Communications on Stochastic Analysis
Vol. 3, No. 2 (2009) 291-311

Serials Publications
www.serialspublications.com

MARKOVIAN PROPERTIES OF THE PAULI-FIERZ MODEL

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ABSTRACT. The Hamiltonian approach of the Pauli-Fierz model is studied by Derezinski and Jaksic (cf. [7]). In this paper, we give the Markovian description of this model. Using the weak coupling limit, we derive the quantum Markovian semigroup of the Pauli-Fierz system from its Hamiltonian description. Also, we give the explicit form of the associated Lindblad generator. As a consequence, we study the properties of the associated quantum master equation: Quantum detailed balance condition and return to equilibrium. Finally, we give the associated quantum Langevin equation which can be obtained by a repeated quantum interaction Hamiltonian.

1. Introduction

This paper is aimed at studying a particular open quantum system known as the Pauli-Fierz model. The approach is based on the theory of quantum Markovian semigroups. Indeed, the Pauli-Fierz model describes the dynamics of a small system, where the observables are bounded operators (matrices) defined on a finite dimensional Hilbert space $\mathcal{H}_S$, submitted to perturbations originated in a free Bose gas which represents the environment. This model can be considered as a generalization of the spin-boson system in the following sense: the degree of freedom of the small system is given by an arbitrary integer.

To describe the dynamics of the environment (also called *reservoir*), linear operators on another Hilbert space $\mathcal{H}_R$ are used. Furthermore, to consider the system-reservoir interaction a coupling is needed. To properly define that coupling, one needs to consider the tensor product $\mathcal{H}_S \otimes \mathcal{H}_R$ as the total Hilbert space.

This is the standard setting used by a number of authors (cf. [7] [12]) to analyze open systems as a closed system on the wider space $\mathcal{H}_S \otimes \mathcal{H}_R$.

The Markov approach uses to describe the system-reservoir evolution by considering a reduced dynamics on the algebra of observables (operators defined on $\mathcal{H}_S$). Such a reduced dynamics can be obtained, for instance, by performing a weak coupling limit yielding to a semigroup of completely positive maps acting on the algebra $\mathcal{B}(\mathcal{H}_S)$ of all endomorphisms of $\mathcal{H}_S$. This semigroup is characterized via its infinitesimal generator $\mathcal{L}$ bearing the name of *Lindblad generator* (cf. [1] [13]).

2000 *Mathematics Subject Classification.* Primary: 60-XX; Secondary: 81-XX.
Key words and phrases. Pauli-Fierz model, weak coupling limit, Lindblad generator, quantum master equation, quantum Langevin equation.
* This research is partially supported by PBCT-ADI13 grant: Laboratorio de Análisis y Estocástico, Chile .

In section 2, we describe some properties of quantum Markovian semigroup and quantum master equation. The Hamiltonian description, the weak coupling limit and the Lindblad generator of the Pauli-Fierz system are treated in section 3. In section 4, we study the Markovian properties of the quantum master equation. We prove that the faithful thermodynamical equilibrium state of the small system is a stationary state. Moreover, the quantum detailed balance condition is checked. Also, we prove the property of approach to equilibrium for the Pauli-Fierz system at a positive temperature β^{-1}. The quantum Langevin equation for the Pauli-Fierz system is studied in section 5.

2. Quantum Markovian Semigroup and Quantum Master Equation

Throughout the paper we consider a Hilbert space $\mathcal{H}_S$ (resp. $\mathcal{H}_R$) to describe the small system (resp. the environment).

The Markovian approach consists on omitting what happens in the environment and to concentrate only on the effective dynamics of the small system. If at time $t = 0$ the initial state of the small system is supposed to be given by a density matrix ρ_0, then the mathematical model for the evolution of the states (of the small system) is described by a completely positive map T_t^*, which is trace preserving and ensures that $\rho_t = T_t^*(\rho_0)$ is also a state. Moreover, this model is one for which the likelihood of a given future state, at any moment, depends only on its present state, and not on any past state. The dual of T_t^*, denoted T_t, is a one parameter semigroup of completely positive normal maps from $\mathcal{B}(\mathcal{H}_S)$ to $\mathcal{B}(\mathcal{H}_S)$, which is continuous for the σ-weak topology (w^*-topology) and satisfying $T_t(I) = I$, for all $t \geq 0$. The semigroup T_t is called *quantum Markovian semigroup* (cf. [1]), which satisfies

$$\mathrm{Tr}(\rho T_t(X)) = \mathrm{Tr}(T_t^*(\rho)X),$$

for all $X \in \mathcal{B}(\mathcal{H}_S)$ and all density matrix ρ.

In quantum mechanics, many properties of physical models follow from the explicit form of the generator of T_t. In particular, if T_t is a uniformly continuous quantum Markovian semigroup, that is

$$\lim_{t\to 0} \|T_t - T_0\| = 0,$$

then its generator is given by

$$\mathcal{L}(X) = i[H, X] + \frac{1}{2}\sum_k \Big(2L_k^* X L_k - L_k^* L_k X - X L_k^* L_k\Big), \tag{2.1}$$

for all $X \in \mathcal{B}(\mathcal{H}_S)$ such that $H = H^*$ and $\sum_k L_k^* L_k$ is strongly convergent (cf. [1] [13]). The operator $\mathcal{L}$ is called a *Lindblad generator* (or Lindbladian).

The predual $\mathcal{L}^*$ of $\mathcal{L}$ is usually used for deriving the so-called *quantum master equation* in a Lindblad form

$$\frac{d}{dt}\rho_t = \mathcal{L}^*(\rho_t), \tag{2.2}$$

where $\rho_t = T_t^*(\rho)$, $t \geq 0$ and ρ is a given state.

The equation (2.2) not only governs the time evolutions of the diagonal elements of the density matrix, but also of variables containing information about the decoherence between the states of the system (non-diagonal elements of the density matrix).

It is worth noticing that, in physics, many master equations for the reduced dynamics of a small system in interaction with an exterior system, should be derived from the underlying Hamiltonian dynamics for the total system using the weak coupling limit technique (cf. [4] [5] [8]).

3. Markovian Dynamics of the Pauli-Fierz Model

In this section, after having given a short description of the Pauli-Fierz system (cf. [6] [7]), for more details), we give a rigorous proof for the weak coupling limit. Moreover, we compute the explicit form of the associated Lindblad generator.

3.1. The model. The Pauli-Fierz system consists of a small system in interaction with a reservoir modeled by free bose gas.

The small system is described by a finite dimensional Hilbert space $\mathcal{H}_S$. Its energy is described by an Hamiltonian H_0 defined on $\mathcal{H}_S$. Moreover, the thermodynamical equilibrium state at inverse temperature β is the Gibbs state

$$\rho_s = \frac{e^{-\beta H_0}}{\mathrm{Tr}(e^{-\beta H_0})}.$$

The reservoir is described by the symmetric Fock space $\Gamma_s(L^2(\mathbb{R}^d)),\ d \geq 2$. Moreover, if we denote by $\omega(k) = |k|,\ k \in \mathbb{R}^d$ the energy of a single boson, then the Hamiltonian of the reservoir is given by the differential second quantization of ω, $d\Gamma(\omega)$ which acts on $\Gamma_s(L^2(\mathbb{R}^d))$. The observables are the Weyl operators $W(f),\ f \in L^2(\mathbb{R}^d)$, where

$$W(f) = \exp\Big(\frac{i}{\sqrt{2}}(a(f) + a^*(f))\Big),$$

where $a(f)$ and $a^*(f)$ are the usual annihilation and creation operators.

The equilibrium state of the reservoir at inverse temperature β is the quasi-free state ω_R defined by

$$\omega_R(W(f)) = \exp\Big[-\frac{\|f\|^2}{4} - \frac{1}{2}\int_{\mathbb{R}^d} |f(k)|^2 \rho(k)\, dk\Big], \tag{3.1}$$

where $\rho(k)$ is related to $\omega(k)$ by Planck radiation law

$$\rho(k) = \frac{1}{e^{\beta\omega(k)} - 1}$$

and f satisfying $(1 + \omega^{-1/2})f \in L^2(\mathbb{R}^d)$. The free Hamiltonian of Pauli-Fierz is the self-adjoint operator defined on $\mathcal{H}_S \otimes D(d\Gamma(\omega))$ by

$$H_{\mathrm{fr}} = H_0 \otimes I + I \otimes d\Gamma(\omega),$$

and the full Hamiltonian of the Pauli-Fierz system is

$$H_\lambda = H_0 + d\Gamma(\omega) + \frac{\lambda}{\sqrt{2}} \sum_{n=1}^{N} (q_n \otimes a^+(f_n) + q_n^* \otimes a^-(f_n)),$$

where $(f_n)_n$ is an orthonormal family of Schwartz functions on $\mathbb{R}^d$ and $q_n \in \mathcal{B}(\mathcal{H}_S)$. Because f_n are Schwartz functions, one has

$$\omega^{-1/2} f_n \in L^2(\mathbb{R}^d),\ n = 1, \dots, N.$$

It follows from [7] that $D(H_{\text{fr}})$ is a core for H_λ. Moreover, H_λ is a self-adjoint operator on $D(H_{\text{fr}})$.

