

LOCAL TIME FOR GAUSSIAN PROCESSES AS AN ELEMENT OF SOBOLEV SPACE

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ABSTRACT. In this paper we consider local time for Gaussian process with values in \mathbb{R}^d . We define it as a limit of the standart approximations in Sobolev space. We also study renormalization of local time, by which we mean the modification of the standart approximations by subtracting a finite number of the terms of its Ito-Wiener expansion. We prove that renormalized local time exists and continuous in Sobolev space under some condition on the covariation of the process (the condition is general and includes the non-renormalized local time case). This condition is also necessary for the existence of local time if we consider renormalized local time at zero for zero-mean Gaussian process. We use our general result to obtain the necessary and sufficient condition for the existence of renormalized local time and self-intersection local time for fractional Brownian motion in \mathbb{R}^d .

1. Introduction.

In this article we consider renormalized local times for zero-mean Gaussian processes in Euclidean space. We derive a condition on the covariation of the process that is sufficient for the existence of local time in certain Sobolev space. This condition also provides the continuity of the local time in Sobolev space under weak convergence of associated measures. We prove that our condition is necessary for the existence of local time at zero in the same Sobolev space. Our approach works for the wide class of Gaussian processes. As an application we consider local time and self-intersection local time (with renormalization) for multidimensional fractional Brownian motion and obtain the conditions on the parameters which are sufficient and necessary for the existence of renormalized local time in Sobolev space. Previously similar results were obtained only partially or for the partial case of Brownian motion.

P.Imkeller, V.Perez-Abreu and J.Vives in [5] studied self-intersection local time for multidimensional Brownian motion. For d -dimensional Brownian motion they proved the convergence of the approximations for self-intersection local time at the point $x \neq 0$ in Sobolev space $D_{2,\alpha}$ for $\alpha < 2 - \frac{d}{2}$. Additionally the authors find Ito-Wiener expansion for self-intersection local time in the form of integral from some polynomial of the process. To define self-intersection local time for d -dimensional Brownian motion at zero the authors use renormalization (the subtraction of mathematical expectation from the approximations). It turns out that

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for dimension $d \geq 3$ renormalized approximations for self-intersection local time for d -dimensional Brownian motion does not converge in any $D_{2,\alpha}$. The authors show that if we want the approximating family to be at least bounded we need to multiply renormalized approximations by some constant which converges to zero fast enough. For $d = 2$ on the contrary we have convergence for $\alpha < 1$ (case $\alpha = 0$ is the classical renormalization result, see [8]).

In this article we obtain the existence result for renormalized local time at zero for fractional Brownian motion. It covers the existence for renormalized local time at zero from [5] but it does not cover results for local time at $x \neq 0$ (we have the same condition as for local time at zero, which is stronger). Additionally our result have the necessity part which is not present in [5].

The reverse problem or the non-existence of local time was considered by S. Albeverio, Y. Hu and X. Y. Zhou [1] for renormalized self-intersection local time for two-dimensional Brownian motion. They proved that this renormalized local time is not an element of $D_{2,1}$ (which means it does not admit stochastic differentiation). This is the special case of our result for fractional Brownian motion.

More general case of self-intersection local time for fractional Brownian motions was considered in [3, 4] by D.Nualart and Y.Hu. In [3] the authors considered the convergence in L_2 for renormalized self-intersection local time of d -dimensional fractional Brownian motion with Hurst parameter H and came up with a sufficient condition of form $dH < \frac{3}{2}$. In [4] same authors studied regularity of renormalized self-intersection local time for the same case of fractional Brownian motion. They found a sufficient condition on d, H, α : $H < \min(\frac{3}{2d}, \frac{2\min(\alpha,1)}{d+2\alpha})$ for this local time to belong in $D_{2,\alpha}$ for $\alpha > 0$. Our necessary and sufficient condition is slightly weaker and covers more general case.

Another type of functional spaces that can be used to define local time in Gaussian case are Hida spaces. For example in [11] H.Watanabe defined self-intersection local time for d -dimensional Brownian motion as a generalized Brownian functional in the sense of Hida.

A.A.Dorogovtzev, V.V.Bakun in [2] defined the class of generalized additive homogenous functionals for Brownian motion as an elements of Sobolev spaces. They also proved that it is possible to define renormalized local time of Brownian motion in special Hida space, if renormalization is understood as a subtraction of finite number of terms from Ito-Wiener expansion.

The main idea of our approach is the careful study of Ito-Wiener expansion for local time approximations. It turns out that it is possible to deal with this expansion for the wide class of Gaussian processes, using multiple integrals of random functions. We find a bound on asymptotics of L_2 -norm of n -th member in this expansion as $n \rightarrow +\infty$ (see (4.5) and lemma 4.1). The inequality (4.1) is the key for deriving this bound and consequently our sufficient condition for the existence of local time. A similar inequality for classic Hermite polynomials was used in [5] for the same purpose. Our condition is also necessary for the existence of local time at zero, since corresponding asymptotics becomes precise.

In the next section we introduce our notation, some basic assumptions and define Ito-Wiener expansion, Sobolev space and local time. After that we derive an

integral representation of Ito-Wiener expansion terms for local time approximations. Then we study the convergence of this integral representation and prove some useful inequalities. We continue with the main section of our paper concerning the existence and continuity of local time. The last section contains results about local time and self-intersection local time for fractional Brownian motion. Some of the results in this article appeared earlier in [9, 10] in particular Theorems 3.3, 5.5, 6.1, 6.2.

2. Basic Definitions and Notations.

Let H be a separable Hilbert space with Gaussian measure μ on its Borel σ -algebra. We consider all random variables as functionals on this space. We are going to introduce Ito-Wiener expansion and Sobolev spaces on H (see [6, 12, 7]). Let $L_2(H, \mu) = \bigoplus_{n=0}^{\infty} G_n$ be orthogonal decomposition of $L_2(H, \mu)$ such that G_n is a sequence of mutually orthogonal subspaces, where each G_n represent polynomials of n -th degree (see [12, 7] for details). We can also write $h = \sum_{n=0}^{\infty} P_n h, h \in L_2(H, \mu)$, where P_n is projector on G_n . The sequence $\{P_n h, n \geq 0\}$ is called an Ito-Wiener expansion for h (it is also called chaos decomposition [7]).

We introduce the family of norms on $\bigcup_{n=0}^{\infty} \bigoplus_{m=0}^n G_n$:

$$\|h\|_{2,\alpha}^2 = \sum_{n=0}^{\infty} (1+n)^\alpha \|P_n h\|_2^2$$

where $\|\cdot\|_2$ is a norm in $L_2(H, \mu)$. Sobolev space $D_{2,\alpha}$ is defined as a completion of $\bigcup_{n=0}^{\infty} \bigoplus_{m=0}^n G_n$ by the norm $\|\cdot\|_{2,\alpha}$. In some cases we can write the explicit form of Ito-Wiener expansion using Hermite polynomials. These polynomials can be defined using well-known integral representation:

$$H_k(x) = e^{x^2/2} \int_{\mathbb{R}} (-iy)^k e^{ixy} (2\pi)^{-1/2} e^{-y^2/2} dy \tag{2.1}$$

where $H_n(x)$ is Hermite polynomial of degree n and has coefficient 1 near x^n .

If $m = (m_1, \dots, m_d)$ is multiindex (by standart rules $m! = m_1! \dots m_d!$ and $|m| = m_1 + \dots + m_d$), then we can define d -dimensional Hermite polynomials using following relation: $H_m(x) = \prod_{k=1}^d H_{m_k}(x_k)$.

Denote by \varkappa a standart Gaussian measure on \mathbb{R}^d (mean for \varkappa is zero and covariation matrix equals identity). The set of d -dimensional Hermite polynomials is orthogonal and dense system in $L_2(\mathbb{R}^d, \varkappa)$. Therefore for any g from $L_2(\mathbb{R}^d, \varkappa)$ we have:

$$g(x) = \sum_{m=(0,\dots,0)}^{(+\infty,\dots,+\infty)} \gamma_m(g) H_m(x)$$

$$\gamma_m(g) = \frac{1}{m!} \int_{\mathbb{R}^d} g(y) H_m(y) (2\pi)^{-d/2} e^{-\|y\|^2/2} dy$$

where the sum is over all d -dimensional multiindices m with nonnegative integer coordinates. This formula can be used to derive Ito-Wiener expansion for some functionals on H .

Let η be a standart d -dimensional Gaussian vector such that each its coordinate belongs to G_1 . For any $g \in L_2(\mathbb{R}^d, \mathcal{K})$ the random variable $g(\eta)$ has following Ito-Wiener expansion:

$$g(\eta) = \sum_{n=0}^{+\infty} \sum_{|m|=n} \gamma_m(g) H_m(\eta)$$

where each sum $\sum_{|m|=n} \gamma_m(g) H_m(\eta)$ belongs to G_n . We can also rewrite this decomposition in terms of projectors on G_n :

$$P_n(g(\eta)) = \sum_{|m|=n} \gamma_m(g) H_m(\eta). \quad (2.2)$$

Let T be a separable metric space, $\{\xi(t), t \in T\}$ – zero-mean Gaussian random process on T with values in \mathbb{R}^d . We suppose that ξ is continuous in probability, and each coordinate of ξ belongs to G_1 for all $t \in T$. Last assumption simplify the study of functionals from ξ using Ito-Wiener expansion, due to the formula (2.2). On the other hand it is not restrictive, i.e. we can always define Gaussian process with given continuous covariation satisfying this assumption. Denote as $K(s, t) = E\xi(s)(\xi(t))^T$ the covariation matrix for ξ . Let ν be a finite measure on σ -algebra of Borel sets in T . We assume that $\det K(t, t) > 0$ ν -a.s. This gives us possibility to work with any powers of $K(t, t)$ (for example with $K^{-1/2}(t, t)$), which are defined ν -a.s.

Denote

$$L(f) = \int_T f(\xi(t)) \nu(dt) \quad (2.3)$$

where f is some Borel measurable and bounded function.

