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GENERALIZED CAUCHY-STIELTJES TRANSFORMS OF SOME BETA DISTRIBUTIONS

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Abstract. We express generalized Cauchy-Stieltjes transforms (GCST) of some particular beta distributions depending on a positive parameter λ as λ-powered Cauchy-Stieltjes transforms (CST) of some probability measures. The CST of the latter measures are shown to be the geometric mean of the CST of the Wigner law together with another one. Moreover, these measures are shown to be absolutely continuous and we derive their densities by proving that they are the so-called Markov transforms of compactly-supported probability distributions. Finally, a detailed analysis is performed on one of the symmetric Markov transforms which interpolates between the Wigner ($\lambda = \infty$) and the arcsine ($\lambda = 1$) distributions. We first write down its moments through a terminating series ${}_3F_2$ and show that they are polynomials in the variable $1/\lambda$, however they are no longer positive integer-valued as the are for $\lambda = 1, \infty$ (for instance $\lambda = 2$). Second, we compute the free cumulants in the case when $\lambda = 2$ and explain how to proceed in the cases when $\lambda = 3, 4$. Problems of finding a deformation of the representation theory of the infinite symmetric group and an interpolating convolution are discussed.

1. Motivation

Let $\lambda > 0$ and μ_λ a probability measure (possibly depending on λ) with finite all order moments. The *generalized Cauchy-Stieltjes transform* (GCST) of μ_λ is defined by

$$\int_{\mathbb{R}} \frac{1}{(z-x)^\lambda} \mu_\lambda(dx)$$

for nonreal complex z lying in a suitable branch [8, 17, 20]. For $\lambda = 1$, it reduces to the (ordinary) *Cauchy-Stieltjes transform* (CST) which has been of great importance during the two last decades for both probabilists and algebraists due the central role it plays in free probability and representation theories [1, 10]. Moreover, CST were extensively studied and they are handable in the sense that for instance, a complete characterization of those functions is known and one has a relatively easy inversion formula due to Stieltjes [10]. However, their generalized versions are more hard to handle as one may realize from the complicated inversion formulas displayed in [8, 17, 20]. In this paper, we adress the problem of relating

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both generalized and ordinary transforms, that is, given μ_λ check whether there exists a **probability** measure ν_λ such that

$$\int_{\mathbb{R}} \frac{1}{(z-x)^\lambda}\mu_\lambda(dx) = \left[\int_{\mathbb{R}} \frac{1}{z-x}\nu_\lambda(dx)\right]^\lambda \tag{1.1}$$

for z in some suitable complex region (we shall consider the principal determination) and characterize ν_λ in the affirmative case. Doing so will lead for instance to the invertibility of

$$z \mapsto \left[\int_{\mathbb{R}} \frac{1}{(z-x)^\lambda}\mu_\lambda(dx)\right]^{1/\lambda}$$

for z belonging to some neighborhood of infinity [2] and as a by-product to a kind of λ-free cumulants generating function for μ_λ, referring to the case $\lambda = 1$ [19]. However, this may not be always possible as we shall see, that is, ν_λ may not be a probability measure for some λ. Moreover, when ν_λ is shown to be a probability measure, it is not easy to check whether it is absolutely continuous or not and more harder will be to write down its density when it is so. This was behind our willing to get an insight into the above problem through GCST a particular class of probability measures μ_λ [7]. The latter probability measures may be mapped via affine transformations into beta distributions

$$\beta_{a,b}(dx) = (2-x)^a(2+x)^b\mathbf{1}_{[-2,2]}(x)dx$$

with parameters $a, b > -1$ depending on λ, and their monic orthogonal polynomials, say $(P_n^\lambda)_n$, are the only ones (up to a conjecture, see [7]) that admit ultraspherical-type generating functions:

$$\sum_{n\geq 0} \frac{(\lambda)_n}{n!}P_n^\lambda(x)z^n = \frac{1}{u_\lambda(z)(f_\lambda(z)-x)^\lambda}, \quad x \in \mathrm{supp}(\mu_\lambda), \tag{1.2}$$

where u_λ, f_λ satisfy some technical conditions in a suitable open complex region near $z = 0$ so that (1.2) makes sense. Due to the orthogonality of $P_n^\lambda, n \geq 0$, one gets after integrating both sides in (1.2) with respect to μ_λ (this is a special case of the multiplicative renormalization, [15]):

$$u_\lambda(z) = \int_{\mathbb{R}} \frac{1}{(f_\lambda(z)-x)^\lambda}\mu_\lambda(dx),$$

for small z. Thus, if f_λ is invertible, the last equality is equivalent to

$$u_\lambda[f_\lambda^{-1}(z)] = \int_{\mathbb{R}} \frac{1}{(z-x)^\lambda}\mu_\lambda(dx)$$

so that $u_\lambda(f_\lambda^{-1})$ is the GCST of μ_λ. Fortunately, as one easily sees from the reminder below, f_λ^{-1} may be identified with the CST (in some neighborhood of infinity) of a semi-circle law of mean and variance depending on λ.