Let

$$\mathcal{D} = \{f \in L^2(\mathbb{R}^d) |\ \omega^{-1/2} f \in L^2(\mathbb{R}^d)\}$$

be the domain of the quadratic form $\rho^{1/2}$. Then, the Araki-Woods representation of the pair $(L^2(\mathbb{R}^d),\ \rho)$ is the triple $(\Gamma_s(L^2(\mathbb{R}^d) \oplus \overline{L^2(\mathbb{R}^d)}),\ \pi_{\rho,\,l},\ \Omega_R)$ (cf. [3] [7]), where

- $\pi_{\rho,\,l} : g \longrightarrow W_{\rho,\,l}(g),\ g \in \mathcal{D}$, with $W_{\rho,\,l}(g) = W((1+\rho)^{1/2} g \oplus \bar{\rho}^{1/2} \bar{g})$,
- Ω_R is the vacuum vector of $\Gamma_s(L^2(\mathbb{R}^d) \oplus \overline{L^2(\mathbb{R}^d)})$.

Therefore, one gets

$$\omega_R(W(f)) = \langle \Omega_R, W_{\rho,l}(f)\Omega_R\rangle\rangle = \langle \Omega_R, W((1+\rho)^{1/2} f \oplus \bar{\rho}^{1/2} \bar{f})\Omega_R\rangle,$$

for all $f \in \mathcal{D}$. In the following, if there is no confusion, we denote $\langle \Omega_R, A_1 \dots A_n \Omega_R\rangle$ by $\omega_R(A_1 \dots A_n)$.

Let

$$L_R = d\Gamma(\omega \oplus -\bar{\omega})$$

be the Liouvillean of the reservoir, which satisfies $L_R \Omega_R = 0$. Then, the free semi-Liouvillean of Pauli-Fierz is given by

$$L_0^{semi} = H_0 + L_R$$

and the full semi-Liouvillean is the operator

$$L_\lambda^{semi} = H_0 + L_R + \lambda \varphi_{AW}(q),$$

where

$$\varphi_{AW}(q) = \frac{1}{\sqrt{2}} \sum_{n=1}^{N} \Big(q_n \otimes a^*((1+\rho)^{1/2} f_n \oplus \bar{\rho}^{1/2} \bar{f}_n) + q_n^* \otimes a((1+\rho)^{1/2} f_n \oplus \bar{\rho}^{1/2} \bar{f}_n) \Big).$$

Now, consider the left Araki-Woods algebra $\mathcal{M}_{\rho,\,l}$ generated by $\{W_{\rho,\,l}(g),\ g \in \mathcal{D}\}$. Recall that f_n are Schwartz functions. Then, it is clear that $(1+\omega)(1+\rho)^{1/2} \in L^2(\mathbb{R}^d)$. Hence, from [7], the map

$$\tau_\lambda^t(A) = e^{itL_\lambda^{semi}} A e^{-itL_\lambda^{semi}}$$

defines a W^*-dynamics on $\mathcal{M}_\rho = \mathcal{B}(\mathcal{H}_S) \otimes \mathcal{M}_{\rho,\,l}$.

3.2. Weak coupling limit of the Pauli-Fierz system. The weak coupling limit is one of the main asymptotic techniques in quantum theory of open systems. We replace the Hamiltonian interaction H_I between a small system and an exterior system by λH_I and consider the limit $\lambda \to 0$. In order to obtain a non-vanishing influence of the exterior system on the small system, we have to rescale time as $\tau = \lambda^2 t$. This means that the perturbations of the small system due to the exterior system have a long time $t = \tau/\lambda^2$. In this limit, we obtain a new dynamics describing the irreversible evolution of the small system affected by the exterior system.

3.2.1. *Abstract theory of the weak coupling limit.* Set $\mathcal{Y}$ a Banach space and $\mathcal{X}$ its dual, i.e: $\mathcal{X} = \mathcal{Y}^*$. Let P be a projection on $\mathcal{X}$ and $e^{it\delta_0}$ a one parameter group of isometries on $\mathcal{X}$ which commutes with P. Put $E = P\delta_0$. It is clear that E is the generator of one parameter group of isometries on Ran$P$. Consider a perturbation Q of δ_0 such that $\mathcal{D}(Q) \supset \mathcal{D}(\delta_0)$.

Now, we introduce the following assumptions:

(1) P is a w^*-continuous projection on $\mathcal{X}$ with norm equal to one,
(2) $e^{it\delta_0}$ a one parameter group of w^*-continuous isometries (C_0^*-group) on $\mathcal{X}$,
(3) For $|\lambda| < \lambda_0$, $i\delta_\lambda = i\delta_0 + i\lambda Q$ is the generator of a one parameter C_0^*-semigroup of contractions.

Consider now the operator

$$K_\lambda(t) = i \int_0^{\lambda^{-2}t} e^{-is(E+\lambda PQP)} PQ e^{is(1-P)\delta_\lambda(1-P)} QP \, ds.$$

For the proof of the following theorem we refer the interested reader to [5].

Theorem 3.1. *Suppose that assumptions* (1), (2) *and* (3) *hold true. Assume that the following hypotheses are satisfied:*

(4) *P is a finite range projection and $PQP = 0$,*
(5) *For all $t_1 > 0$, there exists a constant c such that*

$$\sup_{|\lambda|<1} \sup_{0 \le t \le t_1} \|K_\lambda(t)\| \le c.$$

(6) *There exists an operator K defined on* Ran$P$ *such that*

$$\lim_{\lambda \to 0} K_\lambda(t) = K$$

for all $0 < t < \infty$.

Put

$$K^\sharp = \sum_{e \in spE} 1_e(E) K 1_e(E) = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{itE} K e^{-itE} dt.$$

Then, one has

i) *$e^{itK^\sharp}$ is a semigroup of contractions,*
ii) *For all $t_1 > 0$,*

$$\lim_{\lambda\to 0} \sup_{0 \le t \le t_1} \|e^{-itE/\lambda^2} P e^{it(\delta_0 + \lambda Q)/\lambda^2} P - e^{itK^\sharp}\| = 0.$$

3.2.2. *Application to the Pauli-Fierz system.* Consider the Hilbert space $\mathcal{B}(\mathcal{H}_S)$ equipped by the scalar product $\langle A, B\rangle = \mathrm{Tr}(A^* B)$.

Lemma 3.2. *If $q_n \in \mathcal{B}(\mathcal{H}_S)$, then there exist operators $v_{n,j} \in \mathcal{B}(\mathcal{H}_S)$ such that*

$$q_n = \sum_{j=1}^{l} v_{n,j}, \quad \text{with } e^{itH_0} v_{n,j} e^{-itH_0} = e^{it\omega_{n,j}} v_{n,j}.$$

Proof. Note that $[H_0, .]$ is a self-adjoint operator on $\mathcal{B}(\mathcal{H}_S)$. Then, there exists an orthonormal basis $\{A_1, \dots, A_l\}$ of $\mathcal{B}(\mathcal{H}_S)$ such that

$$[H_0, A_j] = \omega_j A_j, \ \forall j = 1, \dots, l.$$

But, q_n belongs to $\mathcal{B}(\mathcal{H}_S)$. This implies that

$$q_n = \sum_{j=1}^{l} \alpha_{n,j} A_j.$$

Put $v_{n,j} = \alpha_{n,j} A_j$. This yields that $q_n = \sum_{j=1}^{l} v_{n,j}$, where $[H_0, v_{n,j}] = \omega_{n,j} v_{n,j}$, that is

$$e^{itH_0} v_{n,j} e^{-itH_0} = e^{it\omega_{n,j}} v_{n,j}.$$

□

As a consequence of the above lemma, we have $q_n^* = \sum_{j=1}^{l} v_{n,j}^*$ with $[H_0, v_{n,j}^*] = -\omega_{n,j} v_{n,j}$. This also gives $[H_0, v_{n,j} + v_{n,j}^*] = 0$. Hence, any self-adjoint operator $q_n \in \mathcal{B}(\mathcal{H}_S)$ takes the form

$$q_n = \sum_{j=1}^{k} (v_{n,j} + v_{n,j}^*), \quad \text{with} \ \ e^{itH_0} v_{n,j} e^{-itH_0} = e^{it\omega_{n,j}} v_{n,j} \ \forall j = 1, \dots, k. \tag{3.2}$$