We need a family of functions which approximate delta-measure at zero. We restrict ourselves to the following:

$$f_\varepsilon(x) = (2\pi)^{-d/2} \varepsilon^{-d} e^{-\|x\|^2/2\varepsilon^2}. \quad (2.4)$$

Definition 2.1. Suppose μ is finite measure on Borel σ -algebra in \mathbb{R}^d . If the family of random variables $L(f_\varepsilon * \mu)$ converges in $D_{2,\alpha}$ as $\varepsilon \rightarrow 0+$ then its limit is called local time of ξ with regard to μ in $D_{2,\alpha}$. If the limit exists we say that local time exists in $D_{2,\alpha}$.

We call the family $\{L(f_\varepsilon * \mu) : \varepsilon > 0\}$ approximations for local time of ξ with regard to μ . By local time at zero we mean local time of measure with unit weight at zero, i.e. for $\mu = \delta_0$.

3. Ito-Wiener Expansion for Local Time Approximations.

As the first part of our investigation we will find Ito-Wiener expansion for $L(f)$. We need representation for $a_n(f) = P_n(L(f))$ suitable for applications (for example to study convergence of $L(f_\varepsilon)$). We already know (see formula (2.2)) some representation for Ito-Wiener expansion of function from Gaussian random

vector with all components in G_1 . Since values of our random processes are also Gaussian random vectors with components in G_1 we can apply (2.2) to $f(\xi(t))$. In the following theorem we use this result to find representation of $a_n(f)$.

Theorem 3.1. *Let f be a bounded Borel measurable function on \mathbb{R}^d , then*

$$a_n(f) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) \int_T e^{\|K^{-1/2}(t,t)\xi(t)\|^2/2} N_n(I, K^{-1/2}(t,t)x, K^{-1/2}(t,t)\xi(t)) \frac{\nu(dt)}{\sqrt{\det K(t,t)}} dx \quad (3.1)$$

where for any real $d \times d$ -matrix A , vectors $b, c \in \mathbb{R}^d$ and non-negative integer n

$$N_n(A, b, c) = \frac{(-1)^n}{n!} (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (Au, v)^n e^{i(b,u)} e^{i(c,v)} e^{-\|u\|^2/2 - \|v\|^2/2} dudv \quad (3.2)$$

Before we prove this theorem we need to deal with function $N_n(A, b, c)$. It is quite obvious from (2.1) that this double integral can be written as a sum of some Hermite polynomials multiplied by some exponent. Here is an exact formula.

Lemma 3.2. *Suppose we have real $d \times d$ -matrix A and singular value decomposition for it: $A = U\Lambda V^T$, where U and V are real unitary $d \times d$ matrices and Λ is diagonal matrix. Denote as $\lambda \in \mathbb{R}^d$ a vector of diagonal elements in Λ . For any vectors $b, c \in \mathbb{R}^d$ and non-negative integer n :*

$$N_n(A, b, c) = e^{-\|b\|^2/2} e^{-\|c\|^2/2} \sum_{|m|=n} \frac{\lambda^m}{m!} H_m(V^T b) H_m(U^T c) \quad (3.3)$$

Proof. To prove (3.3) it is enough to write singular value decomposition for A and after some transformations apply formula (2.1). \square

Proof of theorem 3.1. By (2.2) with $g(x) = f(K^{1/2}(t,t)x)$ we get that $\nu(t)$ -a.e. (since $K^{-1/2}(t,t)\xi(t)$ is standart Gaussian vector $\nu(dt)$ -a.e.):

$$\begin{aligned} P_n(f(\xi(t))) &= \sum_{|m|=n} \gamma_m(g) H_m(K^{-1/2}(t,t)\xi(t)) = \\ &= \int_{\mathbb{R}^d} f(K^{1/2}(t,t)y) \sum_{|m|=n} \frac{1}{m!} H_m(y) H_m(K^{-1/2}(t,t)\xi(t)) (2\pi)^{-d/2} e^{-\|y\|^2/2} dy = \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(K^{1/2}(t,t)y) e^{\|K^{-1/2}(t,t)\xi(t)\|^2/2} N_n(I, y, K^{-1/2}(t,t)\xi(t)) dy \end{aligned}$$

First of all we need to prove that we can integrate this expression by $\nu(dt)$. We know that ξ has a jointly measurable modification (because it is continuous in probability). Therefore we may assume that ξ and consequently $P_n(f(\xi(\cdot)))$ are

jointly measurable and we only need to prove that this integral exists:

$$\begin{aligned} \int_T |P_n(f(\xi(t)))| \nu(dt) &\leq \int_T \int_{\mathbb{R}^d} |f(K^{1/2}(t, t)y)| \sum_{|m|=n} \frac{|H_m(y)H_m(K^{-1/2}(t, t)\xi(t))|}{m!} \\ &\cdot (2\pi)^{-d/2} e^{-\|y\|^2/2} dy \nu(dt) \leq \sup_{x \in \mathbb{R}^d} |f(x)| \int_T \int_{\mathbb{R}^d} \sum_{|m|=n} \frac{1}{m!} |H_m(K^{-1/2}(t, t)\xi(t))H_m(y)| \\ &\cdot (2\pi)^{-d/2} e^{-\|y\|^2/2} dy \nu(dt) = (2\pi)^{-d/2} \sup_{x \in \mathbb{R}^d} |f(x)| \cdot \\ &\cdot \sum_{|m|=n} \frac{1}{m!} \int_T |H_m(K^{-1/2}(t, t)\xi(t))| \nu(dt) \int_{\mathbb{R}^d} |H_m(y)| e^{-\|y\|^2/2} dy \end{aligned}$$

The integral $\int_T |H_m(K^{-1/2}(t, t)\xi(t))| \nu(dt)$ is finite a.s. because

$$E \int_T |H_m(K^{-1/2}(t, t)\xi(t))| \nu(dt) = \int_T \int_{\mathbb{R}^d} |H_m(x)| \frac{e^{-\|x\|^2/2}}{(2\pi)^{d/2}} dx \nu(dt) < +\infty$$

This proves that the integral of $P_n(f(\xi(t)))$ by $\nu(dt)$ is a.s. finite random variable (and it has finite mean). Additionally we can claim using Fubini theorem that we can exchange integrals by t and by y in the final expression for $\int_T P_n(f(\xi(t))) \nu(dt)$.

The next step is to prove that

$$\int_T f(\xi(t)) \nu(dt) = \sum_{n=0}^{+\infty} \int_T P_n(f(\xi(t))) \nu(dt)$$

where the sum converges in $L_2(H, \mu)$ and random variable $\int_T P_n(f(\xi(t))) \nu(dt)$ is a version of some element of G_n .

The following equality is obvious since $f(\xi(t)) \in L_2(H, \mu)$:

$$\int_T f(\xi(t)) \nu(dt) = \int_T \sum_{n=0}^{+\infty} P_n(f(\xi(t))) \nu(dt)$$

The sum can be swapped with the integral because of the following uniform bound:

$$\begin{aligned} \left\| \sum_{n=0}^N \int_T P_n(f(\xi(t))) \nu(dt) - \int_T \sum_{n=0}^{+\infty} P_n(f(\xi(t))) \nu(dt) \right\|_2 &= \\ &= \left\| \int_T \sum_{n=N+1}^{+\infty} P_n(f(\xi(t))) \nu(dt) \right\|_2 \leq \int_T \left\| \sum_{n=N+1}^{+\infty} P_n(f(\xi(t))) \right\|_2 \nu(dt) \end{aligned}$$

where the last term converges to zero when $N \rightarrow +\infty$ by Lebesgue theorem of dominated convergence because

$$\left\| \sum_{n=N+1}^{+\infty} P_n(f(\xi(t))) \right\|_2 \leq \sup_{x \in \mathbb{R}^d} |f(x)|$$

This also proves that $\int_T P_n(f(\xi(t)))\nu(dt) \in L_2(H, \mu)$.

Now we need to prove that $\int_T P_n(f(\xi(t)))\nu(dt) \in G_n$. But we have for all $\eta \in G_n^\perp$

$$E\eta \int_T P_n(f(\xi(t)))\nu(dt) = \int_T E\eta P_n(f(\xi(t)))\nu(dt) = 0$$

by Fubini theorem and the following inequality:

$$\begin{aligned} \int_T E|\eta P_n(f(\xi(t)))|\nu(dt) &\leq \int_T \sqrt{E|\eta|^2 E|P_n(f(\xi(t)))|^2}\nu(dt) \leq \\ &\leq \sqrt{E|\eta|^2} \int_T \sqrt{E|f(\xi(t))|^2}\nu(dt) \leq \sqrt{E|\eta|^2}\nu(T) \sup_{x \in \mathbb{R}^d} |f(x)| < +\infty \end{aligned}$$

So we proved that

$$\begin{aligned} a_n(f) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(K^{1/2}(t,t)y) \cdot \\ &\quad \cdot \int_T e^{\|K^{-1/2}(t,t)\xi(t)\|^2/2} N_n(I, y, K^{-1/2}(t,t)\xi(t))\nu(dt) dy \end{aligned}$$

We get the desired formula (3.1) after introducing new variable x under integral: $x = K^{1/2}(t,t)y$. □

We can obtain similar representation for covariation of $a_n(h_1)$ and $a_n(h_2)$.

Theorem 3.3. *Let h_1 and h_2 be two bounded Borel measurable functions on \mathbb{R}^d , then*

$$\begin{aligned} Ea_n(h_1)a_n(h_2) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_1(x)h_2(y) \\ &\quad \int_T \int_T N_n(G(s,t), K^{-1/2}(s,s)x, K^{-1/2}(t,t)y) \frac{\nu(ds)}{\sqrt{\det K(s,s)}} \frac{\nu(dt)}{\sqrt{\det K(t,t)}} dx dy \end{aligned} \tag{3.4}$$

where $G(s,t)$ is covariation matrix for two random vectors $K^{-1/2}(s,s)\xi(s)$ and $K^{-1/2}(t,t)\xi(t)$.