2. Reminder and Results

Assume μ_λ has zero mean and unit variance, then μ_λ has a generating function for orthogonal polynomials of ultraspherical-type if and only if it is compactly-supported and belongs to one of the four families corresponding to[1] [7]:

$$\begin{aligned} u_\lambda(z) &= z^\lambda, & f_\lambda(z) &= \frac{\lambda+1}{2}z + \frac{1}{z}, \lambda > 0, \\ u_\lambda(z) &= \frac{z^\lambda}{1-(\lambda/2)z^2}, & f_\lambda(z) &= \frac{\lambda}{2}z + \frac{1}{z}, \lambda > 1/2 \\ u_\lambda(z) &= \frac{z^\lambda}{1 \pm \lambda z/\sqrt{2\lambda-1}}, & f_\lambda(z) &= \frac{\lambda^2}{2\lambda-1}z \pm \frac{1}{\sqrt{2\lambda-1}} + \frac{1}{z}, \lambda > 1/2. \end{aligned}$$

We did not displayed the density of μ_λ for sake of clarity and we will do it later. Nevertheless, it is worth recalling that the first family corresponds to the monic Gegenbauer polynomials $(C_n^\lambda)_n$, the second one corresponds to $(C_n^{\lambda-1})_n$ and is related to the Poisson kernel, while the remaining families correspond to shifted Jacobi polynomials whose paremeters differ by 1. Now, let $G_{m,\sigma}$ be the CST of a semi-circular law of mean m and variance σ [4], then its inverse in composition's sense, say $K_{m,\sigma}$, is given by [4]

$$K_{m,\sigma}(z) := G_{m,\sigma}^{-1}(z) = \sigma^2 z + m + \frac{1}{z}. \tag{2.1}$$

Hence, the reader can check that f_λ fits $K_{m,\sigma}$ for particular means and variances depending on λ, so that f_λ^{-1} fits $G_{m,\lambda}$. Nevertheless, we will only make use of $G := G_{0,1}$ since the elemantary identity holds

$$G_{m,\sigma}(z) = \frac{1}{\sigma} G\left(\frac{x-m}{\sigma}\right).$$

Accordingly, our first main result may be stated for sufficiently large z as

$$\left[\int_{\mathbb{R}} \frac{1}{(z-x)^\lambda}\beta_{a,b}(dx)\right]^{1/\lambda} = [\tilde{u}_\lambda(G(z))]^{1/\lambda} = G^{\alpha(\lambda)}(z)\tilde{G}(z)^{\gamma(\lambda)}, \tag{2.2}$$

where $\alpha(\lambda) + \gamma(\lambda) = 1$ and

$$\begin{aligned} \tilde{u}_\lambda(z) &= z^\lambda, & \alpha(\lambda) &= 1, \\ \tilde{u}_\lambda(z) &= \frac{z^\lambda}{1-z^2}, & \alpha(\lambda) &= 1 - \frac{1}{\lambda}, \tilde{G}(z) = \frac{1}{\sqrt{z^2-4}}, \\ \tilde{u}_\lambda(z) &= \frac{z^\lambda}{1 \pm z}, & \alpha(\lambda) &= 1 - \frac{1}{2\lambda}, \tilde{G}(z) = \frac{1}{z \pm 2} \end{aligned}$$

for the four families respectively. It follows from the Nevannlina's characterization of CST of probability measures that

$$z \mapsto \left[\int_{\mathbb{R}} \frac{1}{(z-x)^\lambda}\beta_{a,b}(dx)\right]^{1/\lambda}$$

defines a CST of some probability measure ν_λ for λ provided that $0 \leq \alpha(\lambda) = 1 - \gamma(\lambda) \leq 1$. Note that under this condition, the CST of ν_λ is the geometric mean of $G, \tilde{G}$ and that one discards the values $\lambda \in]1/2, 1[$ for the second family

[1] The third and fourth families correspond to the plus and minus signs respectively.

for which $\tilde{G}$ is the CST of the arcsine distribution [19].
The second main result shows that, if $0 \leq \alpha(\lambda) = 1 - \gamma(\lambda) \leq 1$, then ν_λ are absolutely continuous probability measures and gives their density. This follows from the fact that ν_λ is the so-called Markov transform of some compactly-supported probability measure τ_λ [10], that is

$$\int_{\mathbb{R}} \frac{1}{z-x}\nu_\lambda(dx) = \exp - \int_{\mathbb{R}} \log(z-x)\tau_\lambda(dx), \tag{2.3}$$

