

REPRESENTATIONS OF THE GEGENBAUER OSCILLATOR ALGEBRA AND THE OVERCOMPLETENESS OF SEQUENCES OF NONLINEAR COHERENT STATES

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ABSTRACT. The main purpose of this paper is to investigate a generalized oscillator algebra, naturally associated to the Gegenbauer polynomials and a related class of nonlinear coherent vectors. We derive their overcompleteness relation, in so doing, the partition of unity in terms of the eigenstates of the sequences of coherent vectors is established. It turns out that, in making up such relation, the positivity of the obtained measure is intimately connected, up to a conjecture, to the parameter appearing in the beta distribution. An example of complex hypercontractivity property for an “ultraspherical” Hamiltonian is developed to illustrate our theory.

1. Introduction

Coherent states representation [36] are ubiquitous in the mathematical physics literature. Yet there seems to be a lack of general theory in the context of representations of nonlinear coherent states. This paper is an attempt to partially fill this gap.

The original coherent states based on the Heisenberg-Weyl group has been extended for a number of Lie groups with square integrable representations [4], and they have many applications in quantum mechanics [24]. In particular, they are used as bases of coherent states path integrals [24] or dynamical wavepackets for describing the quantum systems in semiclassical approximations [31]. This framework has been given in a general and elegant mathematical form through the work of Perelomov [37].

Many definitions of coherent states exist. The first one, so-called Klauder-Gazeau type [23], defines the coherent states as the eigenstates of the annihilation operator A for each individual oscillator mode of the electromagnetic field; namely, a system of coherent states is defined to be a set $\{\Omega(z); z \in \mathfrak{X}\}$ of quantum states in some interacting Fock space Γ [3, 39], parameterized by some set \mathfrak{X} such that: (i) $A\Omega(z) = z\Omega(z)$, $\forall z \in \mathfrak{X}$, (ii) the map $z \mapsto \Omega(z)$ is smooth, and (iii) the system is overcomplete; i.e.

$$\int_{\mathfrak{X}} |\Omega(z)\rangle \langle \Omega(z)| \nu(dz) = I. \quad (1.1)$$

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Physicists usually call property (1.1) completeness relation.

A second definition of coherent states for oscillators, so-called Perelomov-type [37], assumes the existence of a Weyl-type unitary operators

$$W(z) = \exp(zA^\dagger - \bar{z}A)$$

acting on some interacting Fock space Γ . The coherent states parameterized by z are given by

$$\Omega(z) = W(z)\Phi_0, \quad z \in \mathfrak{X}.$$

Here the annihilation operator A and the creation operator A^\dagger are mutually adjoint and Φ_0 is the Fock vacuum.

A third definition is based on the Heisenberg uncertainty relation with the position operator Q and momentum operator P given, as usual, by

$$Q = \frac{i}{2}(A^\dagger + A) \quad \text{and} \quad P = \frac{i}{2}(A^\dagger - A),$$

the coherent states defined above have the minimum-uncertainty value $\Delta Q \Delta P = 1/2$ and maintain this relation in time (temporal stability of coherent states) [5].

Let us stress that coherent states have two important properties. First, they are not orthogonal to each other with respect to the positive measure in (1.1). Second, they provide a resolution of the identity, i.e., they form an overcomplete set of states in the interacting Fock space. In fact it is well known that they form a ‘‘highly overcomplete’’ set in the sense that there are much smaller subsets of coherent states which are also overcomplete. Using them one can express an arbitrary state as a line integral of coherent states [40].

It is therefore clear that knowing that a set of coherent states is overcomplete is not only of theoretical interest; it is also of practical interest in the sense that we are encouraged to search for resolutions of the identity which will make possible the expansion of an arbitrary state in terms of these coherent states. Sometimes it is not easy to find a resolution of the identity and weaker concepts are also sufficient (for example the concept of frames in wavelets). But it is clear that a prerequisite for going down that route is the question of completeness.

In the present paper, considering only the first of the above mentioned definitions, we are concerned by a class of nonlinear coherent states: the Gegenbauer coherent states. These coherent states exhibit nonclassical features and can be realized physically as the stationary states of center-of-mass motion of a trapped ion [33].

The questions considered here are interrelated by four basic notions: orthogonal polynomials, oscillator algebra, overcompleteness relation and, finally, hypercontractivity.

For the Gegenbauer orthogonal polynomials system on real line, we construct an appropriate oscillator algebra such that these polynomials make up the eigenfunctions system of the associated oscillator hamiltonian. The Gegenbauer oscillator algebras are free algebras generated by four operators:

- the annihilation operator A and the creation operator A^\dagger , which are mutually adjoint,

- the self-adjoint preservation operator N , and the unity I , satisfying the following commutation relations:

$$[N, A] = -A, \quad [N, A^\dagger] = A^\dagger, \quad A A^\dagger = \Theta(N), \quad A^\dagger A = \Theta(N + I), \quad (1.2)$$

where Θ is a real analytical function. By definition, we qualify *nonlinear coherent states* those associated with oscillator algebra characterized by (1.2) with Θ being a nonlinear (non constant) real function.

Notice that, when $\Theta(s) = \sigma s + \tau$, we recover the commutation relations of the usual harmonic oscillator algebra. When $\Theta(s) = s(\sigma s + \tau)$, we recover the commutation relations of the generalized harmonic oscillator algebra of Meixner-type. A detailed study of this class is in progress and will appear in a forthcoming paper. In the Gegenbauer class, the function Θ is given by

$$\Theta(s) = \frac{s(s-1+2\beta)}{4(s+\beta)(s-1+\beta)}, \quad s \geq 1, \quad \beta > -\frac{1}{2},$$

see Section 2 and Section 3 below.

Our approach aims at generalizing the pioneering work of Bargmann [11] for the usual harmonic oscillator. It is well-known that the classical Segal-Bargmann transform in Gaussian analysis yields a unitary map of the L^2 space of the Gaussian measure on \mathbb{R} onto the space of L^2 holomorphic functions of the Gaussian measure on \mathbb{C} , see [11, 12, 22, 26]. Later on, based on the work by Accardi-Bożejko [1], Asai [6] has extended the Segal-Bargmann transform to non-Gaussian cases. The crucial point is the introduction of a coherent vector as a kernel function in such a way that a transformed function, which is a holomorphic function on a certain domain, becomes a power series expression. Along this line, Asai-Kubo-Kuo [9] have considered the case of the Poisson measure compared with the case of the Gaussian measure. More recently, Asai [7, 8] has constructed a Hilbert space of analytic L^2 functions with respect to a more general family and give examples including Laguerre, Meixner and Meixner-Pollaczek polynomials. However, the Gegenbauer polynomials, and the overcompleteness property are beyond their scope. To our knowledge, these issues have not been systematically (and intrinsically) addressed in the literature before. In fact, the representations of coherent states are usually developed in Bosonic Bargmann Fock model. To our opinion this is rather unnatural, and the construction presented in this paper is directly connected to the beta-type distribution.

The present paper is organized as follows. In Section 2, we consider the Hilbert space of square integrable functions $L^2(\mathbb{R}, \mu_\beta)$ in which the normalized Gegenbauer polynomial system constitutes an orthonormal basis. Using the Poisson kernel, we define the generalized Fourier transform for this system of polynomials, which allows introducing the position and momentum operators. Then, considering the given L^2 -space as a realization of the Fock space, the creation and annihilation operators can be standardly constructed. Together with the standard number operator in this Fock space, they satisfy commutation relations that generalize the Heisenberg relations and generate a Lie algebra that we naturally call the Gegenbauer oscillator algebra \mathcal{A}_β . In Section 3, following [1] we explicit an equivalent irreducible unitary representation of \mathcal{A}_β on the basis of an adapted one-mode interacting Fock space. Sections 4, 5 and 6 are devoted to the true subject of this

paper. We shall first define a class of nonlinear coherent vectors and we study the associated Bargmann representations. Secondly, we shall derive their over-completeness relation. It turns out that the constructed measure determining a partition of unity for these family of coherent vectors is not always positive. We prove that this is intimately connected, up to a conjecture, to the parameter appearing in the beta distribution. In particular, we found the result proved by Bożejko in a private communication [17] asserting that the answer is negative for the arcsine case. Finally, in Section 6, to illustrate our main results, we give a specific example of complex hypercontractivity property for an “ultraspherical” Hamiltonian.

2. The Gegenbauer Oscillator Algebra

2.1. Gegenbauer polynomials. Let μ_β be the beta-type distribution with parameter $\beta > -\frac{1}{2}$ given by

$$\mu_\beta(dx) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \frac{1}{2})} (1 - x^2)^{\beta - \frac{1}{2}} \chi_{]-1, 1[}(x) dx, \tag{2.1}$$

where Γ is the gamma Euler function and $\chi_{]-1, 1[}$ is the indicator function of $] -1, 1[$. The following three important distributions are particular cases of (2.1).