Note that matrix-valued function G is defined $\nu \times \nu$ a.e. (by our assumptions) and can be expressed in terms of K :

$$G(s,t) = K^{-1/2}(s,s)K(s,t)K^{-1/2}(t,t)$$

Before we continue with the proof of theorem 3.3 we need following simple lemma.

We denote for non-negative integer n and $a, b \in \mathbb{R}^d$:

$$I_n(a,b) = \int_{\mathbb{R}^d} e^{i(a,x)}(b,x)^n (2\pi)^{-d/2} e^{-\|x\|^2/2} dx$$

Lemma 3.4. *If $b \neq 0$*

$$I_n(a, b) = i^n \|b\|^n H_n\left(\frac{(a, b)}{\|b\|}\right) e^{-\|a\|^2/2}$$

Proof. The case $d = 1$ is covered by the formula (2.1), so we can assume that $d \geq 2$. We introduce orthonormal basis $\{e_k, k = 1, \dots, d\}$ in \mathbb{R}^d , such that $e_1 = \frac{b}{\|b\|}$ and $e_2 = \frac{a - e_1(a, e_1)}{\|a - e_1(a, e_1)\|}$. We have $a = (a, e_1)e_1 + \|a - e_1(a, e_1)\|e_2$; $b = e_1\|b\|$. We can write the expression under the integral in $I_n(a, b)$ as follows (denoting $x_i = (e_i, x)$):

$$e^{i(a, x)}(b, x)^n e^{-\|x\|^2/2} = e^{ix_1(a, e_1)} e^{ix_2\|a - e_1(a, e_1)\|} x_1^n \|b\|^n \prod_{i=1}^d e^{-|x_i|^2/2}$$

We can treat $I_n(a, b)$ as an iterated integral by x_1, x_2, \dots . Integrating first by variables x_3, x_4, \dots we get two separate integrals by x_1 and x_2 . Then we can use (2.1):

$$\begin{aligned} I_n(a, b) &= \|b\|^n \int_{\mathbb{R}} e^{i(a, e_1)x_1} x_1^n (2\pi)^{-1/2} e^{-x_1^2/2} dx_1 \\ &\quad \int_{\mathbb{R}} e^{i\|a - e_1(a, e_1)\|x_2} (2\pi)^{-1/2} e^{-x_2^2/2} dx_2 = i^n \|b\|^n H_n\left(\frac{(a, b)}{\|b\|}\right) e^{-\|a\|^2/2} \end{aligned}$$

□

Note that the integral I_n appears in N_n :

$$N_n(A, b, c) = (2\pi)^{-d/2} \frac{(-1)^n}{n!} \int_{\mathbb{R}^d} I_n(c, Au) e^{i(b, u)} e^{-\|u\|^2/2} du \quad (3.5)$$

So we have another representation for N_n . Now we can prove theorem 3.3.

Proof of theorem 3.3. By theorem 3.1 and lemma 3.4:

$$\begin{aligned} a_n(h) &= (2\pi)^{-d} \frac{(-1)^n}{n!} \int_{\mathbb{R}^d} h(x) \int_T \int_{\mathbb{R}^d} i^n \|u\|^n H_n((K^{-1/2}(t, t)\xi(t), u)/\|u\|) \\ &\quad e^{i(K^{-1/2}(t, t)x, u)} e^{-\|u\|^2/2} du \frac{\nu(dt)}{\sqrt{\det K(t, t)}} dx \end{aligned}$$

We get:

$$\begin{aligned} E a_n(h_1) a_n(h_2) &= \\ &= \frac{(-1)^n}{(n!)^2} E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_1(x) h_2(y) \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|u\|^n H_n\left(\frac{(K^{-1/2}(t, t)\xi(t), u)}{\|u\|}\right) \\ &\quad \|v\|^n H_n\left(\frac{(K^{-1/2}(s, s)\xi(s), v)}{\|v\|}\right) e^{i(K^{-1/2}(t, t)x, u)} e^{i(K^{-1/2}(s, s)y, v)} \\ &\quad (2\pi)^{-2d} e^{-\|u\|^2/2 - \|v\|^2/2} dudv \frac{\nu(ds)}{\sqrt{\det K(s, s)}} \frac{\nu(dt)}{\sqrt{\det K(t, t)}} dx dy \end{aligned}$$

If we are able to move mathematical expectation inside all integrals then we get desirable formula by following well-known relation: for any non-negative integer n and two jointly Gaussian random variables η_1, η_2 with unit dispersion: $EH_n(\eta_1)H_n(\eta_2) = n!(E\eta_1\eta_2)^n$. In our case we have

$$\begin{aligned} & \|a\|^n \|b\|^n EH_n((K^{-1/2}(t, t)\xi(t), \frac{a}{\|a\|}))H_n((K^{-1/2}(s, s)\xi(s), \frac{b}{\|b\|})) = \\ & = n!(E(K^{-1/2}(t, t)\xi(t), a)(K^{-1/2}(s, s)\xi(s), b))^n = n!(G(s, t)a, b)^n \end{aligned}$$

Now we need to justify this calculation. By Fubini theorem it is enough to prove that the same integral with modulus inside is finite. Unfortunately this may not be true in our case. We have to treat integrals by x, y, s, t and by u, v separately. First we will prove that mathematical expectation can be exchanged with integrals by x, y, s, t . To do this we use (3.3) and represent the integral by u, v as a product of N_n .

$$\begin{aligned} & E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |h_1(x)h_2(y)| \int_T \int_T \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|u\|^n H_n\left(\frac{(K^{-1/2}(t, t)\xi(t), u)}{\|u\|}\right) \cdot \right. \\ & \quad \cdot \|v\|^n H_n\left(\frac{(K^{-1/2}(s, s)\xi(s), v)}{\|v\|}\right) e^{i(K^{-1/2}(t, t)x, u)} e^{i(K^{-1/2}(s, s)y, v)} \cdot \\ & \quad \cdot (2\pi)^{-2d} e^{-\|u\|^2/2 - \|v\|^2/2} dudv \Big| \frac{\nu(ds)}{\sqrt{\det K(s, s)}} \frac{\nu(dt)}{\sqrt{\det K(t, t)}} dx dy \leq \\ & \quad \leq C_n \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_T \int_T e^{-\|K^{-1/2}(t, t)x\|^2/2 - \|K^{-1/2}(s, s)y\|^2/2} \cdot \\ & \quad \cdot \sum_{|m_1|=n} \sum_{|m_2|=n} \left(\frac{1}{m_1!} \frac{1}{m_2!} |H_{m_1}(K^{-1/2}(t, t)x)| |H_{m_2}(K^{-1/2}(s, s)y)| \cdot \right. \\ & \quad \cdot E |H_{m_1}(K^{-1/2}(t, t)\xi(t))| |H_{m_2}(K^{-1/2}(s, s)\xi(s))| \Big) \cdot \\ & \quad \cdot \frac{\nu(ds)}{\sqrt{\det K(s, s)}} \frac{\nu(dt)}{\sqrt{\det K(t, t)}} dx dy \end{aligned}$$

The integral

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\det K(t, t) \det K(s, s))^{-1/2} |H_{m_1}(K^{-1/2}(t, t)x)| |H_{m_2}(K^{-1/2}(s, s)y)| \\ & \quad e^{-\|K^{-1/2}(t, t)x\|^2/2 - \|K^{-1/2}(s, s)y\|^2/2} dx dy \end{aligned}$$

is bounded for any fixed multiindices m_1, m_2 uniformly on s, t . The integral

$$\int_T \int_T E |H_{m_1}(K^{-1/2}(t, t)\xi(t))| |H_{m_2}(K^{-1/2}(s, s)\xi(s))| \nu(ds) \nu(dt)$$

is finite for any fixed multiindices m_1, m_2 since by Holder inequality

$$E|H_{m_1}(K^{-1/2}(t, t)\xi(t))|H_{m_2}(K^{-1/2}(s, s)\xi(s))| \leq \left(E(H_{m_1}(K^{-1/2}(t, t)\xi(t)))^2 E(H_{m_2}(K^{-1/2}(s, s)\xi(s)))^2 \right)^{1/2}$$

and the expression on the right side does not depend on s, t because $K^{-1/2}(t, t)\xi(t)$ is standart Gaussian vector.

Now all we need to prove is that mathematical expectation of

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|u\|^n \|v\|^n \left| H_n((K^{-1/2}(t, t)\xi(t), u)/\|u\|) H_n((K^{-1/2}(s, s)\xi(s), v)/\|v\|) \right| e^{-\|u\|^2/2 - \|v\|^2/2} dudv$$

is finite. Then by Fubini theorem we can move mathematical expectation under integrals. Again we can use Holder inequality

$$E|H_n((K^{-1/2}(t, t)\xi(t), u)/\|u\|)|H_n((K^{-1/2}(s, s)\xi(s), v)/\|v\|)| \leq \left(E(H_n((K^{-1/2}(t, t)\xi(t), u)/\|u\|))^2 E(H_n((K^{-1/2}(s, s)\xi(s), v)/\|v\|))^2 \right)^{1/2}$$

and the expression on the right side does not depend on s, t, u, v . □

4. The Convergence of Ito-Wiener Expansion.

In [5] the authors used uniform bounds on $H_k(x)$ to find some estimates on $a_n(f)$. One such bound can be proved using (2.1) as shown below:

$$|H_k(x)e^{-x^2/2}| = \left| \int_{\mathbb{R}} (-iy)^k e^{ixy} (2\pi)^{-1/2} e^{-y^2/2} dy \right| \leq \int_{\mathbb{R}} |y|^k (2\pi)^{-1/2} e^{-y^2/2} dy$$

Instead of using this inequality directly we will find bounds for Fourier type integrals in (3.2) using similar idea. As a result we obtain more general inequalities for sums of Hermite polynomials. Let S_d be a sphere of radius 1 in \mathbb{R}^d and σ is uniform surface measure on S_d .