and from Cifarelli and Regazzini's results (see [10] p. 51). For the first and the second families, ν_λ is given by the Wigner distribution and a symmetric deformation of it respectively, while for the remaining ones, it is a nonsymmetric deformation of the Wigner distribution. Since the latter is a universal limiting object (representation theory of the infinite symmetric group, spectral theory of large random matrices, free probability theory), we give a particular interest in ν_λ corresponding to the second family. We first express its moments by means of a terminating ${}_3F_2$ series interpolating between the moments of the Wigner and the arcsine distributions (Catalan and shifted Catalan numbers, [10] p.64). Unfortunately, the moments are no longer positive integer-valued as it is shown for $\lambda = 2$. Nevertheless, they are polynomials in the variable $1/\lambda$. Finally, we use (2.2) to compute the inverse of its CST and the free cumulants generating function in the case $\lambda = 2$ which involve a weighted sum of the Catalan and the shifted Catalan numbers. For $\lambda = 3, 4$, this is a more complicated task and by Galois theory, this is a limitation rather than a restriction since we are led to find a root of a polynomial equation of degree λ.

Remarks 2.1. (1) For the second family μ_λ and for the discarded values of λ such that the condition $0 \leq \alpha(\lambda) = 1 - \gamma(\lambda) \leq 1$ fails to hold,

$$z \mapsto \left[\int_{\mathbb{R}} \frac{1}{(z-x)^\lambda}\mu_\lambda(dx)\right]^{1/\lambda}$$

does not define the CST of a probability distribution.

(2) It follows from (1.1) and (2.3) that

$$\int_{\mathbb{R}} \frac{1}{(z-x)^\lambda}\mu_\lambda(dx) = \exp -\lambda \int_{\mathbb{R}} \log(z-x)\tau_\lambda(dx) = \left[\int_{\mathbb{R}} \frac{1}{z-x}\nu_\lambda(dx)\right]^\lambda.$$

Similar identities already showed up in relation to Bayesian statistics (see [10] p. 59) and in relation to the so-called GGC random variables and Dirichlet means [9].

Throughout the paper, computations are performed up to constants depending on λ, which normalize the finite positive measures involved here to be probability measures. The paper is divided into five sections: the first four sections are devoted to the four families μ_λ while the last one is devoted to the particular interest we give in the probability measure ν_λ corresponding to the second family.

3. Markov Transforms: Symmetric Measures

3.1. First family. On the one hand,

$$\mu_\lambda(dx) \propto \left(1 - \frac{x^2}{2(1+\lambda)}\right)^{\lambda-1/2} \mathbf{1}_{[\pm\sqrt{2(1+\lambda)}]}(x)dx, \lambda > 0,$$

and its image of under the map $x \mapsto \sqrt{(1+\lambda)/2}x$ has the density proportional to

$$\left(4 - x^2\right)^{\lambda-1/2} \mathbf{1}_{[-2,2]}(x).$$

On the other hand, it is easy to see that

$$f_\lambda(z) = \frac{\lambda+1}{2}z + \frac{1}{z} = K_{0,\sqrt{(1+\lambda)/2}}(z)$$

so that

$$f_\lambda^{-1}(z) = G_{0,\sqrt{(1+\lambda)/2}}(z) = \sqrt{\frac{2}{1+\lambda}}G\left(\sqrt{\frac{2}{1+\lambda}}z\right).$$

It follows that

$$\int_{-2}^{2} \frac{1}{(z-x)^\lambda} \left(4 - x^2\right)^{\lambda-1/2} dx \propto= [G(z)]^\lambda = \left[\int_{-2}^{2} \frac{1}{z-x}\sqrt{4-x^2}\frac{dx}{2\pi}\right]^\lambda.$$

Using the fact that the Wigner distribution is the Markov transform of the arcsine distribution (see [10] p.64), one finally gets

Proposition 3.1. *For $\lambda > 0$,*

$$\int_{\mathbb{R}} \frac{1}{(z-x)^\lambda}\mu_\lambda(dx) = \exp -\lambda \int_{\mathbb{R}} \log(z-x)\tau_\lambda(dx) = \left[\int_{\mathbb{R}} \frac{1}{z-x}\nu_\lambda(dx)\right]^\lambda. \quad (3.1)$$

where

$$\begin{aligned}
\mu_\lambda(dx) &\propto \left(4 - x^2\right)^{\lambda-1/2} \mathbf{1}_{[-2,2]}dx, \\
\nu_\lambda(dx) &= \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}dx, \\
\tau_\lambda(dx) &= \frac{1}{\pi\sqrt{4-x^2}}\mathbf{1}_{[-2,2]}dx.
\end{aligned}$$

Remark 3.2. The value $\lambda = 1/2$ corresponds to the uniform measure on $[-2, 2]$. Recently, it was shown that this measure is the only nonatomic probability measure with a square root-type generating funtion [16].