- $\mu_{\frac{1}{2}}(dx) = \frac{1}{2} \chi_{]-1, 1[}(x) dx$: the uniform distribution;
- $\mu_0(dx) = \frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}} \chi_{]-1, 1[}(x) dx$: the arcsine distribution;
- $\mu_1(dx) = \frac{2}{\pi} \sqrt{1 - x^2} \chi_{]-1, 1[}(x) dx$: the famous Wigner semi-circle distribution occurring in random matrices and free probability.

As easily seen, μ_β has finite moments of all orders and the linear span of monomials x^n , $n = 0, 1, \dots$, is dense in $L^2(\mu_\beta) := L^2(\mathbb{R}, \mu_\beta)$. The inner product is defined by

$$\langle f, g \rangle_{L^2(\mu_\beta)} := \int_{\mathbb{R}} \overline{f(x)} g(x) \mu_\beta(dx), \quad f, g \in L^2(\mu_\beta).$$

From [10, 28], we recall the following useful background. The Gram-Schmidt procedure applied to $\{x^n, n = 0, 1, 2, \dots\}$ gives a complete system $\{P_{n,\beta}\}_{n=0}^\infty$ of orthogonal polynomials such that $P_{n,\beta}$ is of degree n with leading coefficient 1 and given explicitly by

$$P_{n,\beta}(x) = \frac{\Gamma(\beta)n!}{2^n \Gamma(n + \beta)} G_{n,\beta}(x), \quad n \geq 0,$$

where $G_{n,\beta}$ is the classical ultraspherical Gegenbauer polynomial [32] defined by

$$G_{n,\beta}(x) = \frac{(-1)^n \Gamma(\beta + \frac{1}{2}) \Gamma(n + 2\beta)}{2^n \Gamma(2\beta) \Gamma(n + \beta) + \frac{1}{2}} \frac{(1 - x^2)^{\frac{1}{2} - \beta}}{n!} D_x^n \left[(1 - x^2)^{n + \beta - \frac{1}{2}} \right].$$

From the general theory of orthogonal polynomials, it is well known, (see [21]), that there exist two sequences of real numbers $\alpha_{n,\beta} \in \mathbb{R}$, $\omega_{n,\beta} \geq 0$, so-called the Szegő-Jacobi parameters such that the following relations hold: for $n \geq 0$,

$$(x - \alpha_{n,\beta}) P_{n,\beta}(x) = P_{n+1,\beta}(x) + \omega_{n,\beta} P_{n-1,\beta}(x), \tag{2.2}$$

$$\langle P_{n,\beta}, P_{m,\beta} \rangle_{L^2(\mu_\beta)} = (\omega_{1,\beta} \omega_{2,\beta} \cdots \omega_{n,\beta}) \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker symbol and by convention

$$\omega_{0,\beta} = 1, \quad P_{-1,\beta} = 0, \quad P_{0,\beta} = 1.$$

Explicitly, we have

$$\omega_{n,\beta} = \frac{n(n-1+2\beta)}{4(n+\beta)(n-1+\beta)}, \quad n \geq 1 \tag{2.3}$$

and, from the fact μ_β is a symmetric measure,

$$\alpha_{n,\beta} = 0, \quad n \geq 0.$$

2.2. Gegenbauer oscillator algebra. Denote $Q_{0,\beta}(x) = 1$ and

$$Q_{n,\beta}(x) = (\omega_{1,\beta} \omega_{2,\beta} \cdots \omega_{n,\beta})^{-\frac{1}{2}} P_{n,\beta}(x), \quad n \geq 1.$$

Then $\{Q_{n,\beta}\}_{n=0}^\infty$ is a complete orthonormal system in $L^2(\mu_\beta)$ and the recurrence relation (2.2) becomes

$$xQ_{n,\beta}(x) = \Lambda_{n,\beta}Q_{n+1,\beta}(x) + \Lambda_{n-1,\beta}Q_{n-1,\beta}(x), \quad n \geq 1 \tag{2.4}$$

with

$$\Lambda_{n,\beta} := (\omega_{n+1,\beta})^{\frac{1}{2}}, \quad n \geq 0.$$

The relation (2.4) indicates a manner by which the position operator \mathbb{X}_β on $L^2(\mu_\beta)$, acts on the basis elements $\{Q_{n,\beta}\}_{n=0}^\infty$:

$$(\mathbb{X}_\beta Q_{n,\beta})(x) := xQ_{n,\beta}(x) = \Lambda_{n,\beta}Q_{n+1,\beta}(x) + \Lambda_{n-1,\beta}Q_{n-1,\beta}(x), \quad n \geq 0. \tag{2.5}$$

Now we want to define the momentum operator \mathbb{P}_β on $L^2(\mu_\beta)$. For this end we use the Poisson kernel $\kappa_\beta^z \in L^2(\mu_\beta) \otimes L^2(\mu_\beta)$ defined by

$$\kappa_\beta^z(x, y) := \sum_{n=0}^\infty z^n Q_{n,\beta}(x) Q_{n,\beta}(y), \quad z \in \mathbb{C}.$$

We define the integral kernel operator $\mathfrak{K}_{\beta,z}$ on $L^2(\mu_\beta)$ by

$$(\mathfrak{K}_{\beta,z}\phi)(y) := \int_{\mathbb{R}} \kappa_\beta^z(x, y)\phi(x)\mu_\beta(dx).$$

It is noteworthy that $\mathfrak{K}_{\beta,i}$ is a unitary operator on $L^2(\mu_\beta)$ (see [41, 42, 43]) and

$$(\mathfrak{K}_{\beta,i})^{-1} = [\mathfrak{K}_{\beta,i}]^* = \mathfrak{K}_{\beta,-i}. \tag{2.6}$$

The unitary operators $\mathfrak{K}_{\beta,i}$ and $\mathfrak{K}_{\beta,-i}$ are called the generalized *Fourier transform* and *inverse Fourier transform*, respectively. We shall denote $\mathfrak{K}_{\beta,i}$ simply by \mathfrak{K}_β .

We define the momentum operator \mathbb{P}_β on $L^2(\mu_\beta)$ by

$$\mathbb{P}_\beta = \mathfrak{K}_\beta^{-1} \mathbb{X}_\beta \mathfrak{K}_\beta.$$

The energy operator is then defined by

$$\mathbb{H}_\beta = \mathbb{X}_\beta^2 + \mathbb{P}_\beta^2. \tag{2.7}$$

Proposition 2.1. *The operators \mathbb{X}_β , \mathbb{P}_β and \mathbb{H}_β act on the basis elements of $L^2(\mu_\beta)$ by*

$$(1) \quad \mathbb{X}_\beta Q_{n,\beta} = \Lambda_{n,\beta}Q_{n+1,\beta} + \Lambda_{n-1,\beta}Q_{n-1,\beta};$$

- (2) $\mathbb{P}_\beta Q_{n,\beta} = i[-\Lambda_{n,\beta} Q_{n+1,\beta} + \Lambda_{n-1,\beta} Q_{n-1,\beta}]$;
 (3) $\mathbb{H}_\beta Q_{n,\beta} = 2(\omega_{n,\beta} + \omega_{n+1,\beta}) Q_{n,\beta}$.

Proof. The statement (1) follows directly from the definition of \mathbb{X}_β . We easily verify that

$$\mathfrak{K}_\beta(z) Q_{n,\beta} = z^n Q_{n,\beta}, \quad z \in \mathbb{C}, \quad n \geq 0.$$

Thus, by using (2.6), one calculate

$$\begin{aligned} \mathbb{P}_\beta Q_{n,\beta} &= \mathfrak{K}_\beta^{-1} [\mathbb{X}_\beta (i^n Q_{n,\beta})] \\ &= i^n \mathfrak{K}_\beta^{-1} [\Lambda_{n,\beta} Q_{n+1,\beta} + \Lambda_{n-1,\beta} Q_{n-1,\beta}] \\ &= i [-\Lambda_{n,\beta} Q_{n+1,\beta} + \Lambda_{n-1,\beta} Q_{n-1,\beta}]. \end{aligned}$$

This proves (2). The identity (3) follows immediately from (1), (2) and (2.7). \square

It is noteworthy that relation (3) tells us that the basis vectors are eigenfunctions of the self-adjoint operator \mathbb{H}_β .