Lemma 4.1. *For all matrices A , vectors b, c and positive integers n, d :*

$$|N_n(A, b, c)| \leq C(n, d) \int_{S_d} \|Au\|^n \sigma(du) \tag{4.1}$$

where

$$C(n, d) = \frac{2^{n+(d-1)/2}}{n!} (2\pi)^{-(d+1)/2} \Gamma\left(\frac{n+d}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \sim \sim (2\pi)^{-d/2} n^{d/2-1}, n \rightarrow +\infty \tag{4.2}$$

If n is even then the right hand side of inequality is equal to $N(A, 0, 0)$.

Proof. By putting modulus inside the integral and setting basis in \mathbb{R}^d with first vector proportional to Au we get:

$$\begin{aligned} |N_n(A, b, c)| &\leq \frac{1}{n!} (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(Au, v)|^n e^{-\|u\|^2/2 - \|v\|^2/2} dudv = \\ &= \frac{1}{n!} (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|Au\|^n |v_1|^n e^{-\|u\|^2/2 - \|v\|^2/2} dudv = \\ &= 2 \frac{1}{n!} (2\pi)^{-(d+1)/2} \int_{S_d} \|Au\|^n \sigma(du) \int_0^\infty r^{n+d-1} e^{-r^2/2} dr \int_0^\infty v_1^n e^{-v_1^2/2} dv = \\ &= C(n, d) \int_{S_d} \|Au\|^n \sigma(du) \end{aligned}$$

If n is even and $b = c = 0$ then right and left hand sides of this inequality are obviously equal.

Now we have to find constant asymptotics:

$$\begin{aligned} C(n, d) &= \frac{2}{n!} (2\pi)^{-(d+1)/2} \int_0^\infty r^{n+d-1} e^{-r^2/2} dr \int_0^\infty v_1^n e^{-v_1^2/2} dv_1 = \\ &= \frac{2^{n+(d-1)/2}}{n!} (2\pi)^{-(d+1)/2} \Gamma\left(\frac{n+d}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \end{aligned}$$

We use Stirling formula in the following form:

$$\Gamma(x) \sim \sqrt{2\pi/x} \left(\frac{x}{e}\right)^x, \quad x \rightarrow +\infty$$

Since $n! = \Gamma(n+1)$ we can apply this formula to obtain asymptotics:

$$\begin{aligned} C(n, d) &= \frac{2^{n+(d-1)/2}}{\Gamma(n+1)} (2\pi)^{-(d+1)/2} \Gamma\left(\frac{n+d}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \sim \\ &= 2^{(d-1)/2} (2\pi)^{-(d+1)/2} 2^{-(d+1)/2} (8\pi)^{1/2} e^{-(d+1)/2} \\ &\quad \sqrt{\frac{1}{n+d}} \left(1 + \frac{d+1}{n+1}\right)^{(n+1)/2} (n+d)^{(d-1)/2} \sim (2\pi)^{-d/2} n^{d/2-1}, \quad n \rightarrow +\infty \end{aligned}$$

□

We want to extend theorem 3.3 for the following object instead of $a_n(f)$:

$$\begin{aligned} a_n(\mu) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \int_T e^{\|K^{-1/2}(t,t)\xi(t)\|^2/2} \\ &\quad N_n(I, K^{-1/2}(t,t)x, K^{-1/2}(t,t)\xi(t)) \frac{\nu(dt)}{\sqrt{\det K(t,t)}} \mu(dx) \quad (4.3) \end{aligned}$$

where μ is some finite measure on Borel σ -algebra in \mathbb{R}^d . If $\mu(dx) = f(x)dx$ then, obviously, $a_n(f) = a_n(\mu)$.

Theorem 4.2. *If*

$$\int_T \frac{\nu(dt)}{\sqrt{\det K(t, t)}} < +\infty \quad (4.4)$$

then for all non-negative integer n and any finite measure μ on Borel σ -algebra in \mathbb{R}^d $a_n(\mu)$ is correctly defined a.s. and belongs to $L_2(\Omega)$. In addition for any two finite measures μ_1 and μ_2

$$Ea_n(\mu_1)a_n(\mu_2) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_T \int_T N_n(G(s, t), K^{-1/2}(s, s)x, K^{-1/2}(t, t)y) \frac{\nu(ds)}{\sqrt{\det K(s, s)}} \frac{\nu(dt)}{\sqrt{\det K(t, t)}} \mu_1(dx) \mu_2(dy) \quad (4.5)$$

and

$$|Ea_n(\mu_1)a_n(\mu_2)| \leq C(n, d)(2\pi)^{-d} \mu_1(\mathbb{R}^d) \mu_2(\mathbb{R}^d) J_n \quad (4.6)$$

where

$$J_n = \int_T \int_T \int_{S_d} \|G(s, t)u\|^n \sigma(du) \frac{\nu(ds)}{\sqrt{\det K(s, s)}} \frac{\nu(dt)}{\sqrt{\det K(t, t)}} \quad (4.7)$$

Proof. The proof of the formula (4.5) repeats proof of theorem 3.3. All calculations are the same after obvious modifications. We only need to make sure we can use Fubini theorem as we did (without condition (4.4) this might not be true). In fact we have to prove two things. The first is that we can change the order of integration by x and t in (4.3). The second is that we can put mathematical expectation inside integrals by s, t and $\mu_1(dx)\mu_2(dy)$ after we write formula for $Ea_n(\mu_1)a_n(\mu_2)$.

Denote

$$b_n(\mu) = \int_{\mathbb{R}^d} \int_T e^{\|K^{-1/2}(t, t)\xi(t)\|^2/2} |N_n(I, K^{-1/2}(t, t)x, K^{-1/2}(t, t)\xi(t))| \frac{\nu(dt)}{\sqrt{\det K(t, t)}} \mu(dx)$$

This is the exact formula for a_n after putting modulus inside the integral (except for some irrelevant constant). We can prove both statements above by showing that $b_n(\mu) \in L_2(\Omega)$ for all finite measures μ .

From (3.3) and using that $H_n(x)e^{-x^2/2}$ is a bounded function of x we get:

$$e^{\|K^{-1/2}(t, t)\xi(t)\|^2/2} |N_n(I, K^{-1/2}(t, t)x, K^{-1/2}(t, t)\xi(t))| \leq C_n \sum_{|m|=n} \frac{1}{m!} |H_m(K^{-1/2}(t, t)\xi(t))| \quad (4.8)$$

The expression $K^{-1/2}(t, t)\xi(t)$ has standart Gaussian distribution and therefore the second moment of the sum in the right hand side does not depend on t . Finally:

$$\begin{aligned}
 E(b_n(\mu))^2 &\leq \\
 &\leq \int_{\mathbb{R}^d} \int_T E \left(e^{\|K^{-1/2}(t,t)\xi(t)\|^2/2} N_n(I, K^{-1/2}(t,t)x, K^{-1/2}(t,t)\xi(t)) \right)^2 \cdot \\
 &\quad \cdot \frac{\nu(dt)}{\sqrt{\det K(t,t)}} \mu(dx) \leq \tilde{C}_n \mu(\mathbb{R}^d) \int_T \frac{\nu(dt)}{\sqrt{\det K(t,t)}} < +\infty
 \end{aligned}$$

The inequality (4.6) is a straightforward application of (4.1) to (4.5). □

Note that the condition (4.4) is sufficient (and necessary if $n = 0$) for J_n to be finite (because $\|G(s, t)\| \leq 1$) and we can replace it in the theorem with $J_n < +\infty$ if $\mu(dx) = f(x)dx$ for some bounded function f , since we can replace (4.5) with (3.4). It is natural to expect that the condition $J_n < +\infty$ is in fact always sufficient for $a_n(\mu) \in L_2$ (it is necessary for even n because if $\mu_1 = \mu_2 = \delta_0$ the inequality (4.6) turns to be an equality, due to lemma 4.1). If it is true then we are able to define $a_n(\mu)$ for some n even in cases, where we can not define $a_0(\mu)$. We want to use this to define renormalized local time. But we can not be sure that the integral in the definition of $a_n(\mu)$ exists. To avoid this problem we need to redefine $a_n(\mu)$.

Denote $T_k = \{t \in T : \det K(t, t) > \frac{1}{k}\}, k = 1, 2, \dots$. Since each T_k is also separable metric space we can apply the same theory for each T_k and define $a_n^k(\mu)$ as an analogue of our previous definition of $a_n(\mu)$ if we replace T with T_k by restricting ν on T_k . It is obvious that the condition (4.4) holds on each T_k , therefore $a_n^k(\mu)$ is correctly defined due to theorem 4.2. Also note that $1_{T_k} \rightarrow 1, k \rightarrow +\infty$ ν -a.e. since $\nu(\{t : \det K(t, t) = 0\}) = 0$. It means that $J_n^k \uparrow J_n, k \rightarrow +\infty$ where J_n^k is an analogue of J_n on T_k . We define $a_n(\mu)$ as follows:

$$a_n(\mu) = L_2 - \lim_{k \rightarrow +\infty} a_n^k(\mu) \tag{4.9}$$

What is essentially done here can be seen as the replacement of a.s. integral in the definition of $a_n(\mu)$ with some analogue in L_2 . Note that if (4.4) holds then the limit above exist a.s. because the integral in the definition of $a_n(\mu)$ exists a.s. Moreover it easy to see using theorem 4.2 that the limit also exists in L_2 , i.e. our new definition may differ from the original definition only if the condition (4.4) does not hold. From now on we assume that $a_n(\mu)$ is defined as L_2 limit by the formula (4.9).