In the following, we suppose that the operators $q_n, \ n = 1, \dots, N$, are self-adjoint operators, which have the decomposition (3.2). Moreover, we assume that they satisfy

$$e^{itH_0} v_{n,j} e^{-itH_0} = e^{it\omega_{n,j}} v_{n,j}, \ \omega_{n,j} > 0, \tag{3.3}$$

where $\omega_{n,j} \neq \omega_{n,j'}$ for all $j \neq j'$. Therefore, the full semi-Liouvillean of Pauli-Fierz becomes

$$L_\lambda^{semi} = H_0 + L_R + \lambda \sum_{n=1}^{N} q_n \otimes \varphi_{AW}(f_n),$$

where $\varphi_{AW}(f_n)$ is the Araki-Woods field associated to f_n

$$\varphi_{AW}(f_n) = \frac{1}{\sqrt{2}} \big(a^*((1+\rho)^{1/2} f_n \oplus \bar{\rho}^{1/2} \bar{f}_n) + a((1+\rho)^{1/2} f_n \oplus \bar{\rho}^{1/2} \bar{f}_n) \big).$$

Consider the following assumptions:

$$\langle f_n, \, e^{it\omega} f_m \rangle \ = \ \delta_{nm} \, \langle f_n, e^{it\omega} f_m \rangle, \tag{3.4}$$

$$\langle f_n, \, e^{it\omega} \rho f_m \rangle \ = \ \delta_{nm} \langle f_n, \, e^{it\omega} \rho f_m \rangle. \tag{3.5}$$

The assumptions (3.4) and (3.5) are satisfied for the functions
$f_n(x) = Y_{l_n, m_n}(\hat{x}) g_n(|x|)$ where the Y_{l_n, m_n} are the spherical harmonics and $l_n \neq l_r$ or $m_n \neq m_r$ for all $n \neq r$.

Now, our aim is to use the abstract theory given in subsection 3.2.1 for proving the weak coupling limit of the Pauli-Fierz system.

Put

$$\begin{aligned} V &= \sum_{n=1}^{N} q_n \otimes \varphi_{AW}(f_n) = \sum_{n=1}^{N} V_n,\ \ V_n = q_n \otimes \varphi_{AW}(f_n), \\ Q &= [V, .] = \sum_{n=1}^{N} Q^{(n)},\ \ Q^{(n)} = [V_n, .], \\ \delta_0 &= [L_0^{semi}, .],\ \ \delta_\lambda = [L_\lambda^{semi}, .] = \delta_0 + \lambda Q, \\ P(B \otimes C) &= \omega_R(C) B \otimes 1_{\Gamma_s(L^2(\mathbb{R}^d)\oplus \overline{L^2(\mathbb{R}^d)})},\ \forall\, B \otimes C \in \mathcal{M}_\rho. \end{aligned}$$

It is clear that Ran$P$ is a finite dimensional subspace of $\mathcal{M}_\rho$ with norm equal to one. Also, to check that hypotheses (1), (2), (3) and (4) of Theorem 3.1 are satisfied. Particularly, we have $PQP = 0$. Moreover, the operator E is given by

$$E = P\delta_0 = \delta_0 P = [H_0, .]P.$$

Set $P_1 = 1 - P$. Then, the operator $K_\lambda(t)$ is written as

$$K_\lambda(t) = i \int_0^{\lambda^{-2}t} e^{-isE} P[V, .] e^{isP_1[L_\lambda^{semi}, .]P_1} [V, .]P\, ds.$$

Put

$$U_t^\lambda = e^{itP_1[L_\lambda^{semi}, .]P_1},\ \ U_t = e^{itP_1[L_0^{semi}, .]P_1}.$$

This yields that

$$K_\lambda(t) = i \int_0^{\lambda^{-2}t} e^{-isE} P[V, .] U_s^\lambda [V, .]P\, ds. \tag{3.6}$$

Actually, we prove the following.

Lemma 3.3. *We have*

$$K_\lambda(t) = i \int_0^{\lambda^{-2}t} e^{-isE} P[V, .] e^{isP_1[L_0^{semi}, .]P_1} [V, .]P\, ds + i \sum_{p\geq 1} (i\lambda)^p R_p(\lambda^{-2}t),$$

where

$$R_p(t) = \int_{0\leq t_p \leq \cdots \leq t_0 \leq t} e^{-it_0 E} P[V, .] U_{t_0} (P_1 Q_1 P_1) \dots (P_1 Q_p P_1) [V, .]P\, dt_p \dots dt_0,$$

with $Q_j = U_{-t_j}[V, .]U_{t_j}$ *for all* $j = 1, \dots, p$.

Proof. We have

$$U_t^\lambda = U_t + i\lambda \int_0^t U_{t-s} P_1 [V, .] P_1 U_s^\lambda\, ds.$$

Therefore, the operator $U_{-t}U_t^\lambda$ satisfies the following equation

$$U_{-t}U_t^\lambda = I + i\lambda \int_0^t (U_{-s} P_1 [V, .] P_1 U_s)(U_{-s} U_s^\lambda)\, ds.$$

Note that the operator U_s commutes with P_1. Hence, one gets

$$U_{-t}U_t^\lambda = I + i\lambda \int_0^t (P_1 U_{-s} [V, .] U_s P_1)(U_{-s} U_s^\lambda)\, ds.$$

This yields that the series iteration of $U_{-t}U_t^\lambda$ is given by

$$U_{-t}U_t^\lambda = I + \sum_{p\geq 1}(i\lambda)^p \int_{0\leq t_p\leq\cdots\leq t_1\leq t}(P_1Q_1P_1)\dots(P_1Q_pP_1)\,dt_p\dots dt_1.$$

This proves that

$$U_t^\lambda = U_t + \sum_{p\geq 1}(i\lambda)^p \int_{0\leq t_p\leq\cdots\leq t_1\leq t}U_t(P_1Q_1P_1)\dots(P_1Q_pP_1)\,dt_p\dots dt_1. \tag{3.7}$$

Then, after substituting (3.7) in (3.6), one obtains

$$\begin{aligned} K_\lambda(t) &= i\int_0^{\lambda^{-2}t} e^{-isE}P[V,.]e^{isP_1[L_0^{semi},.]P_1}[V,.]P\,ds \\ &\quad + i\sum_{p\geq 1}(i\lambda)^p \int_{0\leq t_p\leq\cdots\leq t_0\leq\lambda^{-2}t} e^{-it_0E}P[V,.]U_{t_0}(P_1Q_1P_1) \\ &\qquad \dots(P_1Q_pP_1)[V,.]P\,dt_p\dots dt_0. \end{aligned}$$

This ends the proof. □

The explicit form of $R_p(t)$ can be simplified.

Lemma 3.4. *We have*

$$\begin{aligned} R_p(t) = \sum_{n=1}^{N}\int_{0\leq t_p\leq\cdots\leq t_0\leq t} & P[q_{n,0}\otimes\varphi_{AW}(e^{-it_0\omega}f_n),.]P_1 \\ \dots P_1[q_{n,p}\otimes\varphi_{AW}(e^{-it_p\omega}f_n),.]&P_1[q_{n,p+1}\otimes\varphi_{AW}(e^{-it_{p+1}\omega}f_n),.]Pdt_p\dots dt_0, \end{aligned}$$

where $t_{p+1}=0$ *and* $q_{n,j}=e^{-it_jH_0}q_ne^{it_jH_0}$.