Definition 2.2. The creation and annihilation operators are defined as follows

$$\mathbf{a}_\beta^\dagger = \frac{1}{2} (\mathbb{X}_\beta + i\mathbb{P}_\beta), \quad \mathbf{a}_\beta = \frac{1}{2} (\mathbb{X}_\beta - i\mathbb{P}_\beta).$$

Proposition 2.3.

$$\left(\mathbf{a}_\beta^\dagger\right)^* = \mathbf{a}_\beta, \quad \mathbf{a}_\beta^\dagger Q_{n,\beta} = \Lambda_{n,\beta} Q_{n+1,\beta}, \quad \mathbf{a}_\beta Q_{n,\beta} = \Lambda_{n-1,\beta} Q_{n-1,\beta};$$

$$\left[\mathbf{a}_\beta, \mathbf{a}_\beta^\dagger\right] = \Theta_\beta(\mathbf{n}_\beta + \mathbf{i}_\beta) - \Theta_\beta(\mathbf{n}_\beta); \quad (2.8)$$

$$\left[\mathbf{n}_\beta, \mathbf{a}_\beta^\dagger\right] = \mathbf{a}_\beta^\dagger, \quad \left[\mathbf{n}_\beta, \mathbf{a}_\beta\right] = -\mathbf{a}_\beta, \quad (2.9)$$

where \mathbf{n}_β is the standard number operator acting on basis vectors by

$$\mathbf{n}_\beta Q_{n,\beta} = n Q_{n,\beta}, \quad n \geq 0,$$

\mathbf{i}_β is the identity operator on $L^2(\mu_\beta)$ and Θ_β is the function given by

$$\Theta_\beta(s) = \frac{s(s-1+2\beta)}{4(s+\beta)(s-1+\beta)}. \quad (2.10)$$

Proof. A straightforward verification. \square

The right hand side of (2.8) is uniquely determined by the spectral theorem. In fact, (2.8) can be regarded as a generalization of the usual boson Fock CCR. The label * means the conjugation operation and operators are represented by the same notations as their closures.

Definition 2.4. The Lie algebra generated by the operators $\mathbf{a}_\beta^\dagger, \mathbf{a}_\beta, \mathbf{n}_\beta, \mathbf{i}_\beta$ with commutation relations (2.8) and (2.9) is called *the Gegenbauer oscillator algebra* and will be denoted \mathcal{A}_β .

3. One-mode Interacting Fock Space Representation

In this Section we shall give a unitary equivalent irreducible Fock representation of the algebra \mathcal{A}_β . Let us consider the Hilbert space \mathcal{K} to be the complex numbers which, in physical language, corresponds to a 1-particle space in zero space-time dimension. In this case, for each $n \in \mathbb{N}$, also $\mathcal{K}^{\otimes n}$ is 1-dimensional, so we identify it to the multiples of a number vector denoted by $\Phi^{+(n)}$. The pre-scalar product on $\mathcal{K}^{\otimes n}$ can only have the form:

$$\langle z, w \rangle_{\otimes n} := \lambda_n \bar{z} w, \quad z, w \in \mathbb{C},$$

where the λ_n 's are positive numbers.

According to our setting, we define the sequence $\lambda_\beta = \{\lambda_{n,\beta}\}_{n=0}^\infty$ by

$$\lambda_{n,\beta} := \omega_{0,\beta} \omega_{1,\beta} \cdots \omega_{n,\beta}, \quad n \geq 0.$$

From (2.3), $\lambda_{n,\beta}$ can be rewritten explicitly

$$\lambda_{n,\beta} = \frac{\beta \Gamma(n + 2\beta) \Gamma(\beta)^2 n!}{2^{2n} (n + \beta) \Gamma(2\beta) \Gamma(n + \beta)^2}, \quad n \geq 0. \tag{3.1}$$

Definition 3.1. The *beta-type one-mode interacting Fock space*, denoted Γ_β , is the Hilbert space given by taking quotient and completing the orthogonal direct sum

$$\bigoplus_{n=0}^\infty (\mathcal{K}^{\otimes n}, \langle \cdot, \cdot \rangle_{\otimes n, \beta})$$

where $\langle \cdot, \cdot \rangle_{\otimes n, \beta} \equiv \langle \cdot, \cdot \rangle_{\otimes n}$ with the choice (3.1).

Denote $\Phi_0 = \Phi^{+(0)}$ the vacuum vector and, for $n \geq 1$, $\Phi_n = \Phi^{+(n)}$. For two elements $\Phi = \sum_{n=0}^\infty a_n \Phi_n$, $\Psi = \sum_{n=0}^\infty b_n \Phi_n$ in Γ_β , we have

$$\langle \Phi, \Psi \rangle_{\Gamma_\beta} = \sum_{n=0}^\infty \lambda_{n,\beta} \bar{a}_n b_n. \tag{3.2}$$

The creation operator is the densely defined operator A_β^\dagger on Γ_β satisfying

$$A_\beta^\dagger : \Phi_n \longmapsto \Phi_{n+1}, \quad n \geq 0.$$

The annihilation operator A_β is given, according to the scalar produces (3.2) as the adjoint of A_β^\dagger , by $A_\beta \Phi_0 = 0$ and

$$A_\beta \Phi_{n+1} = \frac{\lambda_{n+1,\beta}}{\lambda_{n,\beta}} \Phi_n, \quad n \geq 0. \tag{3.3}$$

Hence, for any $n \geq 0$,

$$A_\beta A_\beta^\dagger(\Phi_n) = \frac{\lambda_{n+1,\beta}}{\lambda_{n,\beta}} \Phi_n, \quad A_\beta^\dagger A_\beta(\Phi_n) = \frac{\lambda_{n,\beta}}{\lambda_{n-1,\beta}} \Phi_n$$

and the following relations arise

$$A_\beta A_\beta^\dagger = \frac{\lambda_{N_\beta + I_\beta}}{\lambda_{N_\beta}}, \quad A_\beta^\dagger A_\beta = \frac{\lambda_{N_\beta}}{\lambda_{N_\beta - I_\beta}}, \tag{3.4}$$

where I_β is the identity operator on Γ_β , N_β is the number operator satisfying $N_\beta \Phi_n = n\Phi_n$, $n \geq 0$, and the right hand side of (3.4) is uniquely determined by the spectral theorem.

Proposition 3.2. *The Lie algebra generated by A_β^\dagger , A_β , N_β , I_β gives rise to a unitary equivalent irreducible representation of the Gegenbauer oscillator algebra \mathcal{A}_β .*

Proof. From the relations (2.3), (3.1) and (3.4) we read, for any $n \geq 0$,

$$\begin{aligned} [A_\beta, A_\beta^\dagger] \Phi_n &= \left(\frac{\lambda_{n+1,\beta}}{\lambda_{n,\beta}} - \frac{\lambda_{n,\beta}}{\lambda_{n-1,\beta}} \right) \Phi_n \\ &= (\omega_{n+1,\beta} - \omega_{n,\beta}) \Phi_n \\ &= [\Theta_\beta(N_\beta + I_\beta) - \Theta_\beta(N_\beta)] \Phi_n, \end{aligned}$$

where the last equality is in the sense of functional calculus and Θ_β is the function defined in (2.10). In a similar way we found

$$[N_\beta, A_\beta^\dagger] \Phi_n = A_\beta^\dagger \Phi_n, \quad [N_\beta, A_\beta] \Phi_n = -A_\beta \Phi_n, \quad \forall n \geq 0.$$

The irreducibility property is obvious from the completeness of the family $\{\Phi_n\}_{n=0}^\infty$ in Γ_β . \square

As it is expected, we want the Fock space representation of the Gegenbauer oscillator algebra to be unitary equivalent to the one in the $L^2(\mu_\beta)$ -space. This question is answered by Accardi-Bożejko isomorphism [1].

Now we state the result of Accardi and Bożejko : there exists a unitary isomorphism $\mathfrak{U}_\beta : \Gamma_\beta \longrightarrow L^2(\mu_\beta)$ satisfying the following relations:

- (1) $\mathfrak{U}_\beta \Phi_0 = 1$;
- (2) $\mathfrak{U}_\beta A_\beta^\dagger \mathfrak{U}_\beta^{-1} Q_{n,\beta} = \Lambda_{n,\beta} Q_{n+1,\beta}$;
- (3) $\mathfrak{U}_\beta (A_\beta + A_\beta^\dagger) \mathfrak{U}_\beta^{-1} Q_{n,\beta} = x Q_{n,\beta}$.