3.2. Second family. The density of μ_λ reads

$$\mu_\lambda(dx) \propto \left(1 - \frac{x^2}{2\lambda}\right)^{\lambda-3/2} \mathbf{1}_{[-\sqrt{2\lambda},\sqrt{2\lambda}]}(x)dx, \quad \lambda > 1/2,$$

and we map it using $x \mapsto \sqrt{\lambda/2}x$ to

$$\left(4 - x^2\right)^{\lambda-3/2} \mathbf{1}_{[-2,2]}(x)dx, \quad \lambda > 1/2.$$

Now,

$$f_\lambda^{-1}(z) = \sqrt{\frac{2}{\lambda}}G\left(\sqrt{\frac{2}{\lambda}}z\right)$$

so that

$$\int_{-2}^{2} \frac{1}{(z-x)^{\lambda}}\left(4-x^2\right)^{\lambda-3/2} dx \propto \frac{G^{\lambda}(z)}{1-G^2(z)} = \tilde{u}_{\lambda}(G)(z).$$

Using

$$G(z) = \int_{-2}^{2} \frac{1}{z-x}\sqrt{4-x^2}\frac{dx}{2\pi} = \frac{z-\sqrt{z^2-4}}{2} = \frac{2}{z+\sqrt{z^2-4}},$$

for $z \in \mathbb{C} \setminus [-2,2]$ together with $G^2(z)+1 = zG(z)$, then

$$1-G^2(z) = 2 - zG(z) = 2 - \frac{2z}{z+\sqrt{z^2-4}} = \sqrt{z^2-4}G(z).$$

But

$$\frac{1}{\pi}\int_{-2}^{2} \frac{1}{z-x}\frac{dx}{\sqrt{4-x^2}} = \frac{1}{\sqrt{z^2-4}} = \exp -\frac{1}{2}\log[(z-2)(z+2)]$$

for suitable z, therefore

Proposition 3.3.

$$\int_{-2}^{2} \frac{1}{(z-x)^{\lambda}}\left(4-x^2\right)^{\lambda-3/2} dx \propto \frac{G^{\lambda-1}(z)}{\sqrt{z^2-4}} = G^{\lambda-1}(z)G_{\text{arcsin}}(z)$$
$$= \exp -\lambda \int_{\mathbb{R}} \log(z-x)\tau_{\lambda}(dx)$$

where

$$\tau_{\lambda}(dx) = \left(1-\frac{1}{\lambda}\right)\frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}\mathbf{1}_{[-2,2]}(dx) + \frac{1}{\lambda}\frac{\delta_{-2}+\delta_2}{2}(dx),$$

which is not a probability measure unless $\lambda \geq 1$.

Next, let us seek $\nu_{\lambda}, \lambda \geq 1$ such that

$$\exp -\int_{\mathbb{R}} \log(z-x)\tau_{\lambda}(dx) = \int \frac{1}{z-x}\nu_{\lambda}(dx).$$

In fact,

$$T_{\lambda}(z) := G^{1-1/\lambda}(z)G_{\text{arcsin}}^{1/\lambda}(z), \quad z \in \mathbb{C} \setminus [-2,2],$$

defines the CST of a probability measure and this claim is readily checked using Lemma II. 2.2 in [18]. More precisely, one has for $\Im(z) > 0$

$$\arg[G^{1-1/\lambda}(z)G_{\text{arcsin}}^{1/\lambda}(z)] = \left(1-\frac{1}{\lambda}\right)\arg[G(z)] + \frac{1}{\lambda}\arg[G_{\text{arcsin}}(z)] \in]-\pi, 0[$$

so that T_{λ} is of imaginary-type (maps the upper half-plane into the lower half-plane), and

$$\lim_{y\to\infty} iyT_{\lambda}(iy) = \lim_{y\to\infty}[iyG(iy)]^{1-1/\lambda}[iyG_{\text{arcsin}}(iy)]^{1/\lambda}.$$

For $1/2 < \lambda < 1$, one easily gets the inequality

$$2\arg[G_{\text{arcsin}}(z)] < \arg[T_{\lambda}(z)] < \arg[G_{\text{arcsin}}(z)] - \arg[G(z)]$$

and there is no guarantee for ν_{λ} to be a probability distribution. In order to characterize $\nu_{\lambda}, \lambda \geq 1$, the Markov transform of τ_{λ}, we will use results by Cifarelly and Regazzini [10]. In fact, since τ_λ is a compactly-supported probability measure, then ν_λ is absolutely continuous with density proportional to

$$\sin(\pi F_\lambda(x)) \exp - \int_{-2}^{2} \log|x-u| \tau_\lambda(du), \tag{3.2}$$

where

$$\begin{aligned} \pi F_\lambda(x) &:= \tau_\lambda(]-\infty, x]) \\ &= \begin{cases} 0, & \text{if} \quad x < -2, \\ \pi/(2\lambda), & \text{if} \quad x = -2, \\ (1-1/\lambda)[\arcsin(x/2) + \pi/2], & \text{if} \quad x \in [-2, 2[, \\ \pi, & \text{if} \quad x \geq 2. \end{cases} \end{aligned}$$