Proof. Recall that $PU_{-t_0}=P$. This implies that

$$R_p(t) = \int_{0\leq t_p\leq\cdots\leq t_0\leq t} e^{-it_0E}PQ_0(P_1Q_1P_1)\dots(P_1Q_pP_1)Q_{p+1}Pdt_p\dots dt_0.$$

Moreover, one has

$$e^{-it_0E}P = Pe^{-it_0[H_0,.]} = Pe^{-it_0[H_0,.]}e^{-it_0[L_R,.]}.$$

This yields that

$$\begin{aligned} e^{-it_0E}PQ_0P_1 &= Pe^{-it_0[H_0,.]}[V,.]e^{it_0[H_0,.]}e^{it_0[L_R,.]}P \\ &= Pe^{-it_0[H_0,.]}e^{-it_0[L_R,.]}[V,.]e^{it_0[H_0,.]}e^{it_0[L_R,.]}P \\ &= \sum_{n=1}^{N}P[q_{n,0}\otimes\varphi_{AW}(e^{-it_0\omega}f_n),.]P_1. \end{aligned}$$

Now, by using the fact that

$$U_{t_j} = e^{it_j[H_0,.]}e^{it_j[L_R,.]}P_1 + P \quad\text{and}\quad e^{-it_jL_R}\varphi_{AW}(f_n)e^{it_jL_R} = \varphi_{AW}(e^{-it_j\omega}f_n),$$

it is easy to show that

$$R_p(t) = \sum_{n_1,\dots,n_N=1}^{N} \int_{0\le t_p\le\cdots\le t_0\le t} P[q_{n_1,0}\otimes\varphi_{AW}(e^{-it_0\omega}f_{n_1}),.]P_1 \dots P_1[q_{n_N,p+1}\otimes\varphi_{AW}(e^{-it_{p+1}\omega}f_{n_N}),.]P dt_p\dots dt_0.$$

Finally, the result of the above lemma follows immediately from identities (3.4), (3.5) and the explicit form of the projection P. □

As a consequence of the above lemma, one has $K_\lambda(t) = \sum_{n=1}^N K_{\lambda,n}(t)$, where

$$K_{\lambda,n}(t) = i\int_0^{\lambda^{-2}t} e^{-isE}P[V_n,.]e^{isP_1[L_0^{semi},.]P_1}[V_n,.]P\,ds + i\sum_{p\ge1}(i\lambda)^p R_{p,n}(\lambda^{-2}t),$$

$$R_{p,n}(t) = \int_{0\le t_p\le\cdots\le t_0\le t} P[q_{n,0}\otimes\varphi_{AW}(e^{-it_0\omega}f_n),.]P_1 \dots P_1[q_{n,p}\otimes\varphi_{AW}(e^{-it_p\omega}f_n),.]P_1[q_{n,p+1}\otimes\varphi_{AW}(e^{-it_{p+1}\omega}f_n),.]P dt_p\dots dt_0.$$

Theorem 3.5. *Suppose that the following hypothesis is satisfied:*

$$\sup_n \int_0^\infty (1+t^\varepsilon)|h_n(t)|dt < \infty,$$

for some $0<\varepsilon<1$. *Then, the hypotheses of Theorem 3.1 hold true. Moreover, the operator* $K^\sharp$ *is given by*

$$K^\sharp = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{itE}Ke^{-itE}dt,$$

where

$$K = \sum_{n=1}^N K_n, \quad \textit{with } K_n = i\int_0^\infty e^{-isE}P[V_n,.]e^{is[L_0^{semi},.]}[V_n,.]P\,ds.$$

Proof. Firstly, note that it is easy to show that hypotheses (1) − (5) of Theorem 3.1 are satisfied. Moreover, in order to investigate the hypothesis (6) of Theorem 3.1, it suffices to prove that

$$\lim_{\lambda\to0} K_{\lambda,n}(t) = K_n,\ 0<t<\infty. \tag{3.8}$$

But, the proof of the identity (3.8) is similar as the one of Theorem 3.8 in [8]. □

3.2.3. *Lindblad generator of the Pauli-Fierz system.* In this subsection, our purpose is to prove that the operator $\mathcal{L} = iK^\sharp$ takes the form (2.1).

Let us start by proving the following lemma.

Lemma 3.6. *For all $X \in \mathcal{B}(\mathcal{H}_S)$, one has*

$$e^{itH_0}q_n e^{-itH_0}Xe^{i(t-s)H_0}q_n e^{-i(t-s)H_0} = \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j}+\omega_{n,j'})}e^{-is\omega_{n,j'}}v_{n,j}Xv_{n,j'}$$
$$+\sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{-it(\omega_{n,j}+\omega_{n,j'})}e^{is\omega_{n,j'}}v_{n,j}^*Xv_{n,j'}^* + \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j}-\omega_{n,j'})}e^{is\omega_{n,j'}}v_{n,j}Xv_{n,j'}^*$$
$$+\sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j'}-\omega_{n,j})}e^{-is\omega_{n,j'}}v_{n,j}^*Xv_{n,j'} + \sum_{j=1}^{k} e^{is\omega_{n,j}}v_{n,j}Xv_{n,j}^*$$
$$+\sum_{j=1}^{k} e^{-is\omega_{n,j}}v_{n,j}^*Xv_{n,j},$$

$$Xe^{itH_0}q_n e^{-itH_0}e^{i(t-s)H_0}q_n e^{-i(t-s)H_0} = \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j}+\omega_{n,j'})}e^{-is\omega_{n,j'}}Xv_{n,j}v_{n,j'}$$
$$+\sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{-it(\omega_{n,j}+\omega_{n,j'})}e^{is\omega_{n,j'}}Xv_{n,j}^*v_{n,j'}^* + \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j}-\omega_{n,j'})}e^{is\omega_{n,j'}}Xv_{n,j}v_{n,j'}^*$$
$$+\sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j'}-\omega_{n,j})}e^{-is\omega_{n,j'}}Xv_{n,j}^*v_{n,j'} + \sum_{j=1}^{k} e^{is\omega_{n,j}}Xv_{n,j}v_{n,j}^*$$
$$+\sum_{j=1}^{k} e^{-is\omega_{n,j}}Xv_{n,j}^*v_{n,j},$$

$$e^{i(t-s)H_0}q_n e^{-i(t-s)H_0}Xe^{itH_0}q_n e^{-itH_0} = \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j}+\omega_{n,j'})}e^{-is\omega_{n,j}}v_{n,j}Xv_{n,j'}$$
$$+\sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{-it(\omega_{n,j}+\omega_{n,j'})}e^{is\omega_{n,j}}v_{n,j}^*Xv_{n,j'}^* + \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j}-\omega_{n,j'})}e^{-is\omega_{n,j}}v_{n,j}Xv_{n,j'}^*$$
$$+\sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j'}-\omega_{n,j})}e^{is\omega_{n,j}}v_{n,j}^*Xv_{n,j'} + \sum_{j=1}^{k} e^{-is\omega_{n,j}}v_{n,j}Xv_{n,j}^*$$
$$+\sum_{j=1}^{k} e^{is\omega_{n,j}}v_{n,j}^*Xv_{n,j},$$

$$e^{i(t-s)H_0}q_n e^{-i(t-s)H_0} e^{itH_0} q_n e^{-itH_0} X = \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j}+\omega_{n,j'})} e^{-is\omega_{n,j}} v_{n,j} v_{n,j'} X$$

$$+ \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{-it(\omega_{n,j}+\omega_{n,j'})} e^{is\omega_{n,j}} v_{n,j}^* v_{n,j'}^* X + \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j}-\omega_{n,j'})} e^{-is\omega_{n,j}} v_{n,j} v_{n,j'}^* X$$

$$+ \sum_{\substack{j,j'=1\\ j\neq j'}}^{k} e^{it(\omega_{n,j'}-\omega_{n,j})} e^{is\omega_{n,j}} v_{n,j}^* v_{n,j'} X + \sum_{j=1}^{k} e^{-is\omega_{n,j}} v_{n,j} v_{n,j}^* X$$

$$+ \sum_{j=1}^{k} e^{is\omega_{n,j}} v_{n,j}^* v_{n,j} X.$$

Proof. Note that identities (3.2) and (3.3) imply that

$$e^{i(t-s)H_0} q_n e^{-i(t-s)H_0} = \sum_{j=1}^{k} \big(e^{i(t-s)\omega_{n,j}} v_{n,j} + e^{-i(t-s)\omega_{n,j}} v_{n,j}^* \big), \qquad (3.9)$$

$$e^{itH_0} q_n e^{-itH_0} = \sum_{j=1}^{k} \big(e^{it\omega_{n,j}} v_{n,j} + e^{-it\omega_{n,j}} v_{n,j}^* \big). \qquad (3.10)$$

Therefore, using (3.9) and (3.10), it is straightforward to prove the results in the above lemma. □

Now, we prove the following.