Theorem 4.3. *If for some non-negative integer n we have $J_n < +\infty$ then $a_n(\mu)$ is correctly defined as an element of $L_2(\Omega)$ and both formula (4.5) and inequality (4.6) are valid.*

Proof. Since we know that theorem 4.2 holds on each T_k it is enough to prove that the sequence $a_n^k(\mu), k = 1, \dots$ is fundamental in L_2 . The formula (4.5) and inequality (4.6) follow immediately after passing to the limit in similar formulae on each T_k . But fundamentality follows from the inequality (4.6) in the theorem 4.2 applied for $T_k \setminus T_l$. Indeed $a_n^k(\mu) - a_n^l(\mu)$ is analogue of $a_n(\mu)$ on $T_k \setminus T_l$ if $k > l$ and

we have

$$E(a_n^k(\mu) - a_n^l(\mu))^2 \leq C(n, d)(2\pi)^{-d} \int_{T_k \setminus T_l} \int_{T_k \setminus T_l} \int_{S_d} \|G(s, t)u\|^n \sigma(du) \frac{\nu(ds)}{\sqrt{\det K(s, s)}} \frac{\nu(dt)}{\sqrt{\det K(t, t)}}$$

where the right hand side converges to zero if $k, l \rightarrow +\infty$ due to our assumption. □

The next theorem contains main convergence result for a_n .

Theorem 4.4. *If for some n the condition $J_n < +\infty$ holds then a_n is continuous in $L_2(\Omega)$ on the space of finite measures on Borel σ -algebra in \mathbb{R}^d with topology of weak convergence.*

Proof. Suppose that the sequence of measures μ_m converges weakly to μ , then: $\forall u \in \mathbb{R}^d : \hat{\mu}_m(u) \rightarrow \hat{\mu}(u), m \rightarrow +\infty$ where $\hat{\mu}_m(u) = \int_{\mathbb{R}^d} e^{i(x,u)} \mu_m(dx)$ is Fourier transform for μ_m and $\hat{\mu}$ is Fourier transform for μ .

To use the convergence of Fourier transforms we rewrite (3.4) using $\hat{\mu}_m$ instead of μ_m . Simple calculations show that for a pair of finite measures μ_i, μ_j :

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} N(A, x, y) \mu_i(dx) \mu_j(dy) &= \\ &= \frac{(-1)^n}{n!} (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (Au, v)^n \hat{\mu}_i(u) \hat{\mu}_j(v) e^{-\|u\|^2/2 - \|v\|^2/2} dudv \end{aligned}$$

We apply these two formulae to (4.5) (integrals by s, t and x, y in (3.4) can be taken in any order, as we know from the proof of theorem 4.2):

$$\begin{aligned} E a_n(\mu_i) a_n(\mu_j) &= \\ &= \frac{(-1)^n}{n!} (2\pi)^{-2d} \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (G(s, t)u, v)^n \hat{\mu}_i(K^{-1/2}(s, s)u) \hat{\mu}_j(K^{-1/2}(t, t)v) \\ &\quad e^{-\|u\|^2/2 - \|v\|^2/2} dudv \frac{\nu(ds)}{\sqrt{\det K((s, s))}} \frac{\nu(dt)}{\sqrt{\det K((t, t))}} \end{aligned} \tag{4.10}$$

We can estimate this integral (like in the proof of inequality (4.6)):

$$\begin{aligned} \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(G(s, t)u, v)|^n |\hat{\mu}_i(K^{-1/2}(s, s)u)| |\hat{\mu}_j(K^{-1/2}(t, t)v)| \\ e^{-\|u\|^2/2 - \|v\|^2/2} dudv \frac{\nu(ds)}{\sqrt{\det K((s, s))}} \frac{\nu(dt)}{\sqrt{\det K((t, t))}} \leq \\ \leq \mu_i(\mathbb{R}^d) \mu_j(\mathbb{R}^d) J_n C(n, d) \end{aligned}$$

So if $J_n < +\infty$ then by Lebesgue theorem of dominated convergence the expression $E a_n(\mu_i) a_n(\mu_j)$ converges to $E(a_n(\mu))^2$ as $i, j \rightarrow +\infty$. It means that $a_n(\mu_i)$

converges in L_2 as $i \rightarrow +\infty$. Since by similar argument $E a_n(\mu_i) a_n(\mu)$ converges to $E(a_n(\mu))^2$ as $i \rightarrow +\infty$ the limit is in fact equal $a_n(\mu)$. \square

5. The Existence of Local Time and Renormalization.

We know from theorem 4.3 that if the condition $J_n < +\infty$ holds then we have $L_2 - \lim_{\varepsilon \rightarrow 0^+} a_n(f_\varepsilon * \mu) = a_n(\mu)$ (here f_ε is from (2.4)) since measures $(f_\varepsilon * \mu)(x)dx$ converge weakly to μ . The next theorem allows to extend this convergence to $L(f_\varepsilon * \mu)$.

Theorem 5.1. *If for some real α and non-negative integer m*

$$\sum_{n=m}^{+\infty} (1+n)^{\alpha+d/2-1} J_n < +\infty \tag{5.1}$$

then the following function

$$L_m(\mu) = \sum_{n=m}^{+\infty} a_n(\mu)$$

(where the infinite sum is considered as a limit in $D_{2,\alpha}$) is continuous in $D_{2,\alpha}$ on the space of finite measures μ on Borel σ -algebra in \mathbb{R}^d with topology of weak convergence.

Proof. Since $J_n < +\infty$ for $n \geq m$ then by theorem 4.3 random functions a_n are correctly defined for all $n \geq m$ and additionally

$$\begin{aligned} \|L_m(\mu)\|_{2,\alpha}^2 &= \sum_{n=m}^{+\infty} (1+n)^\alpha \|a_n(\mu)\|_2^2 \leq (\mu(\mathbb{R}^d))^2 (2\pi)^{-d} \\ &\cdot \sum_{n=m}^{+\infty} C(n,d) (1+n)^\alpha J_n \leq (\mu(\mathbb{R}^d))^2 \tilde{C}(d) \sum_{n=m}^{+\infty} (1+n)^{\alpha+d/2-1} J_n < +\infty \end{aligned}$$

where we also used formula (4.2) for $C(n,d)$. So $L_m(\mu)$ is well-defined as an element of $D_{2,\alpha}$. If the sequence of finite measures μ_k converges weakly to some finite measure μ then each term in the following sum

$$\|L_m(\mu_k) - L_m(\mu)\|_{2,\alpha}^2 = \sum_{n=m}^{+\infty} (1+n)^\alpha \|a_n(\mu_k) - a_n(\mu)\|_2^2$$

is bounded uniformly on k by terms of the sum in (5.1):

$$\|a_n(\mu_k) - a_n(\mu)\|_2^2 \leq 2((\mu(\mathbb{R}^d))^2 + \sup_k (\mu_k(\mathbb{R}^d))^2) \tilde{C}(d) (1+n)^{d/2-1} J_n$$

Using theorem 4.4 we also get that $\|a_n(\mu_k) - a_n(\mu)\|_2^2$ converges to zero. Consequently $L_m(\mu_k)$ converges to $L_m(\mu)$ in $D_{2,\alpha}$. \square

Theorem above proves the existence of so-called renormalized local time $L_m(\mu)$ (or usual local time if $m = 0$) if the condition (5.1) holds. Our renormalization is different from classical renormalization, where only mathematical expectation is subtracted (see for example [8]). Instead we use more general approach from [2]

and subtract a finite number of terms in Ito-Wiener expansion of local time approximations.

It is possible to derive another form of the condition (5.1).

Lemma 5.2. *Let (E, \mathfrak{F}) be a measurable space with finite measure θ on it and let f be a measurable function on E with values in $[0, 1]$. Denote*

$$K_n = \int_E (f(x))^n \theta(dx)$$

If $\gamma \in \mathbb{R}$ and a sequence of non-negative numbers $\{c_n, n \geq 0\}$ satisfy

$$0 < \liminf_{n \rightarrow +\infty} \frac{c_n}{(n+1)^\gamma} \leq \sup_{n \geq 0} \frac{c_n}{(n+1)^\gamma} < \infty \quad (5.2)$$

then following two conditions are equivalent

$$\begin{aligned} 1) & \sum_{n=0}^{\infty} K_n c_n < +\infty \\ 2) & \int_E p_{-\gamma-1}(1-f(x))\theta(dx) < +\infty \end{aligned}$$

where

$$\forall z \in [0, 1] : p_\beta(z) = \begin{cases} z^\beta, & \beta < 0 \\ 1 - \ln z, & \beta = 0 \\ 1, & \beta > 0 \end{cases}$$

If $c_0 > 0$ then the same statement holds for arbitrary θ (not necessarily finite).

Proof. For positive c_0 we conclude that if $\sum_{n=0}^{\infty} K_n c_n < +\infty$ then $K_0 = \theta(E) < +\infty$ and measure θ is finite. Since $p_\beta(z) \geq 1$ for all $z \in [0, 1]$ we have

$$\theta(E) \leq \int_E p_{-\gamma-1}(1-f(x))\theta(dx)$$

and if the integral is finite, then measure θ is again finite. Therefore if $c_0 > 0$ we may assume that measure θ is finite and it is enough to consider only this case.

Notice that for $\gamma < -1$ the sum and the integral are obviously finite. So we only need to consider the case $\gamma \geq -1$.

We will prove that the sequence c_n can be replaced with any other sequence with same property (5.2). Suppose \tilde{c}_n is a sequence of non-negative numbers that satisfy (5.2) and $\sum_{n=0}^{\infty} K_n c_n < +\infty$. Choose $m \in \mathbb{N}$, such that $\inf_{n \geq m} \frac{c_n}{(n+1)^\gamma} > 0$. It can be done because of (5.2). We have

$$\sum_{n=m}^{\infty} K_n \tilde{c}_n \leq \sup_{n \geq 0} \frac{\tilde{c}_n}{(n+1)^\gamma} \sum_{n=m}^{\infty} K_n (n+1)^\gamma \leq \frac{\sup_{n \geq 0} \frac{\tilde{c}_n}{(n+1)^\gamma}}{\inf_{n \geq m} \frac{c_n}{(n+1)^\gamma}} \sum_{n=m}^{\infty} K_n c_n < +\infty$$

Similarly if $\sum_{n=0}^{\infty} K_n \tilde{c}_n < +\infty$ then the same statement holds for c_n , i.e. we can replace c_n with \tilde{c}_n .