Note that since (3.2) is valid when τ_λ is the arcsine distribution and ν_λ is the Wigner distribution, one deduces that

$$\exp - \int_{-2}^{2} \log|x-u| \frac{1}{\pi} \frac{1}{\sqrt{4-u^2}} du, \quad x \in [-2, 2],$$

does not depend on x (is constant). This striking result may be used to derive the density of ν_λ in our case and in the forthcoming ones since τ_λ is a convex linear combination of the arcsine distribution and a discrete probability measure. In the case in hand, easy computations yield

Proposition 3.4.

$$\frac{\nu_\lambda(dx)}{dx} \propto \cos\left[\left(1 - \frac{1}{\lambda}\right) \arcsin\frac{x}{2}\right] \frac{1}{(4-x^2)^{1/2\lambda}} \mathbf{1}_{]-2,2[}(x), \lambda \geq 1.$$

Note that $\lambda = 1$ corresponds to the arcsine distribution while $\lambda = \infty$ correponds to the Wigner distribution. Thus, ν_λ interpolates between them. The reader may wonder how to compute the normalizing constant or the moments of ν_λ. This will be clear after dealing with the two remaining families μ_λ.

4. Markov Transforms: Nonsymmetric Measures

4.1. Third family. The probability distribution μ_λ has the density

$$\left(1 - \frac{\sqrt{2\lambda-1}x - 1}{2\lambda}\right)^{\lambda-1/2} \left(1 + \frac{\sqrt{2\lambda-1}x - 1}{2\lambda}\right)^{\lambda-3/2},$$

where

$$x \in \left[\frac{1-2\lambda}{\sqrt{2\lambda-1}}, \frac{1+2\lambda}{\sqrt{2\lambda-1}}\right], \quad \lambda > 1/2.$$

Moreover

$$f_\lambda(z) = \frac{\lambda^2}{2\lambda-1} z + \frac{1}{\sqrt{2\lambda-1}} + \frac{1}{z} = K_{1/\sqrt{2\lambda-1}, \lambda/\sqrt{2\lambda-1}}(z).$$

Thus,

$$f_\lambda^{-1}(z) = G_{1/\sqrt{2\lambda-1},\lambda/\sqrt{2\lambda-1}}(z) = \frac{\sqrt{2\lambda-1}}{\lambda} G\left[\frac{\sqrt{2\lambda-1}}{\lambda}\left(z - \frac{1}{\sqrt{2\lambda-1}}\right)\right]$$
$$= \frac{\sqrt{2\lambda-1}}{\lambda} G\left[\frac{\sqrt{2\lambda-1}z-1}{\lambda}\right].$$

Now, the image of μ_λ under the map $x \mapsto (\sqrt{2\lambda-1}x-1)/\lambda$ transforms its density to

$$\left(1-\frac{x}{2}\right)^{\lambda-1/2}\left(1+\frac{x}{2}\right)^{\lambda-3/2}\mathbf{1}_{[-2,2]}(x)$$

and one easily sees that

$$\int_{-2}^{2}\frac{1}{(z-x)^\lambda}\left(4-x^2\right)^{\lambda-3/2}(2-x)\,dx \propto \frac{G^\lambda(z)}{1-G(z)} := \tilde{u}_\lambda(G)(z).$$

Now note that $(1-G(z))^2 = 1+G^2(z)-2G(z) = (z-2)G(z)$ which yields $1-G(z) = \sqrt{z-2}\sqrt{G(z)}$ where the branch of the square root is taken so that $1-G$ is of positive imaginary-type (i.e., maps the upper half plane into itself).

Proposition 4.1.

$$\int_{-2}^{2}\frac{1}{(z-x)^\lambda}\left(4-x^2\right)^{\lambda-3/2}(2-x)\,dx \propto \frac{G^\lambda(z)}{1-G(z)} = \frac{G^{\lambda-1/2}(z)}{\sqrt{z-2}}.$$

In this case τ_λ satisfies for suitable z

$$\int_{\mathbb{R}}\log(z-x)\tau_\lambda(dx) = -\left(1-\frac{1}{2\lambda}\right)\log(G(z)) - \frac{1}{2\lambda}\log\frac{1}{z-2}$$

whence we deduce that

$$\tau_\lambda(dx) = \left(1-\frac{1}{2\lambda}\right)\frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}\mathbf{1}_{[-2,2]}(dx) + \frac{1}{2\lambda}\delta_2(dx)$$

which is a probability measure for all $\lambda > 1/2$. Besides, the same arguments used before show that for suitable z

$$z \mapsto \frac{G^{1-1/(2\lambda)}(z)}{(z-2)^{1/2\lambda}},$$

is the CST of a probability distribution $\nu_\lambda, \lambda > 1/2$ which is absolutely continuous with density given by

Proposition 4.2.