Theorem 3.7. *Suppose that*

$$\sup_n \int_0^\infty \Big[\Big| \int_{\mathbb{R}^d} e^{it\omega} \, (\rho(k)+1) \, |f_n(k)|^2 dk \Big| + \Big| \int_{\mathbb{R}^d} e^{it\omega} \, \rho(k) \, |f_n(k)|^2 dk \Big| \Big] \, ds < \infty.$$

Then, for all $X \in \mathcal{B}(\mathcal{H}_S)$, one has

$$\begin{aligned}\mathcal{L}(X) = \sum_{n=1}^{N} \sum_{j=1}^{k} \quad & i\big(\mathrm{Im}(f_n, f_n)^+_{-\omega_{n,j}} - \mathrm{Im}(f_n, f_n)^-_{\omega_{n,j}}\big) \, [v_{n,j}^* v_{n,j}, \, X] \\ & + i\big(\mathrm{Im}(f_n, f_n)^-_{-\omega_{n,j}} - \mathrm{Im}(f_n, f_n)^+_{\omega_{n,j}}\big) \, [v_{n,j} v_{n,j}^*, \, X] \\ & + \mathrm{Re}(f_n, f_n)^+_{-\omega_{n,j}} \, \big[2 v_{n,j}^* X v_{n,j} - \{v_{n,j}^* v_{n,j}, \, X\}\big] \\ & + \mathrm{Re}(f_n, f_n)^-_{-\omega_{n,j}} \, \big[2 v_{n,j} X v_{n,j}^* - \{v_{n,j} v_{n,j}^*, \, X\}\big],\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Re}(f_n,f_n)^+_{-\omega_{n,j}} &= \frac{\pi e^{\beta\omega_{n,j}}}{e^{\beta\omega_{n,j}}-1}\int_{\mathbb{R}^d}|f_n(k)|^2\delta(\omega(k)-\omega_{n,j})\,dk,\\
\mathrm{Re}(f_n,f_n)^-_{-\omega_{n,j}} &= \frac{\pi}{e^{\beta\omega_{n,j}}-1}\int_{\mathbb{R}^d}|f_n(k)|^2\delta(\omega(k)-\omega_{n,j})\,dk,\\
\mathrm{Im}(f_n,f_n)^+_{-\omega_{n,j}} &= PP\int\frac{\rho(k)+1}{\omega(k)-\omega_{n,j}}|f_n(k)|^2\,dk,\\
\mathrm{Im}(f_n,f_n)^-_{\omega_{n,j}} &= \int_{\mathbb{R}^d}\frac{\rho(k)}{\omega(k)+\omega_{n,j}}\,|f_n(k)|^2\,dk,\\
\mathrm{Im}(f_n,f_n)^+_{\omega_{n,j}} &= \int_{\mathbb{R}^d}\frac{\rho(k)+1}{\omega(k)+\omega_{n,j}}\,|f_n(k)|^2\,dk,\\
\mathrm{Im}(f_n,f_n)^-_{-\omega_{n,j}} &= PP\int\frac{\rho(k)}{\omega(k)-\omega_{n,j}}|f_n(k)|^2\,dk.
\end{aligned}$$

Proof. Recall that $K^\sharp=\sum_{n=1}^N K_n^\sharp$, where

$$K_n^\sharp=\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{itE}K_ne^{-itE}dt.$$

Then, for all $X\in\mathcal{B}(\mathcal{H}_S)$, the term

$$e^{itE}e^{-isE}P[V_n,.]e^{is[L_0^{semi},.]}[V_n,.]Pe^{-itE}(X)$$

is equal to

$$\begin{aligned}
&\Big[e^{itH_0}e^{-isH_0}q_ne^{isH_0}q_ne^{-itH_0}X-e^{itH_0}e^{-isH_0}q_ne^{isH_0}e^{-itH_0}Xe^{itH_0}q_ne^{-itH_0}\Big]\\
&\quad\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))-\Big[e^{itH_0}q_ne^{-itH_0}Xe^{itH_0}e^{-isH_0}q_ne^{isH_0}e^{-itH_0}\\
&\quad-Xe^{itH_0}q_ne^{-isH_0}q_ne^{-itH_0}e^{isH_0}\Big]\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n)),
\end{aligned}$$

which is equal to

$$\begin{aligned}
&\Big[e^{i(t-s)H_0}q_ne^{-i(t-s)H_0}e^{itH_0}q_ne^{-itH_0}X-e^{i(t-s)H_0}q_ne^{-i(t-s)H_0}Xe^{itH_0}q_ne^{-itH_0}\Big]\\
&\quad\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))-\Big[e^{itH_0}q_ne^{-itH_0}Xe^{i(t-s)H_0}q_ne^{-i(t-s)H_0}\\
&\quad-Xe^{itH_0}q_ne^{-itH_0}e^{i(t-s)H_0}q_ne^{-i(t-s)H_0}\Big]\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n)). \qquad (3.11)
\end{aligned}$$

Now, from Lemma 3.6, (3.11) and the following identity

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\nu t}dt=0,\ \nu\in\mathbb{R}^* \qquad (3.12)$$

we prove that, for all $j\neq j'$, the terms

$$\begin{aligned}
&v_{n,j}Xv_{n,j'},\ v_{n,j}^*Xv_{n,j'}^*,\ v_{n,j}^*Xv_{n,j'},\ v_{n,j}Xv_{n,j'}^*,\ Xv_{n,j}v_{n,j'},\ Xv_{n,j}^*v_{n,j'}^*,\\
&Xv_{n,j}v_{n,j'}^*,\ Xv_{n,j}v_{n,j'}^*,\ v_{n,j}v_{n,j'}X,\ v_{n,j}^*v_{n,j'}^*X,\ v_{n,j}v_{n,j'}^*X,\ v_{n,j}^*v_{n,j'}X
\end{aligned}$$

disappear in the explicit form of $K_n^\sharp$. This gives

$$\begin{aligned}
K_n^\sharp = i\sum_{j=1}^{k} \; & -\Big[\int_0^\infty e^{-is\omega_{n,j}}\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))ds \\
& \quad + \int_0^\infty e^{is\omega_{n,j}}\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n))ds\Big]\, v_{n,j}Xv_{n,j}^* \\
& -\Big[\int_0^\infty e^{is\omega_{n,j}}\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))ds \\
& \quad + \int_0^\infty e^{-is\omega_{n,j}}\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n))ds\Big]\, v_{n,j}^*Xv_{n,j} \\
& + \Big[\int_0^\infty e^{is\omega_{n,j}}\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n))ds]\, Xv_{n,j}v_{n,j}^* \\
& + \Big[\int_0^\infty e^{-is\omega_{n,j}}\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n))ds\Big]\, Xv_{n,j}^*v_{n,j} \\
& + \Big[\int_0^\infty e^{-is\omega_{n,j}}\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))ds\Big]\, v_{n,j}v_{n,j}^*X \\
& + \Big[\int_0^\infty e^{is\omega_{n,j}}\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))ds\Big]\, v_{n,j}^*v_{n,j}X. \qquad (3.13)
\end{aligned}$$

Now, let us introduce the well known formula of Schwartz distribution theory

$$\frac{\pm i}{\omega \pm i0} = \pi\delta(\omega) \pm iV_p(\frac{1}{\omega}), \qquad (3.14)$$

where

$$\begin{aligned}
\int f(x)\delta(x)\,dx &= f(0), \\
\int f(x)V_p(\frac{1}{x})\,dx &= \lim_{\varepsilon\to 0}\int_{|x|\ge\varepsilon}\frac{f(x)}{x}\,dx = PP\int\frac{f(x)}{x}\,dx, \\
\int f(x)\frac{1}{x+i0}\,dx &= \lim_{\varepsilon\to 0}\int f(x)\frac{1}{x+i\varepsilon}\,dx
\end{aligned}$$

for all function $f:\mathbb{R}\to\mathbb{R}$, provided that the integrals on the right are well defined and that the limits exist. Note that

$$\begin{aligned}
\omega_R\big(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n)\big) &= \langle\Omega_R, \varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n)\Omega_R\rangle \\
&= \langle(1+\rho)^{1/2}\oplus\bar\rho^{1/2}\bar f_n, e^{is\omega}(1+\rho)^{1/2}\oplus e^{is\omega}\bar\rho^{1/2}\bar f_n\rangle \\
&= \int_{\mathbb{R}^d} e^{is\omega}(1+\rho(k))|f_n(k)|^2dk \\
&\quad + \int_{\mathbb{R}^d} e^{-is\omega}\rho(k)|f_n(k)|^2dk \\
&= \overline{\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n))}.
\end{aligned}$$

This implies that

$$\int_0^\infty e^{-is\omega_{n_j}}\omega_R\big(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n)\big)ds$$
$$= \int_0^\infty \int_{\mathbb{R}^d} e^{is(\omega-\omega_{n_j})}(\rho(k)+1)|f_n(k)|^2\, dkds$$
$$+ \int_0^\infty \int_{\mathbb{R}^d} e^{-is(\omega+\omega_{n_j})}\rho(k)|f_n(k)|^2\, dkds.$$