From the above relations (1)-(3) we observe that Accardi-Bożejko isomorphism is uniquely determined by the correspondences

$$\begin{aligned} \Phi_0 &\longmapsto Q_{0,\beta}; \\ \Phi_n &\longmapsto (\Lambda_{0,\beta} \Lambda_{1,\beta} \cdots \Lambda_{n-1,\beta}) Q_{n,\beta}, \quad n \geq 1. \end{aligned} \tag{3.5}$$

Moreover, the unitary equivalence between the two representations of the Gegenbauer oscillator algebra is obtained through the formulas

$$\mathfrak{U}_\beta A_\beta^\dagger \mathfrak{U}_\beta^{-1} = \mathfrak{a}_\beta^\dagger, \quad \mathfrak{U}_\beta A_\beta \mathfrak{U}_\beta^{-1} = \mathfrak{a}_\beta, \quad \mathfrak{U}_\beta N_\beta \mathfrak{U}_\beta^{-1} = \mathfrak{n}_\beta.$$

4. Bargmann Space and Holomorphic Representation

Let \mathfrak{X} be a connected open subset of \mathbb{C} and ν_β be an absolutely continuous positive radial measure on \mathfrak{X} with continuous Radon-Nikodym derivative

$$\Psi_\beta(|z|^2) = \frac{\nu_\beta(dz)}{dz},$$

where dz is the Lebesgue measure on \mathbb{C} . Our basic Hilbert space is the space $\mathcal{H}(\mathfrak{X}) \cap L^2(\nu_\beta)$ of square integrable analytic functions, ($\mathcal{H}(\mathfrak{X})$ is the fréchet space of all analytic functions in \mathfrak{X}). Clearly the space $\mathcal{H}(\mathfrak{X}) \cap L^2(\nu_\beta)$ is a closed subspace of $L^2(\nu_\beta)$.

In the remainder of this Section, we take the following “Bargmann-Segal” condition on ν_β ,

$$\langle z^n, z^m \rangle_{L^2(\nu_\beta)} = \lambda_{n,\beta} \delta_{n,m}, \quad \forall n, m = 0, 1, 2, \dots \tag{4.1}$$

Later on, we shall delve a little more in detailed study of the existence of such positive radial measure ν_β with condition (4.1).

Under the assumption (4.1), we denote $\mathcal{H}L_\beta^2(\mathfrak{X})$ the Hilbert space $\mathcal{H}(\mathfrak{X}) \cap L^2(\nu_\beta)$. $\mathcal{H}L_\beta^2(\mathfrak{X})$ is the (weighted) Bargmann space associated to the measure μ_β . It is easily seen that point evaluations $\delta_z : f \mapsto f(z)$, $z \in \mathfrak{X}$ are continuous and $\{(\lambda_{n,\beta})^{-1/2} z^n\}_{n=0}^\infty$ is a complete orthonormal basis for $\mathcal{H}L_\beta^2(\mathfrak{X})$. Therefore, the reproducing kernel is

$$K_\beta(z, w) = \sum_{n=0}^\infty \frac{\bar{z}^n w^n}{\|z^n\|_{\mathcal{H}L_\beta^2(\mathfrak{X})}^2} = \sum_{n=0}^\infty \frac{\bar{z}^n w^n}{\lambda_{n,\beta}}. \tag{4.2}$$

The convergence radius of the series

$$L_\beta(|z|^2) := K_\beta(z, z) = \sum_{n=0}^\infty \frac{|z|^{2n}}{\lambda_{n,\beta}} \tag{4.3}$$

is $R = \frac{1}{\sqrt{2}}$, for any $\beta > -\frac{1}{2}$. Hence, denoting $\mathfrak{X}_{\sqrt{2}} = \{z \in \mathbb{C}; |z| < \frac{1}{\sqrt{2}}\}$, the space $\mathcal{H}L_\beta^2(\mathfrak{X}_{\sqrt{2}})$ is given by

$$\mathcal{H}L_\beta^2(\mathfrak{X}_{\sqrt{2}}) = \left\{ F(z) = \sum_{n=0}^\infty a_n z^n, \text{ holomorphic on } \mathfrak{X}_{\sqrt{2}}, \sum_{n=0}^\infty \lambda_{n,\beta} |a_n|^2 < \infty \right\}.$$

The map

$$\mathcal{S}_\beta : \Phi_n \in \Gamma_\beta \mapsto z^n \in \mathcal{H}L_\beta^2(\mathfrak{X}_{\sqrt{2}})$$

can be uniquely extended to a unitary isomorphism. Moreover, the Lie algebra generated by $\mathcal{S}_\beta A_\beta^\dagger \mathcal{S}_\beta^{-1}$, $\mathcal{S}_\beta A_\beta \mathcal{S}_\beta^{-1}$ and $\mathcal{S}_\beta N_\beta \mathcal{S}_\beta^{-1}$ gives rise to a unitary equivalent irreducible representation of the Gegenbauer oscillator algebra \mathcal{A}_β . This representation is called Bargmann (or holomorphic) representation of the Gegenbauer oscillator algebra.

5. Gegenbauer Coherent States

As indicated in the introduction, we consider in this paper, the Klauder-Gazeau type coherent states, $\Omega_\beta(z)$, which are considered to satisfy the following conditions:

- for any z , $\Omega_\beta(z)$ is an eigenvector of A_β , i.e. $A_\beta \Omega_\beta(z) = z \Omega_\beta(z)$;
- normalization: for any z , $\|\Omega_\beta(z)\|_{\Gamma_\beta} = 1$;
- continuity with respect to the complex number z ;

- overcompleteness, i.e., there exist a positive radial measure $\nu_\beta(dz) = \Psi_\beta(|z|^2)dz$ for which we have the resolution of identity

$$\int_{\mathfrak{X}_{\sqrt{2}}} |\Omega_\beta(z)\rangle \langle \Omega_\beta(z)| \Psi_\beta(|z|^2) dz = I_\beta.$$

5.1. The Gegenbauer coherent vectors $\Omega_\beta(z)$.

Theorem 5.1. *The family of Gegenbauer coherent vectors $\Omega_\beta(z)$ is given by*

$$\Omega_\beta(z) = (L_\beta(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\lambda_{n,\beta}} \Phi_n, \quad z \in \mathfrak{X}_{\sqrt{2}}, \tag{5.1}$$

where L_β is defined in (4.3).

Proof. For $z \in \mathfrak{X}_{\sqrt{2}}$, let $\Omega_\beta(z) = \sum_{n=0}^{\infty} a_n \Phi_n$. By using (3.3) we have

$$\begin{aligned} A_\beta \Omega_\beta(z) &= \sum_{n=0}^{\infty} a_n A_\beta \Phi_n \\ &= \sum_{n=1}^{\infty} a_n \frac{\lambda_{n,\beta}}{\lambda_{n-1,\beta}} \Phi_{n-1} \\ &= \sum_{n=0}^{\infty} a_{n+1} \frac{\lambda_{n+1,\beta}}{\lambda_{n,\beta}} \Phi_n. \end{aligned}$$

From the equality $A_\beta \Omega_\beta(z) = z \Omega_\beta(z)$, we get

$$a_{n+1} = \frac{\lambda_{n+1,\beta}}{\lambda_{n,\beta}} z a_n, \quad \forall n \geq 0.$$

Hence, since $\lambda_{0,\beta} = 1$, we have

$$a_n = a_0 \frac{z^n}{\lambda_{n,\beta}}$$

and therefore

$$\Omega_\beta(z) = a_0 \sum_{n=0}^{\infty} \frac{z^n}{\lambda_{n,\beta}} \Phi_n.$$

By the normalization property of $\Omega_\beta(z)$, we shall choose $a_0 \in \mathbb{C}$ in such a way that

$$1 = \langle \Omega_\beta(z), \Omega_\beta(z) \rangle_{\Gamma_\beta} = |a_0|^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\lambda_{n,\beta}} = |a_0|^2 L_\beta(|z|^2),$$

which gives $|a_0| = (L_\beta(|z|^2))^{1/2}$. In conclusion, the eigenvectors of A_β are

$$\Omega_\beta(z) = (L_\beta(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\lambda_{n,\beta}} \Phi_n, \quad z \in \mathfrak{X}_{\sqrt{2}}$$

as desired. □

Through Accardi-Bożejko isomorphism (3.5) one can define the Gegenbauer coherent vectors $\Omega_\beta(z; x)$ in $L^2(\mu_\beta)$:

$$\Omega_\beta(z; x) = (L_\beta(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\lambda_{n,\beta}}} Q_{n,\beta}(x), \quad z \in \mathfrak{X}_{\sqrt{2}}, \quad |x| < 1. \quad (5.2)$$

In the following, for future use, we give representation of $\Omega_\beta(z; x)$ in terms of hypergeometric functions. Recall that the Jacobi Polynomials with parameters $a > -1, b > -1$ is defined by

$$P_n^{(a,b)}(x) = \binom{n+a}{n} {}_2F_1 \left(\begin{matrix} -n, n+a+b+1 \\ a+1 \end{matrix} \middle| \frac{1}{2}(1-x) \right) \quad (5.3)$$

where the hypergeometric function ${}_2F_1 \left(\begin{matrix} u, v \\ w \end{matrix} \middle| t \right)$ is given by

$${}_2F_1 \left(\begin{matrix} u, v \\ w \end{matrix} \middle| t \right) = \sum_{n=0}^{\infty} \frac{(u)_n (v)_n}{(w)_n} \frac{t^n}{n!}$$

and, for real number γ , the Pochhammer symbol $(\gamma)_n$ is defined by

$$(\gamma)_0 = 1 \quad \text{and} \quad (\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}, \quad n \geq 1.$$