$$\frac{\nu_\lambda(dx)}{dx} \propto \sin\left[\left(1-\frac{1}{2\lambda}\right)\left(\arcsin\frac{x}{2}+\frac{\pi}{2}\right)\right]\frac{1}{(2-x)^{1/2\lambda}}\mathbf{1}_{]-2,2[}(x), \quad \lambda > 1/2.$$

4.2. Fourth family. The density of μ_λ is given by

$$\left(1-\frac{\sqrt{2\lambda-1}x+1}{2\lambda}\right)^{\lambda-3/2}\left(1+\frac{\sqrt{2\lambda-1}x+1}{2\lambda}\right)^{\lambda-1/2} \tag{4.1}$$

where

$$x \in \left[\frac{-1-2\lambda}{\sqrt{2\lambda-1}}, \frac{-1+2\lambda}{\sqrt{2\lambda-1}}\right], \quad \lambda > 1/2.$$

The density of the image of μ_λ under the map $x \mapsto (\sqrt{2\lambda-1}x+1)/\lambda$ reads

$$\left(1-\frac{x}{2}\right)^{\lambda-3/2}\left(1+\frac{x}{2}\right)^{\lambda-1/2}\mathbf{1}_{[-2,2]}(x) \propto \left(4-x^2\right)^{\lambda-3/2}(2+x)\,\mathbf{1}_{[-2,2]}(x).$$

The same scheme used to deal with the third family gives

$$\tau_\lambda(dx) = \left(1-\frac{1}{2\lambda}\right)\frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}\mathbf{1}_{[-2,2]}(dx) + \frac{1}{2\lambda}\delta_{-2}(dx) \tag{4.2}$$

and that the CST of ν_λ is given by

$$\frac{G^{1-1/(2\lambda)}(z)}{(z+2)^{1/2\lambda}}.$$

However, the density of ν_λ is somewhat different from the one in the previous case:

$$\frac{\nu_\lambda(dx)}{dx} \propto \cos\left[\left(1-\frac{1}{2\lambda}\right)\arcsin\frac{x}{2}\right]\frac{1}{(2+x)^{1/2\lambda}}\mathbf{1}_{]-2,2[}(x), \lambda > 1/2. \tag{4.3}$$

Proposition 4.3. *Let $\lambda > 1/2$. Then*

$$\int_{\mathbb{R}} \frac{1}{(z-x)^\lambda}\mu_\lambda(dx) = \exp -\lambda\int_{\mathbb{R}}\log(z-x)\tau_\lambda(dx) = \left[\int_{\mathbb{R}}\frac{1}{z-x}\nu_\lambda(dx)\right]^\lambda$$

where $\mu_\lambda, \tau_\lambda, \nu_\lambda$ are displayed in (4.1), (4.2) *and* (4.3).

Remarks 4.4. (1) For the four families, the measure τ_λ may be mapped to

$$\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\mathbf{1}_{[-1,1]}(dx) + M_\lambda\delta_{-1}(dx) + N_\lambda\delta_1(dx)$$

for some positive constants M_λ, N_λ. A more wider class of measures including the above one were considered in [13].

(2) For $0 < a < 1, b = 1-a$, one can always define an operation

$$(\nu_1, \nu_2) \mapsto \nu/G_\nu = G^a_{\nu_1}G^b_\nu,$$

where ν_1, ν_2, ν are probability measures. However, every probability measure will be idompotent and the operation is not commutative unless $a = b = 1/2$.

5. On the Moments of the Second Markov Transform

It is known that the Wigner distribution is a universal limiting distribution: it is the spectral distribution of large rescaled random matrices from the so called Wigner ensemble, the limiting distribution of the rescaled Plancherel transition of the growth process for Young diagrams [11] and more generally of the rescaled transition measure of rectangular diagrams associated with roots of some adjacent orthogonal polynomials [12]. It also plays a crucial role in free probability theory where it appears as the central limiting distribution of the sum of free random variables [1, 19]. Note also that the standard arcsine distribution is the central limiting distribution for the t-convolution with $t = 1/2$ [3, 5, 14] and is the limiting distribution of the so-called Shrinkage process [11].

As a matter a fact, it is interesting to find a parallel to the above facts when the Wigner law is replaced by the Markov transform ν_λ corresponding to the second family. More interesting is to define a convolution operation that interpolates the t-convolutions for $t = 1/2$ ($\lambda = 1$) and $t = 1$ ($\lambda = \infty$, free convolution) having ν_λ as a central limiting distribution [6]. Since ν_λ is symmetric and compactly-supported, it is entirely determined by its even moments and we claim

Proposition 5.1. *The normalizing constant of* ν_λ *is given by*

$$c_\lambda = 2^{1-1/\lambda}\sqrt{\pi}\frac{\Gamma(1-1/(2\lambda))\Gamma(3/2-1/(2\lambda))}{\Gamma(2-1/\lambda)}$$

and the even moments of ν_λ *may be expressed as*

$$m_{2n}^{\lambda} := \int_{-2}^{2} x^{2n}\nu_\lambda(dx) = 2^{2n}{}_3F_2\left(\begin{matrix}-n, 1-1/(2\lambda), 3/2-1/(2\lambda)\\ 2-1/\lambda, 1\end{matrix}; 1\right).$$

Moreover, the moments are polynomials in the variable $1/\lambda$.