According to the assumption of the above theorem, one has

$$\int_0^\infty e^{-is\omega_{n_j}}\omega_R\big(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n)\big)ds$$
$$= \lim_{\varepsilon\to 0^+}\int_0^\infty \int_{\mathbb{R}^d} e^{is(\omega-\omega_{n_j}+i\varepsilon)}(\rho(k)+1)|f_n(k)|^2\, dkds$$
$$+ \lim_{\varepsilon\to 0^+}\int_0^\infty \int_{\mathbb{R}^d} e^{-is(\omega+\omega_{n_j}-i\varepsilon)}\,\rho(k)\,|f_n(k)|^2\, dkds.$$

Note that for all $\varepsilon > 0$, the functions

$$(s,k)\mapsto e^{is(\omega-\omega_{n_j}+i\varepsilon)}(\rho(k)+1)|f_n(k)|^2$$
$$(s,k)\mapsto e^{-is(\omega+\omega_{n_j}-i\varepsilon)}\,\rho(k)\,|f_n(k)|^2$$

are integrable on $[0,\infty)\times\mathbb{R}^d$. Then, by Fubini Theorem, one obtains

$$\int_0^\infty e^{-is\omega_{n_j}}\omega_R\big(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n)\big)ds$$
$$= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^d} \frac{i}{\omega-\omega_{n_j}+i\varepsilon}(\rho(k)+1)|f_n(k)|^2\, dk$$
$$+ \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^d} \frac{-i}{\omega+\omega_{n_j}-i\varepsilon}\,\rho(k)\,|f_n(k)|^2\, dk.$$

It is worthwhile to note that the test functions f_n are Schwartz functions. Therefore, we apply formula (3.14) to get

$$\int_0^\infty e^{-is\omega_{n_j}}\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))\, ds$$
$$= \mathrm{Re}(f_n,f_n)^+_{-\omega_{n_j}} + i\mathrm{Im}(f_n,f_n)^+_{-\omega_{n_j}} - i\mathrm{Im}(f_n,f_n)^-_{\omega_{n_j}}$$
$$= \overline{\int_0^\infty e^{is\omega_{n_j}}\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n))\, ds}.$$

In the same way, we prove that

$$\int_0^\infty e^{-is\omega_{n_j}}\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n))\,ds$$

$$= \mathrm{Re}(f_n,f_n)^-_{-\omega_{n_j}} + i\mathrm{Im}(f_n,f_n)^-_{-\omega_{n_j}} - i\mathrm{Im}(f_n,f_n)^+_{\omega_{n_j}}$$

$$= \overline{\int_0^\infty e^{is\omega_{n_j}}\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))\,ds},$$

$$\int_0^\infty e^{is\omega_{n_j}}\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n))\,ds$$

$$= \mathrm{Re}(f_n,f_n)^+_{-\omega_{n_j}} - i\mathrm{Im}(f_n,f_n)^+_{-\omega_{n_j}} + i\mathrm{Im}(f_n,f_n)^-_{\omega_{n_j}}$$

$$= \overline{\int_0^\infty e^{-is\omega_{n_j}}\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))\,ds},$$

$$\int_0^\infty e^{is\omega_{n_j}}\omega_R(\varphi_{AW}(f_n)\varphi_{AW}(e^{is\omega}f_n))\,ds$$

$$= \mathrm{Re}(f_n,f_n)^-_{-\omega_{n_j}} + i\mathrm{Im}(f_n,f_n)^+_{\omega_{n_j}} - i\mathrm{Im}(f_n,f_n)^-_{-\omega_{n_j}}$$

$$= \overline{\int_0^\infty e^{-is\omega_{n_j}}\omega_R(\varphi_{AW}(e^{is\omega}f_n)\varphi_{AW}(f_n))\,ds}.$$

Therefore, for all $X \in \mathcal{B}(\mathcal{K})$, one obtains

$$\begin{aligned}
\mathcal{L}_n(X) &= iK_n^\sharp(X)\\
&= \sum_{j=1}^k i\big(\mathrm{Im}(f_n,f_n)^+_{-\omega_{n_j}} - \mathrm{Im}(f_n,f_n)^-_{\omega_{n_j}}\big)\,[v^*_{n_j}v_{n_j},\,X]\\
&\quad + i\big(\mathrm{Im}(f_n,f_n)^-_{-\omega_{n_j}} - \mathrm{Im}(f_n,f_n)^+_{\omega_{n_j}}\big)\,[v_{n_j}v^*_{n_j},\,X]\\
&\quad + \mathrm{Re}(f_n,f_n)^+_{-\omega_{n_j}}\,\big[2v^*_{n_j}Xv_{n_j} - \{v^*_{n_j}v_{n_j},\,X\}\big]\\
&\quad + \mathrm{Re}(f_n,f_n)^-_{-\omega_{n_j}}\,\big[2v_{n_j}Xv^*_{n_j} - \{v_{n_j}v^*_{n_j},\,X\}\big].
\end{aligned}$$

Note that $K^\sharp = \sum_{n=1}^N K_n^\sharp$. This implies that $\mathcal{L} = i\sum_{n=1}^N K_n^\sharp = \sum_{n=1}^N \mathcal{L}_n$. This ends the proof. □

4. Markovian Properties of the Quantum Master Equation Associated to the Pauli-Fierz System

In this section, we study the Markovian properties of the master equation associated to the Pauli-Fierz system. We show that the quantum detailed balance condition is satisfied with respect to the thermodynamical equilibrium state ρ_s. Moreover, we prove the property of return to equilibrium.

4.1. Quantum master equation. Let ρ be a density matrix in $\mathcal{B}(\mathcal{H}_S)$. Then, the master equation associated to the Pauli-Fierz system is given by

$$\begin{aligned}\frac{d\rho(t)}{dt} = \sum_{n=1}^{N}\sum_{j=1}^{k} \quad & -i\big(\mathrm{Im}(f_n, f_n)^+_{-\omega_{n,j}} - \mathrm{Im}(f_n, f_n)^-_{\omega_{n,j}}\big)\,[v^*_{n,j}v_{n,j},\, \rho(t)] \\ & -i\big(\mathrm{Im}(f_n, f_n)^-_{-\omega_{n,j}} - \mathrm{Im}(f_n, f_n)^+_{\omega_{n,j}}\big)\,[v_{n,j}v^*_{n,j},\, \rho(t)] \\ & +\mathrm{Re}(f_n, f_n)^+_{-\omega_{n,j}}\,\big[2v_{n,j}\rho(t)v^*_{n,j} - \{\rho(t),\, v^*_{n,j}v_{n,j}\}\big] \\ & +\mathrm{Re}(f_n, f_n)^-_{-\omega_{n,j}}\,\big[2v^*_{n,j}\rho(t)v_{n,j} - \{\rho(t),\, v_{n,j}v^*_{n,j}\}\big].\end{aligned}$$

Recall that $\mathcal{H}_S$ is a finite dimensional Hilbert space and H_0 is a self-adjoint operator on $\mathcal{H}_S$. Let c be a positive real such that the spectrum of H_0, $\sigma(H_0) \subset [-c,\, c]$. Note that

$$e^{-\beta x}1_{[-c,\, c]}(x) = \int_{\mathbb{R}} e^{itx}\Big(\int_{\mathbb{R}} e^{-its}e^{-\beta s}1_{[-c,\, c]}(s)ds\Big)dt.$$

Then, by using spectral Theorem, one has $1_{[-c,\, c]}(H_0) = I$. Moreover, by functional calculus, one gets

$$e^{-\beta H_0} = \int_{\mathbb{R}} e^{itH_0}\Big(\int_{\mathbb{R}} e^{-its}e^{-\beta s}1_{[-c,\, c]}(s)ds\Big)dt.$$

Identity (3.3) implies that

$$e^{-\beta H_0}v_{n,j} = e^{\beta\omega_{n,j}}v_{n,j}e^{-\beta K},\ e^{-\beta H_0}v^*_{n,j} = e^{-\beta\omega_{n,j}}v^*_{n,j}e^{-\beta H_0},$$

for all $n = 1, \dots, N;\ j = 1, \dots, k$. This gives

$$\begin{aligned} & e^{-\beta H_0}v^*_{n,j}v_{n,j} = v^*_{n,j}v_{n,j}e^{-\beta H_0},\ e^{-\beta H_0}v_{n,j}v^*_{n,j} = v_{n,j}v^*_{n,j}e^{-\beta H_0}, \\ & v_{n,j}e^{-\beta H_0}v^*_{n,j} = e^{-\beta\omega_{n,j}}v_{n,j}v^*_{n,j}e^{-\beta H_0},\ v^*_{n,j}e^{-\beta H_0}v_{n,j} = e^{\beta\omega_{n,j}}v^*_{n,j}v_{n,j}e^{-\beta H_0},\end{aligned}$$

for all $n = 1, \dots, N$; $j = 1, \dots, k$. Hence, the thermodynamical equilibrium state ρ_s at inverse temperature β of the small system satisfies

$$[v^*_{n,j}v_{n,j},\, \rho_s] = [v_{n,j}v^*_{n,j},\, \rho_s] = 0,\ \forall n = 1, \dots, N;\ j = 1, \dots, k. \tag{4.1}$$