If we replace the integer n by a real number $s > -1$ in the left hand side of (5.3) we obtain the Jacobi function

$$P_s^{(a,b)}(x) = \frac{\Gamma(s+a+1)}{\Gamma(s+1)\Gamma(a+1)} {}_2F_1 \left(\begin{matrix} -s, s+a+b+1 \\ a+1 \end{matrix} \middle| \frac{1}{2}(1-x) \right). \quad (5.4)$$

It is well known that the diagonal case, $a = b = \beta - \frac{1}{2}$, gives the Gegenbauer polynomials $G_{n,\beta}(x)$ up to normalization

$$\begin{aligned} G_{n,\beta}(x) &= \frac{(2\beta)_n}{(\beta + \frac{1}{2})_n} P_n^{(\beta-\frac{1}{2}, \beta-\frac{1}{2})}(x) \\ &= \frac{(2\beta)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n+2\beta \\ \beta + \frac{1}{2} \end{matrix} \middle| \frac{1}{2}(1-x) \right), \end{aligned} \quad (5.5)$$

where, we assume $\beta > 0$.

For $\beta > 0$, from Eq. (3.1) we read

$$\lambda_{n,\beta} = \frac{(2\beta)_n (2^{-n} n!)}{(\beta)_n (\beta+1)_n}. \quad (5.6)$$

It then follows, from (4.3),

$$\begin{aligned} L_\beta(|z|^2) &= \sum_{n=0}^{\infty} \frac{(\beta)_n (\beta+1)_n}{(2\beta)_n} \frac{(2|z|^2)^n}{n!} \\ &= {}_2F_1 \left(\begin{matrix} \beta, \beta+1 \\ 2\beta \end{matrix} \middle| 2|z|^2 \right). \end{aligned} \quad (5.7)$$

For $\beta = 0$, (arcsine case - Tchebychev polynomials of first kind), we easily have

$$L_0(|z|^2) = \frac{1}{1 - 2|z|^2}, \quad z \in \mathfrak{X}_{\sqrt{2}}. \quad (5.8)$$

From (5.3) and (5.5) the normalized Gegenbauer polynomials can be rewritten as

$$Q_{n,\beta}(x) = \sqrt{\frac{(2\beta)_n(n+\beta)}{\beta 2^n n!}} {}_2F_1\left(\begin{matrix} -n, n+2\beta \\ \beta + \frac{1}{2} \end{matrix} \middle| \frac{1}{2}(1-x)\right). \quad (5.9)$$

Now, for $\beta > 0$, substituting (5.7), (3.1) and (5.9) in (5.2), we get,

$$\begin{aligned} \Omega_\beta(z; x) &= \left[{}_2F_1\left(\begin{matrix} \beta, \beta+1 \\ 2\beta \end{matrix} \middle| 2|z|^2\right) \right]^{-1/2} \\ &\times \frac{{}_2F_1\left(\begin{matrix} \frac{\beta+1}{2}, \frac{\beta+2}{2} \\ \beta + \frac{1}{2} \end{matrix} \middle| \frac{2(x^2-1)z^2}{(1-\sqrt{2}xz)^2}\right)}{(1-\sqrt{2}xz)^{\beta+1}}. \end{aligned} \quad (5.10)$$

For the case $\beta = 0$, we have the generating function

$$\sum_{n=0}^{\infty} t^n Q_{n,0}(x) = \frac{1}{1 - 2tx + t^2}, \quad |t| < 1.$$

Put $t = \sqrt{2}z$. Then from (5.8) we deduce

$$\Omega_0(z; x) = \frac{(1 - 2|z|^2)^{1/2}}{1 - 2\sqrt{2}xz + 2z^2}, \quad z \in \mathfrak{X}_{\sqrt{2}}, \quad |x| < 1.$$

Summing up the above calculus, the following theorem is proved.

Theorem 5.2. *In $L^2(\mu_\beta)$, the Gegenbauer coherent vectors are give by*

(1) *For $\beta = 0$, (the arcsine case),*

$$\Omega_0(z; x) = \frac{(1 - 2|z|^2)^{1/2}}{1 - 2\sqrt{2}xz + 2z^2}, \quad z \in \mathfrak{X}_{\sqrt{2}}, \quad |x| < 1.$$

(2) *For $\beta > 0$,*

$$\begin{aligned} \Omega_\beta(z; x) &= \left[{}_2F_1\left(\begin{matrix} \beta, \beta+1 \\ 2\beta \end{matrix} \middle| 2|z|^2\right) \right]^{-1/2} \\ &\times \frac{{}_2F_1\left(\begin{matrix} \frac{\beta+1}{2}, \frac{\beta+2}{2} \\ \beta + \frac{1}{2} \end{matrix} \middle| \frac{2(x^2-1)z^2}{(1-\sqrt{2}xz)^2}\right)}{(1-\sqrt{2}xz)^{\beta+1}}. \end{aligned}$$

Remark 5.3. By using regularity properties of the hypergeometric series [28] and identity (5.7), we can easily prove the continuity of the map

$$\mathfrak{X}_{\sqrt{2}} \ni z \longmapsto \Omega_\beta(z) \in \Gamma_\beta$$

defined by (5.1).

5.2. Overcompleteness of the Gegenbauer coherent vectors. Now we shall investigate the Bargmann-Segal condition (4.1) in Section 4. This leads to the problem of constructing a positive radial measure $\nu_\beta(dz) = \Psi_\beta(|z|^2) dz$ in the partition of unity

$$\int_{\mathfrak{X}_{\sqrt{2}}} |\Omega_\beta(z)\rangle \langle \Omega_\beta(z)| \nu_\beta(dz) = I_\beta = \sum_{n=0}^\infty \left| \frac{\Phi_n}{\sqrt{\lambda_{n,\beta}}} \right\rangle \left\langle \frac{\Phi_n}{\sqrt{\lambda_{n,\beta}}} \right| \tag{5.11}$$

where, for vectors u, v, x , the one-rank projection $|u\rangle\langle v|$ is defined by $|u\rangle\langle v|x = \langle v, x\rangle u$.

For $z \in \mathfrak{X}_{\sqrt{2}}$, write $z = re^{i\theta}$ with $0 \leq r < \frac{1}{\sqrt{2}}, 0 \leq \theta \leq 2\pi$, then

$$\begin{aligned} \sum_{n=0}^\infty \left| \frac{\Phi_n}{\sqrt{\lambda_{n,\beta}}} \right\rangle \left\langle \frac{\Phi_n}{\sqrt{\lambda_{n,\beta}}} \right| &= \int_{\mathfrak{X}_{\sqrt{2}}} \Psi_\beta(|z|^2) |\Omega_\beta(z)\rangle \langle \Omega_\beta(z)| dz \\ &= \sum_{m=0}^\infty \sum_{n=0}^\infty \left\{ \frac{1}{2\lambda_{n,\beta}\lambda_{m,\beta}} \int_0^{1/\sqrt{2}} \left(\frac{\Psi_\beta(r^2)}{L_\beta(r^2)} \right) r^{n+m} d(r^2) \right. \\ &\quad \left. \times \int_0^{2\pi} e^{i(n-m)\theta} d\theta \right\} |\Phi_n\rangle \langle \Phi_m| \\ &= \sum_{n=0}^\infty \left\{ \frac{\pi}{\lambda_{n,\beta}} \int_0^{1/2} x^n \left(\frac{\Psi_\beta(x)}{L_\beta(x)} \right) dx \right\} \left| \frac{\Phi_n}{\sqrt{\lambda_{n,\beta}}} \right\rangle \left\langle \frac{\Phi_n}{\sqrt{\lambda_{n,\beta}}} \right| \end{aligned}$$

where we marked the change of variable $x = r^2$. Put $V_\beta(x) = \frac{\Psi_\beta(x)}{L_\beta(x)}$, one can deduce that the overcompleteness of the family of Gegenbauer coherent vectors is equivalent to the classical moment problem:

$$\int_0^{1/2} x^n V_\beta(x) dx = \frac{\lambda_{n,\beta}}{\pi}, \quad n = 0, 1, 2, \dots \tag{5.12}$$

Now we are in position to prove our two main theorems.