Proof. : make the change de variable $x \mapsto 2\sin x$ in the integral

$$\int_{-2}^{2} x^{2n}\cos\left[\left(1-\frac{1}{\lambda}\right)\arcsin\frac{x}{2}\right][4-x^2]^{-1/(2\lambda)}dx,\ n \geq 0,$$

to obtain

$$2^{2n+2-1/\lambda}\int_0^{\pi/2}[\sin x]^{2n}\cos\left[\left(1-\frac{1}{\lambda}\right)x\right][\cos x]^{1-1/\lambda}dx,\ n \geq 0.$$

Then expand

$$[\sin x]^{2n} = (1-\cos^2 x)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k[\cos x]^{2k}$$

and use the formula (see [20], p. 177)

$$\int_0^{\pi/2}\cos[(p-q)x][\cos x]^{p+q-2}dx = \frac{\pi}{2^{p+q-1}}\frac{\Gamma(p+q-1)}{\Gamma(p)\Gamma(q)},\ p+q > 1, \tag{5.1}$$

with $p+q = 2k+3-1/\lambda, p-q = 1-1/\lambda$, to get

$$\int_{-2}^{2} \frac{x^{2n}}{[4-x^2]^{1/(2\lambda)}} \cos\left[\left(1-\frac{1}{\lambda}\right)\arcsin\frac{x}{2}\right] dx \\ = 2^{2n}\pi \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(2k+2-1/\lambda)}{\Gamma(k+2-1/\lambda)k!}\left(-\frac{1}{4}\right)^k .$$

This may be expressed through ${}_3F_2$ hypergeometric series as follows: write the binomial coefficient as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = (-1)^k \frac{(-n)_k}{k!}$$

and use the duplication formula to rewrite

$$\sqrt{\pi}\Gamma\left(2k+2-\frac{1}{\lambda}\right) = 2^{2k+1-1/\lambda}\Gamma\left(k+1-\frac{1}{2\lambda}\right)\Gamma\left(k+3/2-\frac{1}{2\lambda}\right).$$

It follows that

$$\int_{-2}^{2} \frac{x^{2n}}{[4-x^2]^{-1/2\lambda}} \cos\left[\left(1-\frac{1}{\lambda}\right)\arcsin\frac{x}{2}\right] dx \\ = c_\lambda 2^{2n} {}_3F_2\left(\begin{matrix}-n, 1-1/(2\lambda), 3/2-1/(2\lambda) \\ 2-1/\lambda, 1\end{matrix}; 1\right).$$

To prove the last claim, let $y = 1/\lambda$ and expand:

$$\begin{aligned}\left(1-\frac{y}{2}\right)_k &= \left(k-\frac{y}{2}\right)\left(k-1-\frac{y}{2}\right)\cdots\left(1-\frac{y}{2}\right) \\ &= \left(-\frac{1}{2}\right)^k (y-2)(y-4)\cdots(y-2k+2)(y-2k),\end{aligned}$$

and similarly

$$\begin{aligned}\left(\frac{3}{2}-\frac{y}{2}\right)_k &= \left(-\frac{1}{2}\right)^k (y-3)(y-5)\cdots(y-2k-1), \\ (2-y)_k &= (-1)^k(y-2)(y-3)\cdots(y-k-1).\end{aligned}$$

The proof ends after forming the ratio

$$\frac{(1-y/2)_k(3/2-y/2)_k}{(2-y)_k} = \left(-\frac{1}{4}\right)^k (y-k-2)\cdots(y-2k)(y-2k-1)$$

so that

$$m_{2n}^{\lambda} := p_n(y) = \sum_{k=0}^{n} \binom{n}{k} 4^{n-k} \frac{(y-k-2)\cdots(y-2k)(y-2k-1)}{k!}.$$

□

Remark 5.2. The exponential generating series of $(p_n)_n$ is given by

$$\sum_{n\geq 0} p_n(y)\frac{z^n}{n!} = \sum_{0\leq k\leq n} \frac{4^{n-k}}{(n-k)!k!}\frac{(y-k-2)\cdots(y-2k)(y-2k-1)}{k!}z^n$$
$$= e^{4z}\sum_{k\geq 0}\frac{(y-k-2)\cdots(y-2k)(y-2k-1)}{k!}\frac{z^k}{k!}$$
$$= 2^{1-y}e^{4z}{}_2F_2\left(\begin{matrix}1-y/2,(3-y)/2,\\ 2-y,1\end{matrix};4z\right).$$