We want to take as c_n coefficients near q^n in Taylor expansion of $p_{-\gamma-1}(1-q)$. For this we have to prove that they satisfy (5.2).

Consider the case $\gamma = -1$:

$$p_{-\gamma-1}(1-q) = 1 - \ln(1-q) = 1 + \sum_{n=1}^{\infty} \frac{1}{n} q^n$$

where the sum converges for $q \in (-1, 1)$. The condition (5.2) is obviously satisfied for $c_n = 1/n, n = 1, \dots$

Now consider $\gamma > -1$, then

$$p_{-\gamma-1}(1-q) = (1-q)^{-\gamma-1} = \sum_{n=0}^{\infty} \left(\prod_{k=1}^n \left(\frac{\gamma}{k} + 1 \right) \right) q^n$$

where the sum again converges for $q \in (-1, 1)$. We have

$$c_n = \prod_{k=1}^n \left(\frac{\gamma}{k} + 1 \right) = \exp\left(\sum_{k=1}^n \ln\left(\frac{\gamma}{k} + 1\right)\right) \sim C \exp\left(\sum_{k=1}^n \frac{\gamma}{k}\right) \sim \tilde{C} n^\gamma, n \rightarrow \infty$$

and (5.2) is satisfied.

We place the chosen values for c_n in $\sum_{n=0}^{\infty} K_n c_n$ and use non-negativity to swap the sum and the integral:

$$\sum_{n=0}^{\infty} K_n c_n = \int_E \sum_{n=0}^{\infty} (f(x))^n c_n \theta(dx) = \int_E p_{-\gamma-1}(1-f(x)) \theta(dx)$$

(we also used that $p_{-\gamma-1}(1-q) = \sum_{n=0}^{\infty} q^n c_n$ for all $q \in [0, 1]$). We proved that the integral and the sum for the chosen values of c_n are in fact equal, so the integral is finite if and only if the sum is finite. □

Now it is possible to rewrite (5.1).

Theorem 5.3. *The condition (5.1) is equivalent to*

$$\int_T \int_T \int_{S_d} \|G(s, t)u\|^m \cdot p_{-\alpha-d/2}(1 - \|G(s, t)u\|) \sigma(du) \frac{\nu(ds)}{\sqrt{\det K(s, s)}} \frac{\nu(dt)}{\sqrt{\det K(t, t)}} < +\infty \quad (5.3)$$

Proof. It is enough to apply lemma 5.2 with

$$E = S_d \times T \times T,$$

$$\theta(dudsdt) = \|G(s, t)u\|^m \sigma_d(du) \frac{\nu(ds)}{\sqrt{\det K(s, s)}} \frac{\nu(dt)}{\sqrt{\det K(t, t)}},$$

$$f(u, s, t) = \|G(s, t)u\|,$$

$$c_n = (1 + n + m)^{\alpha+d/2-1}$$

□

We obtained a sufficient condition for the existence of local time. In fact if $\mu = \delta_0$ then this condition is also necessary. We are going to prove that and even more: if (5.3) is not true then there is no convergence for local time approximations.

Denote

$$L_m(f) = \sum_{n=m}^{+\infty} a_n(f)$$

Note that f_ε is even function from any coordinate and formula (3.1) shows that $a_n(f_\varepsilon) = 0$ for odd n . It means that for odd m we have $L_m(f_\varepsilon) = L_{m+1}(f_\varepsilon)$ so it make sense to consider only even m .

Theorem 5.4. *If for some even m the family $L_m(f_\varepsilon)$ converge in $D_{2,\alpha}$ as $\varepsilon \rightarrow 0+$ or if $L_m(\delta_0)$ exists as an element of $D_{2,\alpha}$ then the condition (5.3) holds.*

Proof. From (4.10) it is easy to see that

$$\begin{aligned} \|L_m(f_\varepsilon)\|_{2,\alpha}^2 &= \sum_{n=m/2}^{\infty} \frac{(1+2n)^\alpha}{(2\pi)^{-2d}(2n)!} \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (G(s, t)u, v)^{2n} e^{-\varepsilon^2 \|K^{-1/2}(s,s)u\|^2} \\ &\cdot e^{-\varepsilon^2 \|K^{-1/2}(t,t)v\|^2/2} e^{-\|u\|^2/2 - \|v\|^2/2} dudv \frac{\nu(ds)}{\sqrt{\det K((s, s))}} \frac{\nu(dt)}{\sqrt{\det K((t, t))}} \end{aligned}$$

The expression under the integral is always non-negative and monotonously convergent as $\varepsilon \rightarrow 0+$ Consequently for all n the integral converges to (up to the constant)

$$\int_T \int_T N_{2n}(G(s, t), 0, 0) \frac{\nu(ds)}{\sqrt{\det K((s, s))}} \frac{\nu(dt)}{\sqrt{\det K((t, t))}} = C(n, d) J_{2n}$$

and we can estimate the supremum of the whole sum:

$$+\infty > \sup_{\varepsilon>0} \|L_m(f_\varepsilon)\|_{2,\alpha}^2 \geq \sum_{n=m/2}^{\infty} (1+2n)^\alpha (2\pi)^{-d} C(n, d) J_{2n}$$

From theorem 4.3 we can see that if $L(\delta_0)$ is an element of $D_{2,\alpha}$ then:

$$\|L(\delta_0)\|_{2,\alpha} = \sum_{n=m/2}^{\infty} (1+2n)^\alpha (2\pi)^{-d} C(n, d) J_{2n} < +\infty$$

So in both cases we obtained that the sum in (5.1) is finite if we take only terms with even n . But since $\|G(s, t)x\| \leq 1$, we have that $J_{2n+1} \leq J_{2n}$ and consequently $J_{2n+1}(2n+1)^{\alpha+d/2-1} \leq C J_{2n}(2n+1)^{\alpha+d/2-1}$, i.e. condition (5.3) holds. □

If we join Theorems 5.1, 5.3 and 5.4, we obtain following result.

Theorem 5.5. *The family $L_m(f_\varepsilon)$ converge in $D_{2,\alpha}$ as $\varepsilon \rightarrow 0+$ if and only if condition (5.3) holds.*

If the family $L_m(f_\varepsilon)$ converge in $D_{2,\alpha}$ we call its limit renormalized local time at zero and say that renormalized local time at zero exists in $D_{2,\alpha}$. Note that this limit is in fact equal to $L_m(\delta_0)$.

6. Local Time for Fractional Brownian Motion.

We are going to investigate the case of local time and self-intersection local time for fractional Brownian motion. In both cases all our assumptions on the process ξ are easy to check and we want to find all values of parameters such that condition (5.3) holds, because by theorem 5.5 this condition is equivalent to the existence of local time at zero.

Let $X(t), t > 0$ be d -dimensional fractional Brownian motion, i.e. we have d independent copies of Gaussian process with zero mean and covariation

$$r_H(s, t) = (|s|^{2H} + |t|^{2H} - |t - s|^{2H})/2$$

where $H \in (0, 1)$ is Hurst parameter.

First we consider local time of $X(t)$ itself.

Theorem 6.1. *Let $T = [0, 1]; \nu(dt) = dt; \xi(t) = X(t)$. The condition (5.3) holds if and only if*

$$d < \frac{1}{H}, \quad \alpha < \frac{1}{2H} - \frac{d}{2}.$$

If $T = [1/2, 1]$ then (5.3) is equivalent to

$$\alpha < \frac{1}{2H} - \frac{d}{2}.$$

We choose two different T to show that one of the conditions on the parameters is connected to the behaviour of the covariation of the process at the point $t = 0$. We can see that the condition $d < \frac{1}{H}$ is not needed if we exclude some neighbourhood of $t = 0$ from T .

Proof. We have $K(s, t) = r_H(s, t)I$ and

$$G(s, t) = r_H(s, t)(r_H(s, s)r_H(t, t))^{-1/2}I = \frac{|s|^{2H} + |t|^{2H} - |t - s|^{2H}}{2t^H s^H}I$$

so the integral by u in the condition (5.3) disappears and for $\alpha > -\frac{d}{2}$ we get the following integral:

$$\int_0^1 \int_0^1 \left(\frac{s^{2H} + t^{2H} - |s - t|^{2H}}{2s^H t^H} \right)^m \cdot \left(\frac{|s - t|^{2H} - (s^H - t^H)^2}{2s^H t^H} \right)^{-\alpha - d/2} (st)^{-dH} dt ds$$

We use polar coordinates: $s = r \sin \phi, t = r \cos \phi$ and reduce the domain of the integration. If we extend the domain of the integration we will obtain similar integrals, so conditions for the finiteness are equivalent.

$$\int_0^1 \int_0^{\pi/2} \left(\frac{(\cos \phi)^{2H} + (\sin \phi)^{2H} - |\cos \phi - \sin \phi|^{2H}}{2(\cos \phi \sin \phi)^H} \right)^m \cdot \left(\frac{|\cos \phi - \sin \phi|^{2H} - (\cos^H \phi - \sin^H \phi)^2}{2 \cos^H \phi \sin^H \phi} \right)^{-\alpha-d/2} \frac{(\cos \phi \sin \phi)^{-dH}}{r^{2dH-1}} d\phi dr$$

Our integral becomes product of two integrals. The integral by r is finite, if (and only if) $dH < 1$. The integral by ϕ can be infinite only near $\phi = 0, \phi = \frac{\pi}{2}$ and $\phi = \frac{\pi}{4}$, because the expression under the integral is continuous and finite outside some neighbourhood of these points. Since

$$\frac{|\cos \phi - \sin \phi|^{2H} - (\cos^H \phi - \sin^H \phi)^2}{2 \cos^H \phi \sin^H \phi} = 1 - \frac{\sin^H \phi}{2 \cos^H \phi} + \frac{|\cos \phi - \sin \phi|^{2H} - \cos^{2H} \phi}{2 \cos^H \phi \sin^H \phi} \sim 1 - \frac{2H \cos^{2H-1} \phi \sin \phi}{2 \cos^H \phi \sin^H \phi} \rightarrow 1$$

as $\phi \rightarrow 0+$, then the integral by ϕ is finite in the neighbourhood of $\phi = 0$ if $dH < 1$ (because only one important multiplier under the integral is $(\sin \phi)^{-dH}$). The behavior of the integral in the neighbourhood of $\phi = \frac{\pi}{2}$ is similar. Now we investigate the behaviour in the neighbourhood of $\phi = \frac{\pi}{4}$:

$$|\cos \phi - \sin \phi|^{2H} - (\cos^H \phi - \sin^H \phi)^2 \sim 2^H \left| \phi - \frac{\pi}{4} \right|^{2H} - H 2^{2H-1} \left(\phi - \frac{\pi}{4} \right)^2 \sim \sim 2^H \left| \phi - \frac{\pi}{4} \right|^{2H}, \phi \rightarrow \frac{\pi}{4}.$$

From this relation we get sufficient condition for the finiteness of the integral: $2H(-\alpha - \frac{d}{2}) > -1$. It is also necessary, because the expression taken to power m can not refine this singularity.