Theorem 5.4. *The radial measure ν_0 in the partition of unity satisfied by the Tchebychev (of the first kind) coherent vectors $\{\Omega_0(z)\}_{z \in \mathfrak{X}_{\sqrt{2}}}$ is given by*

$$\nu_0(dz) = \frac{1 - |z|^2}{1 - 2|z|^2} \left[\frac{8}{\pi} \delta_0(1 - 2|z|^2) - \frac{4}{\pi} \delta_0(2|z|^2) \right] dz, \quad z \in \mathfrak{X}_{\sqrt{2}}. \tag{5.13}$$

Proof. We recall that

$$\Omega_0(z) = [L_0(|z|^2)]^{-1/2} \sum_{n=0}^\infty \frac{z^n}{\lambda_n} \Phi_n$$

with

$$L_0(|z|^2) = \frac{1 - |z|^2}{1 - 2|z|^2} \quad \text{and} \quad \lambda_n \equiv \lambda_{n,0} = \begin{cases} 1 & \text{if } n = 0 \\ 2^{1-2n} & \text{otherwise} \end{cases}$$

In this case, after changing $t = 2x$, (5.12) becomes

$$\int_0^1 t^n V_0\left(\frac{t}{2}\right) dt = \begin{cases} \frac{2}{\pi} & \text{if } n = 0 \\ \frac{4}{\pi} & \text{if } n \geq 1 \end{cases}$$

The solution of this problem is obviously

$$V_0(x) = \frac{8}{\pi} \delta_0(2x - 1) - \frac{4}{\pi} \delta_0(2x), \quad x = \frac{t}{2}.$$

Finally, from the relation $W_0(x) = L_0(x)V_0(x)$ with $x = |z|^2$, we come to the desired identity (5.13). \square

Theorem 5.5. *Let $\beta > 0$. The radial measure ν_β in the partition of unity satisfied by the Gegenbauer coherent vectors $\{\Omega_\beta(z)\}_{z \in \mathfrak{X}_{\sqrt{2}}}$ is given by*

$$\begin{aligned} \nu_\beta(dz) &= \frac{2\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})} {}_2F_1\left(\begin{matrix} \beta, \beta+1 \\ 2\beta \end{matrix} \middle| 2|z|^2\right) \\ &\times \left\{ \left[(1-\beta)|z|^{4\beta-2} P_{1-\beta}^{(0,2\beta-1)}(4|z|^2-1) \right. \right. \\ &\quad \left. \left. - 2(\beta+1)|z|^{4\beta} P_{1-\beta}^{(1,2\beta)}(4|z|^2-1) \right] + 2\delta_0(2|z|^2-1) \right\} dz. \end{aligned} \quad (5.14)$$

Proof. In view of (5.6) we rewrite (5.12) as

$$\begin{aligned} \int_0^{1/2} x^n V_\beta(x) dx &= \frac{1}{\pi} \frac{n!(2\beta)_n}{(\beta)_n(\beta+1)_n} \\ &= \frac{1}{\pi} \frac{\Gamma(n+1)\Gamma(n+2\beta)}{\Gamma(n+\beta)\Gamma(n+\beta+1)} \frac{\Gamma(\beta)\Gamma(\beta+1)}{2^n\Gamma(2\beta)}. \end{aligned}$$

Hence

$$\int_0^{1/2} (2x)^n \frac{\pi\Gamma(2\beta)}{2\Gamma(\beta)\Gamma(\beta+1)} V_\beta(x) d(2x) = \frac{\Gamma(n+1)\Gamma(n+2\beta)}{\Gamma(n+\beta)\Gamma(n+\beta+1)}.$$

Put $t = 2x$ and $U_\beta(t) = \frac{\pi\Gamma(2\beta)}{2\Gamma(\beta)\Gamma(\beta+1)} V_\beta\left(\frac{t}{2}\right)$. Then we get

$$\int_0^1 t^n U_\beta(t) dt = \frac{\Gamma(n+1)\Gamma(n+2\beta)}{\Gamma(n+\beta)\Gamma(n+\beta+1)}, \quad n = 0, 1, 2, \dots \quad (5.15)$$

From the book [25], formula (7.512.4), we have the identity

$$\int_0^1 (1-x)^{n+2\beta-1} {}_2F_1\left(\begin{matrix} \beta-1, \beta+1 \\ 1 \end{matrix} \middle| x\right) dx = \frac{\Gamma(n+2\beta)\Gamma(n+2\beta+3)}{\Gamma(n+\beta+2)\Gamma(n+\beta)}.$$

Changing the variable $u = 1-x$ in this identity

$$\int_0^1 u^{n+2\beta-1} {}_2F_1\left(\begin{matrix} \beta-1, \beta+1 \\ 1 \end{matrix} \middle| 1-u\right) du = \frac{\Gamma(n+2\beta)\Gamma(n+2\beta+3)}{\Gamma(n+\beta+2)\Gamma(n+\beta)}. \quad (5.16)$$

From (5.4) we know that

$${}_2F_1\left(\begin{matrix} \beta-1, \beta+1 \\ 1 \end{matrix} \middle| \frac{1}{2}(1-x)\right) = P_{1-\beta}^{(0,2\beta-1)}(x),$$

then (5.16) becomes

$$\int_0^1 u^{n+2\beta-1} P_{1-\beta}^{(0,2\beta-1)}(1-u) du = \frac{\Gamma(n+1)\Gamma(n+2\beta)}{(n+\beta+1)\Gamma(n+\beta)\Gamma(n+\beta+1)}.$$

On the other hand, from the identity

$$P_s^{(a,b)}(1) = \frac{\Gamma(s+a+1)}{\Gamma(s+1)\Gamma(a+1)},$$

we have $P_{1-\beta}^{(0,2\beta-1)}(1) = 1$. Thus

$$\lim_{u \rightarrow 1^-} [u^{\beta-1} P_{1-\beta}^{(0,2\beta-1)}(2u-1)] = 1.$$

The derivative of the hypergeometric function ${}_2F_1$ is given by (see [32])

$$\frac{d}{dz} \left[{}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) \right] = \frac{ab}{c} {}_2F_1 \left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix} \middle| z \right),$$

we then obtain

$$\frac{d}{du} [P_{1-\beta}^{(0,2\beta-1)}(2u-1)] = (\beta+1)P_{-\beta}^{(1,2\beta)}(2u-1). \tag{5.17}$$

Now define

$$\begin{aligned} \phi_\beta(u) &:= -\frac{d}{du} [u^{\beta-1} P_{1-\beta}^{(0,2\beta-1)}(2u-1)] \\ &= (1-\beta)u^{\beta-2} P_{1-\beta}^{(0,2\beta-1)}(2u-1) - u^{\beta-1} \frac{d}{du} [P_{1-\beta}^{(0,2\beta-1)}(2u-1)]. \end{aligned} \tag{5.18}$$

Substituting (5.17) in (5.18) we get

$$\phi_\beta(u) = (1-\beta)u^{\beta-2} P_{1-\beta}^{(0,2\beta-1)}(2u-1) - (1+\beta)u^{\beta-1} P_{-\beta}^{(1,2\beta)}(2u-1).$$

Observe that the function ϕ_β verifies

$$\int_u^1 \phi_\beta(y) dy = u^{\beta-1} P_{1-\beta}^{(0,2\beta-1)}(2u-1) - 1, \tag{5.19}$$

then, (5.18) and (5.19) imply

$$\int_0^1 u^{n+\beta} \left(\int_u^1 \phi_\beta(y) dy + 1 \right) du = \frac{\Gamma(n+1)\Gamma(n+2\beta)}{(n+\beta+1)\Gamma(n+\beta)\Gamma(n+\beta+1)}.$$

A simple integration by parts gives

$$\int_0^1 u^{n+\beta+1} \phi_\beta(u) du + 1 = \frac{\Gamma(n+1)\Gamma(n+2\beta)}{\Gamma(n+\beta)\Gamma(n+\beta+1)}.$$

Comparing this last equality with (5.15) to get

$$U_\beta(u) = u^{\beta+1} \phi_\beta(u) + 2\delta_0(u-1).$$

By using the formula $\frac{\Gamma(2\beta)}{\pi\Gamma(\beta)\Gamma(\beta+1)} = 2^{2\beta-1}$ (see [25]) we obtain

$$V_\beta(t) = \frac{\Gamma(\beta+1)}{\sqrt{\pi} 2^{2\beta-1}\Gamma(\beta+\frac{1}{2})} [2^{\beta+1}t^{\beta+1}\phi_\beta(2t) + 2\delta_0(2t-1)]$$

from which, setting $t = 2x$, we have

$$V_\beta(t) = \frac{\Gamma(\beta + 1)}{\sqrt{\pi} 2^{2\beta-1} \Gamma(\beta + \frac{1}{2})} \left[(1 - \beta) 2^{2\beta-1} t^{2\beta-1} P_{1-\beta}^{(0, 2\beta-1)}(4t - 1) - (1 + \beta) 2^{2\beta} t^{2\beta} P_{-\beta}^{(1, 2\beta)}(4t - 1) + 2\delta_0(2t - 1) \right].$$

Taking into account (5.7), the radial measure in the partition of unity (5.11) is given by the formula (5.14). \square

Remark 5.6. In view of Theorem 5.5, for $\beta = 1$ we have $\nu_1(dz) = \Psi_1(|z|^2)dz$ with

$$\Psi_1(s) = \frac{4}{\pi} \frac{1}{1 - 2s} \delta_0(2s - 1).$$

Clearly ν_1 is an absolutely continuous positive measure on $\mathfrak{X}_{\sqrt{2}}$, so that the over-completeness property is satisfied, and then, Klauder-Gazeau type coherent states associated to tchebychev polynomials of second kind (or for Wigner semi-circle distribution) are well defined.