5.1. Combinatorics. For $\lambda = 1$, one recovers the moments of the arcsine distribution given by (use duplication formula)

$$\binom{2n}{n} = 2^{2n}\frac{(1/2)_n}{(1)_n} = 2^{2n}{}_2F_1\left(\begin{matrix}-n,1/2\\ 1\end{matrix};1\right).$$

When $\lambda = \infty$, one recovers the moments of the Wigner distribution:

$$\frac{1}{n+1}\binom{2n}{n} = 2^{2n}\frac{(1/2)_n}{(2)_n} = 2^{2n}{}_2F_1\left(\begin{matrix}-n,3/2\\ 2\end{matrix};1\right).$$

Both moments are integers and it is known that they count the shifted and the ordinary Catalan paths respectively (see [10], p. 64). Unfortunately, m_{2n}^{λ} is not positive integer-valued in general, nevertheless one gets for $\lambda = 2$

$$m_{2n}^2 = 2^{2n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\sqrt{\pi}}\frac{\Gamma(2k+3/2)}{\Gamma(2k+2)} = 2^{2n-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{4k+2}{2k+1}\frac{1}{2^{4k}}$$

so that the few first even moments are equal to $1, 3/2, 31/8, 187/16, 4859/128$.

5.2. Inverses of CST for $\lambda = 2,3,4$. Let $\lambda = n \geq 1$ be a positive integer. Recall that

$$[G_n(z)]^n := \left[\int_{\mathbb{R}}\frac{1}{(z-x)}\nu_n(dx)\right]^n = \tilde{u}_n(G(z)),$$

where as before G is the CST of the Wigner distribution. For the first family, $G_n = G$ for all n so that $K_n = K$ and $K(z) = z + 1/z$ for z in a neighborhood of zero. For the remaining families, the inverse of G_n is given by

$$G_n^{-1}(z) := K_n(z) = K(\tilde{u}_n^{-1}(z^n)), \qquad K := G^{-1},$$

for small $|z|$, subject to the condition

$$K_n(z) = \frac{1}{z} + R_n(z) = \frac{1}{z} + \sum_{k\geq 0} r_k z^k$$

where R_n is an entire function known in free probability theory as the *free cumulants generating function* of ν_n, $(r_k)_k$ is the sequence of the free cumulants, r_0, r_1 are the mean and the variance of ν_n respectively. For the second family, one has to invert

$$z \mapsto \tilde{u}_n(z) = \frac{z^n}{1-z^2}$$

for small $|z|$ in the lower half unit disc (image of G). We are thus led to find one root of the polynomial $z^n + z^2w - w$ for complex numbers z, w in a neighborhood of zero. This task is very complicated since Galois theory asserts that the roots cannot be expressed by means of radicals unless $n \leq 4$. For $n = 2$, easy computations show that

$$K_2(z) = \frac{2z^2+1}{z\sqrt{1+z^2}} = \frac{1}{z} + \sum_{k\geq 0} \frac{(1/2)_k}{(k+1)!}(-1)^k(k+3/2)z^{2k+1}.$$

Thus one sees that $r_0 = 0$, $r_1 = 3/2$ which agrees with our above computations, and that

$$r_{2k+1} = (-1)^k \left[\frac{(1/2)_k}{(1)_k} + \frac{1}{2}\frac{(1/2)_k}{(2)_k}\right].$$

Thus,

$$(-1)^k 2^{2k+1} r_{2k+1} = 2^{2k+1}\frac{(1/2)_k}{(1)_k} + 2^{2k}\frac{(1/2)_k}{(2)_k} = 2\binom{2k}{k} + \frac{1}{k+1}\binom{2k}{k}.$$

For $n = 4$, one has a quadratic polynomial and setting $v = z^2$, one is led to $v^2 + wv - w = 0$ so that

$$v = \frac{-w + \sqrt{w^2+4w}}{2}.$$

The case $z = 3$ is more complicated and we supply one way to adress it: make the substitution $z = Z - w/3$ for suitable Z to get

$$z^3 + wz - w = Z^3 - \frac{w^2}{3}Z + \frac{2}{27}w^3 - w.$$

The last polynomial has the same form as

$$(a+b)^3 - 3ab(a+b) - (a^3+b^3) = 0$$

which hints to look for a root of the form $Z = a + b$ where

$$ab = \frac{w^2}{3}, \quad a^3 + b^3 = w\left[\frac{2}{27}w^2 - 1\right],$$

and computations are left to the curious reader.

Remark 5.3. The above line of thinking remains valid for the nonsymmetric Markov transforms for which

$$\tilde{u}_n(z) = \frac{z^n}{1 \pm z}.$$

For $\lambda = 3$, one already has the appropriate form of the polynomial and there is no need to make the above substitution. However, the case $\lambda = 4$ needs more developed techniques since the polynomial is of degree four and is not quadratic.

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