta(\mathrm{Pr}_g) = \infty$ implies $\sqrt[4]{A}g = 0$. Therefore we get $A\mathrm{Pr}_H = 0 = \mathrm{Pr}_H A$ for any $A \in L^1(\eta)$. σ-weak denseness of $L^1(\eta)$ implies $\mathrm{Pr}_H = 0$ or $H = \{0\}$. Therefore, $\mathrm{dom}(q_\eta)$ is dense in $\mathcal{H}$ and q_η is clearly a positive quadratic form on this domain.

Assume this quadratic form is defined on the whole $\mathcal{H}$, without restricting generality it should hold $q_\eta(f) \geq \|f\|^2$. Then $\mathcal{H}$ equipped with the inner product corresponding to q_η is again a Hilbert space and the identity mapping onto $\mathcal{H}$ is continuous. By the open mapping theorem, its inverse is continuous and therefore bounded. Hence q_η is bounded and there exists $Q \in \mathfrak{L}(\mathcal{H})$ with $\eta = \eta_Q$. The assertion about trace-class is immediate. □

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