On the other hand, from (5.13), we should point out that ν_0 is not a positive measure, and therefore, the overcompleteness relation is not satisfied. So, the Klauder-Gazeau type coherent states associated to tchebychev polynomials of first kind doesn't exist. Such singularity property of the arcsine distribution was firstly pointed out by M. Bożejko in [17] and it was our motivation to consider the main problem of this paper.

For the general case, by using Maple 8, (package `Sumtools` [20, 35]), which contains summation algorithms for hypergeometric series, one can prove that for $\beta = 2^{k-1}$, $k \geq 0$ (k integer) the associated Klauder-Gazeau type coherent states are well defined. In particular, the answer is positive for Klauder-Gazeau Legendre coherent states ($\beta = 1/2$). Similarly, we prove that Gegenbauer coherent vectors fails to satisfy the overcompleteness property for $\beta = 2^{-k+1}$, $k > 2$ (k integer). Unfortunately, we have not found a proof at present for the other cases. However, the particular case $\beta = 2^{k-1}$, $k \geq 0$ is enough for the application we shall investigate in the next Section.

6. Complex Hypercontractivity

The quantum mechanical harmonic oscillator is essentially the Weyl representation of the Lie algebra associated to the Euclidean motion group. In Fock-Bargmann model, it can be described by the quadruple [11]

$$\{\mathcal{HL}^2(\mathbb{C}, \nu), \partial, \partial^\dagger, H\}$$

where

$$\mathcal{HL}^2(\mathbb{C}, \nu) = \left\{ f : \mathbb{C} \rightarrow \mathbb{C}, \text{ holomorphic, } \|f\|_2 := \left(\int_{\mathbb{C}} |f(z)|^2 \nu(dz) \right)^{1/2} < \infty \right\},$$

$$\partial f(z) = \frac{\partial}{\partial z} f(z), \quad \partial^\dagger f(z) = z f(z), \quad Hf(z) = z \frac{\partial}{\partial z} f(z),$$

and $\nu(dz) = \frac{1}{\pi} e^{-z\bar{z}} dz$ is the complex one-dimensional Gaussian measure. They satisfy the canonical commutation relations (CCR)

$$[\partial, \partial^\dagger] = I, \quad [\partial, I] = 0, \quad [\partial^\dagger, I] = 0 \tag{6.1}$$

and Wigner commutation relations (WCR)

$$[H, \partial] = -\partial, \quad [H, \partial^\dagger] = \partial^\dagger. \tag{6.2}$$

The Hamiltonian $H = \partial^\dagger \partial = z \frac{\partial}{\partial z}$ is diagonalized by the orthonormal basis $\{\frac{1}{\sqrt{n!}} z^n\}_{n=0}^\infty$ and has spectrum $\{0, 1, 2, \dots\}$. It is remarkable that the semigroup $\{T_t := e^{-tH}, t \geq 0\}$ enjoys the following complex hypercontractivity property: for t satisfying $e^{-2t} \leq \frac{p}{q}$, T_t is a contraction from $\mathcal{HL}^p(\mathbb{C})$ to $\mathcal{HL}^q(\mathbb{C})$, where, for an integer $p \geq 1$, $\mathcal{HL}^p(\mathbb{C})$ is the Banach space defined by

$$\mathcal{HL}^p(\mathbb{C}) = \left\{ f : \mathbb{C} \rightarrow \mathbb{C}, \text{ holomorphic, } \|f\|_p := \left(\int_{\mathbb{C}} |f(z)|^p \nu(dz) \right)^{1/p} < \infty \right\}.$$

This complex hypercontractivity plays an important role in the study of the Boson fields theory [27].

In the remainder of this paper we shall study a relativistic analogue for quantum mechanical Gegenbauer harmonic oscillator with ultraspherical phase space of one degree freedom $(\mathfrak{X}_{\sqrt{2}}, \nu_\beta(dz))$. According to Theorem 5.5 and Remark 5.6, from now on we suppose $\beta = 2^{k-1}$, $k \geq 0$ (k integer) so that ν_β is a positive measure.

For $p \geq 1$, let

$$\mathcal{HL}_\beta^p(\mathfrak{X}_{\sqrt{2}}) = \left\{ f : \mathfrak{X}_{\sqrt{2}} \rightarrow \mathbb{C}, \text{ holomorphic, } \|f\|_p := \left(\int_{\mathfrak{X}_{\sqrt{2}}} |f(z)|^p \nu_\beta(dz) \right)^{1/p} < \infty \right\},$$

then $\mathcal{HL}_\beta^p(\mathfrak{X}_{\sqrt{2}})$ is a Banach space and $\mathcal{HL}_\beta^2(\mathfrak{X}_{\sqrt{2}})$ is a reproducing kernel Hilbert space with kernel given by (4.2). Recall that an orthonormal basis for $\mathcal{HL}_\beta^2(\mathfrak{X}_{\sqrt{2}})$ is

$$\left\{ \phi_{n,\beta} := (\lambda_{n,\beta})^{-1/2} z^n, \quad n \geq 0 \right\}.$$

The geometric symmetry group of $\mathfrak{X}_{\sqrt{2}}$ [18] is

$$\mathcal{G}_{\sqrt{2}} = JSU(1, 1)J^{-1}$$

where

$$SU(1, 1) = \left\{ A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\} \quad \text{and} \quad J = \begin{pmatrix} \frac{1}{\sqrt{\sqrt{2}}} & 0 \\ 0 & \sqrt{\sqrt{2}} \end{pmatrix}.$$

Consider the following irreducible projective representation of $\mathcal{G}_{\sqrt{2}}$ (see [37])

$$(\Xi_\beta(X)f)(z) = \frac{f(X^{-1}z)}{(a - cz)^{2\beta+1}}, \quad X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}_{\sqrt{2}}, \quad f \in \mathcal{HL}_\beta^2(\mathfrak{X}_{\sqrt{2}}),$$

where $X^{-1}z$ is defined via the action of $\mathcal{G}_{\sqrt{2}}$ on $\mathfrak{X}_{\sqrt{2}}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

The matrix Lie algebra [19] of $\mathcal{G}_{\sqrt{2}}$ is generated by

$$B^- = J \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J^{-1}, \quad B^+ = J \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} J^{-1}, \quad M = J \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} J^{-1}$$

which are essentially Pauli matrices. Then, the associated representation, via discrete series [14, 15], is given by

$$\begin{aligned} -i \Xi_\beta^+ &:= \frac{d}{dt} \Big|_{t=0} \left(\Xi_\beta(e^{tB^+}) \right) = -i \left(\sqrt{2}(2\beta + 1)z + \sqrt{2}z^2 \frac{\partial}{\partial z} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial z} \right) \\ -i \Xi_\beta^- &:= \frac{d}{dt} \Big|_{t=0} \left(\Xi_\beta(e^{tB^-}) \right) = \sqrt{2}(2\beta + 1)z + \sqrt{2}z^2 \frac{\partial}{\partial z} - \frac{1}{\sqrt{2}} \frac{\partial}{\partial z} \\ \Xi_\beta^\circ &:= \frac{d}{dt} \Big|_{t=0} \left(\Xi_\beta(e^{tM}) \right) = -i \left((2\beta + 1) + 2z \frac{\partial}{\partial z} \right). \end{aligned}$$

On $\mathcal{HL}_\beta^2(\mathfrak{X}_{\sqrt{2}})$, Ξ_β^+ and Ξ_β^- are understood as position operator and momentum operator, respectively. Then, the creation and the annihilation operators are defined (modulo the factor $\sqrt{2}$), respectively, by

$$\begin{aligned} \partial_\beta^\dagger &:= \frac{1}{\sqrt{2}}(\Xi_\beta^+ - i\Xi_\beta^-) = 2(2\beta + 1)z + 2z^2 \frac{\partial}{\partial z} \\ \partial_\beta &:= \frac{1}{\sqrt{2}}(\Xi_\beta^+ + i\Xi_\beta^-) = \frac{\partial}{\partial z}. \end{aligned}$$

Set $N = z \frac{\partial}{\partial z}$ we get

$$[\partial_\beta, \partial_\beta^\dagger] = 2(2\beta + 1) + 2(2\beta + 1)N, \quad [N, \partial_\beta] = -\partial_\beta, \quad [N, \partial_\beta^\dagger] = \partial_\beta^\dagger.$$

Accordingly $\{\mathcal{HL}_\beta^2(\mathfrak{X}_{\sqrt{2}}), \partial_\beta, \partial_\beta^\dagger, N\}$ may be viewed as the Gegenbauer harmonic oscillator on $\mathfrak{X}_{\sqrt{2}}$. It is noteworthy that $\{\partial_\beta, \partial_\beta^\dagger\}$ does not satisfy the CCR (6.1), while $\{\partial_\beta, \partial_\beta^\dagger, N\}$ satisfy the WCR (6.2).