Because $\mathrm{Re}(f_n, f_n)^+_{-\omega_{n,j}}e^{-\beta\omega_{n,j}} = \mathrm{Re}(f_n, f_n)^-_{-\omega_{n,j}}$, one obtains

$$\begin{aligned} & \mathrm{Re}(f_n, f_n)^+_{-\omega_{n,j}}\,[2v_{n,j}\rho_s v^*_{n,j} - \{\rho_s,\, v^*_{n,j}v_{n,j}\}] \\ & +\mathrm{Re}(f_n, f_n)^-_{-\omega_{n,j}}[2v^*_{n,j}\rho_s v_{n,j} - \{\rho_s,\, v_{n,j}v^*_{n,j}\}] \\ & = \mathrm{Re}(f_n, f_n)^+_{-\omega_{n,j}}[2e^{-\beta\omega_{n,j}}v_{n,j}v^*_{n,j}\rho_s - 2v^*_{n,j}v_{n,j}\rho_s] \\ & \quad +\mathrm{Re}(f_n, f_n)^-_{-\omega_{n,j}}[2e^{\beta\omega_{n,j}}v^*_{n,j}v_{n,j}\rho_s - 2v_{n,j}v^*_{n,j}\rho_s] \\ & = 2\big(\mathrm{Re}(f_n, f_n)^+_{-\omega_{n,j}}e^{-\beta\omega_{n,j}} - \mathrm{Re}(f_n, f_n)^-_{-\omega_{n,j}}\big)v_{n,j}v^*_{n,j}\rho_s \\ & \quad +2\big(e^{\beta\omega_{n,j}}\mathrm{Re}(f_n, f_n)^-_{-\omega_{n,j}} - \mathrm{Re}(f_n, f_n)^+_{-\omega_{n,j}}\big)v^*_{n,j}v_{n,j}\rho_s = 0\end{aligned}$$

This proves that ρ_s is a stationary state.

4.2. Quantum detailed balance condition. One of the first definition of quantum detailed balance is stated in [11] and others characterizations are given in [1] [10]. The connection of the quantum detailed balance with the KMS condition is investigated in [11] and with the Markov dilations is studied in [10]. The definition given below is taken from [1].

Definition 4.1. Let Θ be the generator of a quantum Markovian semigroup on $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is an Hilbert space, written as

$$\Theta = -i\,[H, .] + \Theta_0,$$

with H is a self adjoint operator on $\mathcal{H}$. We say that Θ satisfies quantum detailed balance condition with respect to a faithful stationary state ρ if

i) $[H, \rho] = 0$,
ii) $\langle \Theta_0(A),\, B\rangle_\rho = \langle A,\, \Theta_0(B)\rangle_\rho$, for all $A,\, B \in D(\Theta_0)$, where $\langle A,\, B\rangle_\rho = \ \mathrm{Tr}(\rho A^* B)$.

Now, we prove the following.

Theorem 4.2. *The quantum Markovian semigroup $T_t = e^{t\mathcal{L}}$ associated to the Pauli-Fierz system satisfies quantum detailed balance condition with respect to the thermodynamical equilibrium state ρ_s of the small system.*

Proof. Note that it is proved in the previous subsection that the faithful thermodynamical faithful state ρ_s of the small system is a stationary state for the quantum Markovian semigroup associated to the Pauli-Fierz system.

Put

$$\begin{aligned} H_1 &= \sum_{n=1}^{N}\sum_{j=1}^{k} \big(\mathrm{Im}(f_n, f_n)^+_{-\omega_{n,j}} - \mathrm{Im}(f_n, f_n)^-_{\omega_{n,j}}\big) v^*_{n,j} v_{n,j} \\ &\qquad + \big(\mathrm{Im}(f_n, f_n)^-_{-\omega_{n,j}} - \mathrm{Im}(f_n, f_n)^+_{\omega_{n,j}}\big) v_{n,j} v^*_{n,j}, \\ \mathcal{L}^D_\omega(X) &= \sum_{n=1}^{N}\sum_{j=1}^{k} \mathrm{Re}(f_n, f_n)^+_{-\omega_{n,j}} \big[2v^*_{n,j} X v_{n,j} - \{v_{n,j} v^*_{n,j},\, X\}\big] \\ &\qquad + \mathrm{Re}(f_n, f_n)^-_{-\omega_{n,j}} \big[2v^*_{n,j} X v_{n,j} - \{v_{n,j} v^*_{n,j},\, X\}\big]. \end{aligned}$$

Hence, from relation (4.1), it is clear that $[H,\, \rho_s] = 0$. Moreover, it is easy to show that $\mathcal{L}^D_\omega$ is a self-adjoint operator for the scalar product $\langle , \rangle_{\rho_s}$. □

4.3. Return to equilibrium. In the Markovian approach of open systems, it is of interest to know whether for a given Lindblad generator $\mathcal{L}$ there exists a independent time final destination state $\rho^{(\infty)}$. This may be a stationary state, that is a state satisfying $\mathcal{L}^*(\rho) = 0$, or a Gibbs state...

A quantum Markovian semigroup T_t is called *relaxing* iff

$$w^* - \lim_{t\to\infty} \rho(t) = \rho^{(\infty)},$$

where $\rho(t) = T^*_t(\rho)$ for a given density matrix ρ. This raises questions concerning existence and particular properties of $\rho^{(\infty)}$ related to $\mathcal{L}^*$ or to the initial condition

$\rho(0)$. The most widely used situation is the following

$$w^* - \lim_{t\to\infty} \rho(t) = \rho^{(\infty)},\ \forall \rho(0). \tag{4.2}$$

In this case, we say that the quantum Markovian semigroup T_t has the properties of return to equilibrium.

A sufficient condition, which ensures that (4.2) is attained, is an immediately consequence of a Fagnola-Rebolledo result (cf. [9]).

Theorem 4.3. *Suppose that $\mathcal{H}$ is a finite dimensional Hilbert space. Let T be a quantum Markovian semigroup on $\mathcal{B}(\mathcal{H})$, which has a stationary faithful state and with generator has the form (2.1). If*

$$\{L_k, L_k^*, H,\ k \geq 1\}' = \{L_k, L_k^*,\ k \geq 1\}',$$

then T has the property of return to equilibrium.

Now, we prove the following.

Theorem 4.4. *Suppose that $\mathrm{Re}(f_n, f_n)^{\pm}_{-\omega_{n,j}} > 0$ and $\mathrm{Im}(f_n, f_n)^{\pm}_{\pm\omega_{n,j}}$ are given by real numbers, for all $j = 1, \ldots, k$; $n = 1, \ldots, N$. Then, for all $0 < \beta < \infty$, the quantum Markovian semigroup of the Pauli-Fierz system has the property of return to equilibrium.*

Proof. Put

$$\begin{aligned}
H_1 &= \sum_{n=1}^{N}\sum_{j=1}^{k}\Big[\big(\mathrm{Im}(f_n, f_n)^{-}_{\omega_{n,j}} - \mathrm{Im}(f_n, f_n)^{+}_{-\omega_{n,j}}\big)v_{n,j}^* v_{n,j} \\
&\qquad + \big(\mathrm{Im}(f_n, f_n)^{-}_{-\omega_{n,j}} - \mathrm{Im}(f_n, f_n)^{+}_{\omega_{n,j}}\big)v_{n,j} v_{n,j}^*\Big], \\
L_{n,j,1} &= (2\,\mathrm{Re}(f_n, f_n)^{+}_{-\omega_{n,j}})^{1/2} v_{n,j}, \\
L_{n,j,2} &= (2\,\mathrm{Re}(f_n, f_n)^{-}_{-\omega_{n,j}})^{1/2} v_{n,j}^*, \\
G_0 &= -\frac{1}{2}\sum_{n=1}^{N}\sum_{j=1}^{k}(L_{n,j,1}^* L_{n,j,1} + L_{n,j,2}^* L_{n,j,2}) - iH_1.
\end{aligned} \tag{4.3}$$

It is straightforward to show that

$$\begin{aligned}
&\{v_{m,j'},\ v_{m,j'}^*,\ v_{n,j}v_{n,j}^*,\ v_{n,j}^* v_{n,j};\ 1 \leq m, n, \leq N,\ 1 \leq j, j' \leq k\}' \\
&= \{v_{m,j'},\ v_{m,j'}^*;\ 1 \leq m \leq N,\ 1 \leq j' \leq k\}'.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\{L_{n,j,p},\ L_{n,j,p}^*;\ p = 1, 2,\ 1 \leq n \leq N,\ 1 \leq j \leq k\}' \\
&= \{H_1,\ L_{n,j,p},\ L_{n,j,p}^*;\ p = 1, 2,\ 1 \leq n \leq N,\ 1 \leq j = 1 \leq k\}'.
\end{aligned}$$

Because the thermodynamical equilibrium state ρ_s is a faithful stationary state for the quantum Markovian semigroup associated to the Pauli-Fierz system, then by Theorem 4.3, one can conclude. □

5. Quantum Langevin Equation Associated to the Pauli-Fierz System

In this section we recall the main results on the Hudson-Parthasarathy equation (or quantum Langevin equation). Moreover, we give the quantum Langevin equation associated to the Pauli-Fierz system.