Note that the case $\alpha \leq -\frac{d}{2}$ (when condition (5.3) has slightly different form) can be treated similarly since the only change is disappearance of singularity near $\phi = \pi/4$.

Now let $T = [1/2, 1]$. In this case points $s = 0$ and $t = 0$ lie outside of the domain of the integration, so after we apply the same transformations as above, we obtain that sets $\{r = 0\}, \{\phi = 0\}$ and $\{\phi = \frac{\pi}{2}\}$ are outside the closure of the domain of the integration. The condition for finiteness of integral here is $\alpha < \frac{1}{2H} - \frac{d}{2}$. □

Now we turn to the interesting case of self-intersection for fractional Brownian motion.

Theorem 6.2. *Let $T = [0, 1]^2; \nu(dt) = dt_1 dt_2; \xi(t) = X(t_1) - X(t_2)$. The condition (5.3) holds if and only if*

$$\alpha < \frac{1}{H} - \frac{d}{2}, \quad d < \frac{3}{2H}, \quad m > \frac{dH - 1}{1 - H}.$$

If $T = [0, 1/3] \times [2/3, 1]$ then (5.3) is equivalent to $\alpha < \frac{1}{H} - \frac{d}{2}$.

This theorem allows us to generalize results from [4]. Together with theorem 5.5 it gives necessary and sufficient conditions for the existence of self-intersection local time (at zero) for fractional Brownian motion. The second case in the theorem illustrates that conditions $d < \frac{3}{2H}$ and $m > \frac{dH-1}{1-H}$ are not needed if we exclude some neighbourhood of the set $\{s = t\}$ from T .

We need to describe the behaviour of the following function:

$$D_\gamma(x, y, z) = |x + y|^\gamma + |x + z|^\gamma - |x|^\gamma - |x + y + z|^\gamma \tag{6.1}$$

Lemma 6.3. (1) For all $\gamma \in (0, 1) \cup (1, 2]$ there exist positive constants C_1 and C_2 such that if $3|y| < |x|, 3|z| < |x|$, then

$$C_1|yz||x|^{\gamma-2} \leq |D_\gamma(x, y, z)| \leq C_2|yz||x|^{\gamma-2}$$

If $\gamma = 1$ then for same x, y, z we have $D_\gamma(x, y, z) = 0$.

(2) If $\gamma \in (1, 2)$ then

$$|D_\gamma(x, y, z)| \leq 2^{1-\gamma}(|y| + |z|)^\gamma - ||y| - |z||^\gamma$$

For $y, z > 0$ the equality holds if and only if $x = -(y + z)/2$.

(3) If $\gamma \in (0, 1)$ then

$$|D_\gamma(x, y, z)| \leq |y|^\gamma + |z|^\gamma - ||y| - |z||^\gamma$$

For $y, z > 0$ the equality holds if and only if $x = -y, -z$.

(4) If $\gamma = 1$ then

$$|D_\gamma(x, y, z)| \leq |y| + |z| - ||y| - |z||$$

For $y, z > 0$ the equality holds if and only if $x \in [-\max(y, z), -\min(y, z)]$.

Proof. First inequality is equivalent to

$$C_1|uv| \leq |D_\gamma(1, u, v)| \leq C_2|uv|; |u| < \frac{1}{3}, |v| < \frac{1}{3}$$

Because u, v is small enough by our assumption it is possible to use the following representation:

$$D_\gamma(x, y, z) = -\gamma(\gamma - 1) \int_0^{|y|} \int_0^{|z|} (x + \text{sign}(y)v + \text{sign}(z)w)^{\gamma-2} dv dw$$

which is valid for all $x > \max(0, -y, -z, -y - z)$ (here we denote $\text{sign}(y) = 1$ if $y \geq 0$ and $\text{sign}(y) = -1$ if $y < 0$). We have

$$\begin{aligned} |D_\gamma(1, u, v)| &= \left| \gamma(\gamma - 1) \int_0^{|u|} \int_0^{|v|} (1 + \text{sign}(u)y + \text{sign}(v)w)^{\gamma-2} dy dw \right| \leq \\ &\leq |\gamma(\gamma - 1)uv| \sup_{|y| < 1/3; |w| < 1/3} (1 + y + w)^{\gamma-2} \leq 3^{2-\gamma} |\gamma(\gamma - 1)| |uv| \end{aligned}$$

and similarly

$$|D_\gamma(1, u, v)| \geq \left(\frac{5}{3}\right)^{2-\gamma} |\gamma(\gamma - 1)| |uv|$$

It is easy to see that for $\gamma = 1$ we have $D_\gamma(1, u, v) = 0$ under the same assumptions on u, v .

To prove other inequalities it is enough to consider the case $0 < y \leq z$. Indeed if we have both y and z negative then we can introduce new variables $\tilde{x} = -x, \tilde{y} = -y, \tilde{z} = -z$ and obtain the same inequality for positive variables. If $y < 0$ and $z > 0$ then another change of variables helps: $\tilde{x} = x + y, \tilde{y} = -y, \tilde{z} = z$. All other cases are similar because of the symmetry with respect to the exchange $y \leftrightarrow z$. Moreover if $y = 0$ then all inequalities are trivial equalities, so we may suppose that $y \neq 0$. Now we can prove these inequalities by finding maximum and minimum of $D_\gamma(x, y, z)$ for fixed y and z . Using the convexity of the power function we can find the sign of the derivative of $D_\gamma(x, y, z)$ by x (at points where this derivative exists). We also know that $D_\gamma(x, y, z)$ is continuous and that $D_\gamma(x, y, z) \rightarrow 0, |x| \rightarrow \infty$ (from the first inequality). We discover that for $\gamma \in [1, 2)$ the function has its minimum at $x = \frac{-y-z}{2}$ and always negative. The right side of inequality is exactly $|D_\gamma(\frac{-y-z}{2}, y, z)|$, so it is proved. For $\gamma \in (0, 1]$ we find maximum at $x = 0; x = -y - z$ and minimum at $x = -y; x = -z$ (the values in both points of each pair are the same). Comparing the modulus of the function values at these points we get $|D_\gamma(x, y, z)| \leq |D_\gamma(-y, y, z)|$. The maximum is achieved only at points $x = \frac{-y-z}{2}$ for $\gamma \in (1, 2)$ and in $x = -y; x = -z$ for $\gamma \in (0, 1)$. For $\gamma = 1$ the maximum is achieved on the interval $x \in [-z, -y]$. Lemma is proved. \square

Proof of theorem 6.2. The covariation of the process is given by $K(s, t) = k(s, t)I$, where $k(s, t) = \frac{1}{2}(|s_1 - t_2|^{2H} + |s_2 - t_1|^{2H} - |s_1 - t_1|^{2H} - |s_2 - t_2|^{2H})$ and therefore $G(s, t) = g(s, t)I$, where $g(s, t) = \frac{k(s, t)}{\sqrt{k(s, s)k(t, t)}} = \frac{D_{2H}(s_1 - t_1, s_2 - s_1, t_1 - t_2)}{2|s_1 - s_2|^H |t_1 - t_2|^H}$ with $D_\gamma(x, y, z)$ as in 6.1. The integral from (5.3) for $\alpha > -\frac{d}{2}$ has form

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 |g(s, t)|^m |1 - |g(s, t)||^{-\alpha - \frac{d}{2}} |s_1 - s_2|^{-dH} |t_1 - t_2|^{-dH} dt_1 dt_2 ds_1 ds_2$$

There is no need to investigate the case $\alpha \leq -\frac{d}{2}$ separately, because, as we will show later, the conditions on the parameters that do not include α are necessary for the finiteness of the integral on the part of the domain, where $1 - |g(s, t)|$ is greater then some positive constant. By introducing new variables $x = s_1 - t_1, y = s_2 - s_1, z = t_1 - t_2$ and extending the domain of the integration we obtain

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \left| \frac{D_{2H}(x, y, z)}{2|yz|^H} \right|^m \left| 1 - \frac{|D_{2H}(x, y, z)|}{2|yz|^H} \right|^{-\alpha - \frac{d}{2}} |y|^{-dH} |z|^{-dH} dx dy dz$$

This new integral is finite if and only if the starting integral is finite (because we can obtain a similar integral by reducing the domain instead of extending). Now we change variables $x = ru, y = r \cos \phi, z = r \sin \phi$ and reduce the domain of the integration using additional constraint $y^2 + z^2 < 1$ (we again get the similar integral by extending the domain so this procedure is also two-way). Integrating by r we obtain a necessary condition for the finiteness: $dH < \frac{3}{2}$.