In the present context, the Hamiltonian is defined by

$$H_\beta = \partial_\beta^\dagger \partial_\beta = 2(2\beta + 1)z \frac{\partial}{\partial z} + 2z^2 \frac{\partial^2}{\partial z^2}.$$

By direct computation, we have

$$H_\beta \phi_{n,\beta} = 2n(n + 2\beta)\phi_{n,\beta}, \quad n = 0, 1, 2, \dots$$

Then, H_β is closely related to the ultraspherical operator

$$T_\beta := -2(1 - x^2) \frac{d^2}{dx^2} + (2\beta + 1)x \frac{d}{dx}$$

acting on $L^2(\mu_\beta)$. It is well-known that T_β is diagonalized by $\{Q_{n,\beta}\}$:

$$T_\beta Q_{n,\beta} = 2n(n + 2\beta)Q_{n,\beta},$$

so that, H_β and T_β have the same spectrum.

The map

$$\Upsilon_\beta : \phi_{n,\beta} \in \mathcal{HL}_\beta^2(\mathfrak{X}_{\sqrt{2}}) \longmapsto Q_{n,\beta} \in L^2(\mu_\beta)$$

can be uniquely extended to a unitary isomorphism denoted Υ_β again. We see that, under Υ_β , T_β goes over into H_β , namely, H_β and T_β are intertwined by Υ_β :

$$H_\beta = \Upsilon_\beta^{-1} T_\beta \Upsilon_\beta.$$

Let $P = \{P^t = e^{-tH_\beta}, t > 0\}$ be the semigroup generated by H_β . Motivated by the applications indicated in [27] we prove the following theorem.

Theorem 6.1. *If $e^{-\beta t} \leq 2^{-1/8}$, then P^t is a contraction from $\mathcal{HL}_\beta^2(\mathfrak{X}_{\sqrt{2}})$ to $\mathcal{HL}_\beta^4(\mathfrak{X}_{\sqrt{2}})$.*

Proof. Recall that $\{\phi_{n,\beta} = (\lambda_{n,\beta})^{-1/2} z^n; n \geq 0\}$ is an orthonormal basis for $\mathcal{HL}_\beta^2(\mathfrak{X}_{\sqrt{2}})$. Moreover, clearly $P^t \phi_{n,\beta} = e^{-2n(n+2\beta)t} \phi_{n,\beta}$. Then, for $f(z) = \sum_{n=0}^{\infty} \gamma_n \phi_{n,\beta} \in \mathcal{HL}_\beta^2(\mathfrak{X}_{\sqrt{2}})$, we have

$$(P^t f)(z) = \sum_{n=0}^{\infty} \gamma_n e^{-2n(n+2\beta)t} \phi_{n,\beta}.$$

Therefore, one can estimate

$$\begin{aligned} \|P^t f\|_4^4 &= \int_{\mathfrak{X}_{\sqrt{2}}} |(P^t f)(z)|^4 \nu_\beta(dz) = \int_{\mathfrak{X}_{\sqrt{2}}} \left((P^t f)(z) \overline{(P^t f)(z)} \right)^2 \nu_\beta(dz) \\ &= \int_{\mathfrak{X}_{\sqrt{2}}} \left\{ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} e^{-2(n_1^2+n_2^2)t} e^{-4\beta t(n_1+n_2)} \gamma_{n_1} \overline{\gamma_{n_2}} \phi_{n_1,\beta} \overline{\phi_{n_2,\beta}} \right\}^2 \nu_\beta(dz) \\ &= \int_{\mathfrak{X}_{\sqrt{2}}} \sum_{n_1, n_2, n_3, n_4=0}^{\infty} e^{-2t(\sum_{k=1}^4 n_k^2)} e^{-4\beta t(\sum_{k=1}^4 n_k)} \gamma_{n_1} \gamma_{n_2} \overline{\gamma_{n_3} \gamma_{n_4}} \\ &\quad \times \phi_{n_1,\beta} \phi_{n_2,\beta} \overline{\phi_{n_3,\beta} \phi_{n_4,\beta}} \nu_\beta(dz) \\ &= \int_{\mathfrak{X}_{\sqrt{2}}} \sum_{n_1, n_2, n_3, n_4=0}^{\infty} e^{-2t(\sum_{k=1}^4 n_k^2)} e^{-4\beta t(\sum_{k=1}^4 n_k)} \gamma_{n_1} \gamma_{n_2} \overline{\gamma_{n_3} \gamma_{n_4}} \\ &\quad \times (\lambda_{n_1,\beta} \lambda_{n_2,\beta} \lambda_{n_3,\beta} \lambda_{n_4,\beta})^{-1/2} z^{(n_1+n_2)} \overline{z}^{(n_3+n_4)} \nu_\beta(dz) \end{aligned}$$

Then, by using the obvious inequality $e^{-2t(\sum_{k=1}^4 n_k^2)} < 1$ and the assumption $e^{-\beta t} \leq 2^{-1/8}$, we get

$$\begin{aligned} \|P^t f\|_4^4 &\leq \sum_{n=0}^{\infty} e^{-8\beta tn} \sum_{n_1+n_2=n} \sum_{n_3+n_4=n} \gamma_{n_1} \gamma_{n_2} \overline{\gamma_{n_3} \gamma_{n_4}} \\ &\quad \times (\lambda_{n_1,\beta} \lambda_{n_2,\beta} \lambda_{n_3,\beta} \lambda_{n_4,\beta})^{-1/2} \lambda_{n,\beta} \\ &= \sum_{n=0}^{\infty} e^{-8\beta tn} \left| \sum_{n_1+n_2=n} \gamma_{n_1} \gamma_{n_2} \left(\frac{\lambda_{n,\beta}}{\lambda_{n_1,\beta} \lambda_{n_2,\beta}} \right)^{1/2} \right|^2 \\ &\leq \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n \left\{ \sum_{n_1+n_2=n} \frac{n!}{n_1! n_2!} \left(\frac{(n_1+\beta)(n_2+\beta)\Gamma(2\beta)\Gamma(n+2\beta)}{(n+\beta)\beta\Gamma(\beta)\Gamma(n_1+2\beta)\Gamma(n_2+2\beta)} \right) \right. \\ &\quad \left. \times \left(\frac{\Gamma(n_1+\beta)\Gamma(n_2+\beta)}{\Gamma(\beta)\Gamma(n+\beta)} \right)^2 \left(\sum_{n_1+n_2=n} |\gamma_{n_1} \gamma_{n_2}|^2 \right) \right\} \\ &\leq \sum_{n=0}^{\infty} \left(\sum_{n_1+n_2=n} \frac{n!}{n_1! n_2!} \left(\frac{1}{2} \right)^{n_1} \left(\frac{1}{2} \right)^{n_2} \right) \left(\sum_{n_1+n_2=n} |\gamma_{n_1} \gamma_{n_2}|^2 \right) \\ &= \|f\|_2^4. \end{aligned}$$

This completes the proof. \square

7. Concluding Remarks

In the present paper we have defined a new type of an oscillator for which the class of Gegenbauer polynomials play the same role as the Hermite polynomials play for standard boson oscillator. By solving the appropriate classical moment problem, we defined a distinguished set of coherent states of the Klauder-Gazeau type. Moreover, to illustrate our theory, we developed an example of complex hypercontractivity property for an ultraspherical Hamiltonian naturally associated to the family of orthogonal polynomials in consideration.

Unfortunately, the class of radial measure given by Eq. (5.14) is rather complicated in the general case. A complete description of the Gegenbauer coherent states of Klauder-Gazeau type in terms of the parameter appearing in the beta distribution is still open problem. A second open problem is indicated in Section 6 concerning the complex hypercontractivity property for quantum mechanical Gegenbauer harmonic oscillator with ultraspherical phase space of one degree freedom $(\mathfrak{X}_{\sqrt{2}}, \nu_{\beta}(dz))$. The research in this line is in progress.

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