Now the integral has the following form:

$$\int_{-\infty}^{+\infty} \int_0^{2\pi} \left| \frac{D_{2H}(u, \cos \phi, \sin \phi)}{2|\cos \phi \sin \phi|^H} \right|^m \left| 1 - \frac{|D_{2H}(u, \cos \phi, \sin \phi)|}{2|\cos \phi \sin \phi|^H} \right|^{-\alpha - \frac{d}{2}} \cdot |\cos \phi|^{-dH} |\sin \phi|^{-dH} (\max(1, |u|))^{2dH-3} d\phi du$$

We restrict ourselves to the case $\phi \in (0, \frac{\pi}{4})$. Other parts of the domain can be treated similarly after change of variables in the integral (for example we can use $(u, \phi) \rightarrow (u, \pi/2 - \phi)$, $(u, \phi) \rightarrow (u + \sin \phi, -\phi)$, $(u, \phi) \rightarrow (u + \cos \phi, \pi - \phi)$). We split the domain of the integration $E = \{\phi \in (0, \frac{\pi}{4}); u \in \mathbb{R}^d\}$ into several parts:

$$\begin{aligned} E_1 &= \{|u| > M\} \cap E \\ E_2 &= \{|u| < M; \phi \geq \varepsilon, \frac{\pi}{4} - \phi \geq \varepsilon\} \cap E \\ E_3 &= \{|u| < M, |u| \geq \varepsilon, |1 + u| \geq \varepsilon; \phi < \varepsilon\} \cap E \\ E_4 &= \{|u|^2 + |\sin \phi|^2 < 4\varepsilon^2\} \cap E \\ E_5 &= \{|1 + u|^2 + |\sin \phi|^2 < 4\varepsilon^2\} \cap E \\ E_6 &= \{|u| < M, |u + \frac{1}{\sqrt{2}}| \geq \varepsilon; \frac{\pi}{4} - \phi < \varepsilon\} \cap E \\ E_7 &= \{|\frac{u}{\cos \phi} + 1|^2 + |tg \phi - 1|^2 < 16\varepsilon^2\} \cap E \end{aligned}$$

It is easy to check that the union of these sets covers E . We consider the integral on each E_i separately and prove that three conditions on parameters from the theorem statement are sufficient for the finiteness of the integral on each E_i and necessary at least on one of E_i .

We choose ε to be small enough (it is sufficient to take $\varepsilon < \frac{1}{16}$). We also choose M for $H \neq \frac{1}{2}$ such that for $|u| > M$ we have $C_2|u|^{2H-2} < \frac{1}{2}$ where C_2 is constant from first inequality from lemma 6.3. We use this inequality on E_1 and obtain that the finiteness of our integral is equivalent to the finiteness of two integrals $\int_M^\infty u^{m(2H-2)+2dH-3} du$ and $\int_0^{\pi/4} |\sin \phi|^{m(1-H)-dH} d\phi$. Both integrals are finite if and only if $m > \frac{dH-1}{1-H}$. If $H = \frac{1}{2}$ we can choose M such that $D_{2H}(u, \cos \phi, \sin \phi) = 0$ on E_1 . In this case it is easy to see that one of the conditions $dH < 1$ or $m > 0$ (one of these is always true if the conditions on the parameters from the theorem statement hold) is sufficient for the integral over E_1 to be finite. Now we will consider other parts of E one by one, but first we need to establish some inequalities using lemma 6.3.

We will prove that function $1 - \frac{|D_{2H}(u, \cos \phi, \sin \phi)|}{2|\cos \phi \sin \phi|^H}$ is greater then some positive constant on sets E_2, E_3, E_4, E_5, E_6 . From lemma 6.3 we get for $H \in (0, \frac{1}{2})$:

$$\begin{aligned} \frac{|D_{2H}(u, \cos \phi, \sin \phi)|}{2|\cos \phi \sin \phi|^H} &\leq \frac{|\cos \phi|^{2H} + |\sin \phi|^{2H} - |\cos \phi - \sin \phi|^{2H}}{2|\cos \phi \sin \phi|^H} = \\ &= 1 - \frac{1}{2}(1 + |ctg \phi - 1|^H - |ctg \phi|^H)(1 - |tg \phi|^H + |1 - tg \phi|^H) \leq 1 \end{aligned}$$

For $H \in (\frac{1}{2}, 1)$

$$\begin{aligned} \frac{|D_{2H}(u, \cos \phi, \sin \phi)|}{2|\cos \phi \sin \phi|^H} &\leq 2^{-2H} \frac{|\cos \phi + \sin \phi|^{2H} - |\cos \phi - \sin \phi|^{2H}}{|\cos \phi \sin \phi|^H} = \\ &= 2^{-2H} (|tg^2 \phi + ctg^2 \phi + 2|^H - |tg^2 \phi + ctg^2 \phi - 2|^H) \leq 1 \end{aligned}$$

For $H = \frac{1}{2}$ both inequalities are true. By simple calculations (using conditions from lemma 6.3) equality in these inequalities holds if and only if $\phi = \pi/4, u = -\frac{1}{\sqrt{2}}$. But function $1 - \frac{|D_{2H}(u, \cos \phi, \sin \phi)|}{2|\cos \phi \sin \phi|^H}$ is continuous on the closure of E (from inequalities above this function converges to 1 if $\phi \rightarrow 0+$) so outside some neighbourhood of $\phi = \pi/4, u = -\frac{1}{\sqrt{2}}$ on E it is greater then some positive constant. Consequently this expression taken to the power $-\alpha - d/2$ can be omitted in the integral.

For E_2 function under the integral is bounded so integral is always finite. For E_3, E_4, E_5 we have to deal only with the singularity of $|\sin \phi|^{-dH}$ (possibly refined by the renormalization multiplier).

For E_3 we have the following bound $|D_{2H}(u, \cos \phi, \sin \phi)| \leq C|\phi|$, where C is a constant which depends only on ε (can be proved by replacing the difference with an integral of a derivative). The condition $m > \frac{dH-1}{1-H}$ is sufficient for the finiteness of the integral. Note that for E_4, E_5 the similar inequality for $|D_{2H}(u, \cos \phi, \sin \phi)|$ is not true (for $2H < 1$), because the derivative by x of $|u+x|^{2H}$ and $|u+\cos \phi+x|^{2H}$ at zero blows up if $u = 0$ or $u = -\cos \phi$ respectively. Instead we have to use a change of the coordinates. For E_3 we can also prove that if $H = 1/2$ then condition $m > \frac{dH-1}{1-H}$ is necessary. It is enough to note that $|D_{2H}(u, \cos \phi, \sin \phi)| = 2 \sin \phi$ on $0 \leq \phi < \varepsilon; u \in (-\cos \phi, -\sin \phi)$.

For E_4 let $\sin \phi = r \sin \theta, u = r \cos \theta$. We obtain the following integral

$$\begin{aligned} &\int_0^{2\varepsilon} \int_0^{2\pi} \left(|r \cos \theta + r \sin \theta|^{2H} + |r \cos \theta + \sqrt{1 - (r \sin \theta)^2}|^{2H} - |r \cos \theta|^{2H} - \right. \\ &\quad \left. - |r \cos \theta + r \sin \theta + \sqrt{1 - (r \sin \theta)^2}|^{2H} \right)^m |\sin \theta|^{-dH - mH} r^{1 - dH - mH} d\theta dr \end{aligned}$$

Here we already dropped multipliers under the integral which are bounded above and below. The expression taken to power m is bounded by the following expression $C r^{\min(2H, 1)} |\sin \theta|$ (to prove it we again use derivative). Thus our integral is bounded by the product of two integrals $\int_0^{2\varepsilon} r^{\min(2H, 1)m + 1 - dH - mH} dr$ and

$\int_0^{2\pi} |\sin \theta|^{m - dH - mH} d\theta$. The conditions for their finiteness are $m(\min(2H, 1) - H) + 2 - dH > 0$ and $m(1 - H) - dH + 1 > 0$. Recall that we already have the conditions $dH < \frac{3}{2}$ and $m > \frac{dH-1}{1-H}$ as necessary and that $m \geq 0$. We can see that these conditions are sufficient for this case. The case of E_5 can be treated similarly. We let $\sin \phi = r \sin \theta, u + 1 = r \cos \theta$ and obtain the similar bound on the function under the integral.

For E_6 the function under the integral is bounded (as we have proved above). For E_7 the only unbounded multiplier in the integral is $\left|1 - \frac{|D_{2H}(u, \cos\phi, \sin\phi)|}{2|\cos\phi\sin\phi|^H}\right|^{-\alpha - \frac{d}{2}}$ and we have

$$2|\cos\phi\sin\phi|^H - |D_{2H}(u, \cos\phi, \sin\phi)| = \\ = |\cos\phi|^{2H}(2|1+w|^H - ||1+v+w|^{2H} + |v-1|^{2H} - |v|^{2H} - |v+w|^{2H}|)$$

where $w = tg\phi - 1$, $v = \frac{u}{\cos\phi} + 1$. We introduce w and v as a new variables of the integration and note that the expression above is equivalent to $|v|^{2H} + |v+w|^{2H}$ (multiplied by a constant), when $v^2 + w^2 \rightarrow 0+$. Using polar coordinates we obtain two integrals $\int_0^{4\varepsilon} r^{1-2H(\alpha + \frac{d}{2})} dr$ and $\int_0^{2\pi} (|\cos\theta|^{2H} + |\cos\theta + \sin\theta|^{2H})^{-\alpha - \frac{d}{2}} d\theta$. The second integral is always finite and the first one gives us necessary and sufficient condition $\alpha + \frac{d}{2} < \frac{1}{H}$.

If we set $T = [0, 1/3] \times [2/3, 1]$ then after a similar transformations we can get the same integral but on a different domain. This domain can be treated exactly like the union of E_2, E_6, E_7 above (for example the singularities near $s_1 = s_2$ and $t_1 = t_2$ and consequently near $\sin\phi = 0$ and $\cos\phi = 0$ are outside the domain of the integration) and we have the condition on α as necessary and sufficient for the finiteness of the integral. \square

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