Let $\mathcal{Z}$ be a separable Hilbert space for which we fix an orthonormal basis $\{z_k,\ k \geq 1\}$. We denote by $\Gamma_s(L^2(\mathbb{R}_+, \mathcal{Z})$, the symmetric Fock space constructed over the Hilbert space $L^2(\mathbb{R}_+, \mathcal{Z})$, which describes a boson exterior system. It is worthwhile to note that the space $L^2(\mathbb{R}_+, \mathcal{Z})$ is identified to the Hilbert space $L^2(\mathbb{R}_+) \otimes \mathcal{Z}$. Let $\mathcal{H}_0$ be the Hilbert space describing a quantum system, which is called *initial space*. Then, the combined system is described by the Hilbert space $\mathcal{H}_0 \otimes \Gamma_s(L^2(\mathbb{R}_+, \mathcal{Z}))$.

Consider the *system operators* $H,\ R_k$ and S_{kl}, $k,\ l \geq 1$, which are bounded operators in $\mathcal{H}_0$. Moreover, we assume that these operators satisfy

$$H = H^*,\ \sum_j S^*_{jk} S_{jl} = \sum_j S_{kj} S^*_{lj} = \delta_{kl} \tag{5.1}$$

and the sum $\sum_k R^*_k R_k$ converges strongly to a bounded operator.

Now, for any $f \in L^2(\mathbb{R}_+, \mathcal{Z})$, we define its exponential vector by

$$e(f) = \sum_{n \geq 0} \frac{f^{\otimes n}}{\sqrt{n!}}.$$

The basic quantum noises $A^+_i(t),\ A_i(t),\ \Lambda_{ij}(t)$ are the operators

$$\begin{aligned} A_i(t) &= a(1_{(0,t)} \otimes z_i), \\ A^+_i(t) &= a^*(1_{(0,t)} \otimes z_i), \\ \Lambda_{ij}(t) &= \Lambda(\pi_{(0,t)} \otimes |z_i\rangle\langle z_j|), \end{aligned}$$

where $i,\ j \geq 1$, $1_{(0,t)}$ is the indicator function over $(0,t)$, while $\pi_{(0,t)}$ is the multiplication operator by $1_{(0,t)}$ in $L^2(\mathbb{R}_+)$. With the notation $f_k(t) = \langle z_k,\ f(t)\rangle$, the quantum noises satisfy

$$\begin{aligned} A_i(t)e(f) &= \int_0^t f_i(s) ds\, e(f), \\ \langle e(g),\ A^+_i(t)e(f)\rangle &= \int_0^t \overline{g_i(s)} ds\, \langle e(g),\ e(f)\rangle, \\ \langle e(g),\ \Lambda^j_i(t)e(f)\rangle &= \int_0^t \overline{g_i(s)} f_j(s) ds\, \langle e(g),\ e(f)\rangle, \end{aligned}$$

where $A_0(t) = tI$ and $h_0(s) = 1$ for all $h \in L^2(\mathbb{R}_+, \mathcal{Z})$.

The Hudson-Parthasarathy equation can be viewed as a stochastic generalization of the Schrodinger equation

$$\begin{cases} dU_t = \big\{ \sum_k R_k dA^+_k(t) + \sum_{kl}(S_{kl} - \delta_{kl}) d\Lambda_{kl}(t) \\ \qquad\qquad - \sum_{kl} R^*_k S_{kl} dA_l(t) + G dt \big\} U_t \\ U_0 = I, \end{cases} \tag{5.2}$$

where $G = -iH - \frac{1}{2}\sum_k R^*_k R_k$.

For the proof of the following theorem, we refer the interested reader to [14].

Theorem 5.1. *Suppose that the system operators H, R_k, S_{kl} satisfy (5.1). Then, there exists a unique strongly continuous unitary adapted process U_t, which is solution of equation (5.2).*

For more physical point of view, these equations describe the evolution of quantum systems affected by a source of noise. Moreover, it is well known that any Hudson Parthasarathy equation, where the system operator satisfy (5.1), deletes a quantum Markovian semigroup in the following sense

$$\langle \Omega, U_t(X \otimes I)U_t^*\Omega\rangle = e^{t\mathcal{L}}(X),$$

where Ω is the vacuum vector of the Fock space $L^2(\mathbb{R}_+, \mathcal{Z})$ and

$$\mathcal{L}(X) = i[H, X] + \frac{1}{2}\sum_k \Big(2R_k^*XR_k - R_k^*R_kX - XR_k^*R_k\Big),$$

for all $X \in \mathcal{B}(\mathcal{H}_0)$.

From the above, the quantum Langevin equation associated to the Pauli-Fierz system is given by

$$\begin{cases} dU_t = \big\{ \sum_{p=1}^2 \sum_{n=1}^N \sum_{j=1}^k L_{n,j,p} dA^+_{n,j,p}(t) - L^*_{n,j,p} dA_{n,j,p}(t) + G_0 dt\big\} U_t \\ U_0 = I, \end{cases} \tag{5.3}$$

where the operators $G_0,\ L_{n,j,p}$ are defined by relation (4.3).

Following Attal-Pautrat (cf. [2]), the equation (5.3) can be obtained by a repeated quantum interaction Hamiltonian. The reservoir is modeled by an infinite chain of identical copies $\mathcal{H} = \mathbb{C}^{N+1} \otimes \mathbb{C}^{k+1} \otimes \mathbb{C}^3$ for which we fix an orthonormal basis $\mathcal{B} = \{e_{n,j,p};\ 0 \le n \le N,\ 0 \le j \le k,\ 0 \le p \le 2\}$, where the vacuum state is $\Omega = e_{0,0,0}$. Then, an orthonormal basis of $\mathcal{B}(\mathcal{H})$ is given by the family $\{a^{m,i,q}_{n,j,p};\ 0 \le, m, n \le N,\ 0 \le i, j \le k,\ 0 \le p, q \le 2\}$ such that

$$a^{m,i,q}_{n,j,p} e_{m',i',q'} = \delta_{(m,i,q),(m',i',q')} e_{n,j,p}.$$

The during time interaction of the small system $\mathcal{H}_S$ and a copy $\mathcal{H}$ is denoted by h and the unitary evolution is defined on $\mathcal{H}_S \otimes \mathcal{H}$ by $U = e^{-ihH}$, where

$$H = H_1 \otimes I + I \otimes H_R + \frac{1}{\sqrt{h}} \sum_{p=1}^{2}\sum_{n=1}^{N}\sum_{j=1}^{k} \Big[V_{n,j,p} \otimes a^{0,0,0}_{n,j,p} + V^*_{n,j,p} \otimes a^{n,j,p}_{0,0,0}\Big],$$

with H_R is the Hamiltonian of a copy $\mathcal{H}$ such that $H_R\Omega = 0,\ V_{n,j,p} = iL_{n,j,p}$, where $L_{n,j,p}$ and H_1 are given by (4.3).

The unitary evolution of the small system $\mathcal{H}_S$ in interaction with the l-th copy $\mathcal{H}_l$ of $\mathcal{H}$, in the reservoir chain, is the operator U_l which acts as U on $\mathcal{H}_S \otimes \mathcal{H}_l$ and as the identity on the other copies. Then, the discrete evolution equation associated to this model is defined on $\mathcal{H}_S \otimes \bigotimes_{\mathbb{N}^*} \mathcal{H}$ by

$$\begin{cases} u_{l+1} = U_{l+1}\, u_l \\ u_0 = I. \end{cases} \tag{5.4}$$

Now, our aim is to compute the coefficients $U^{m,i,q}_{n,j,p}$;$0 \le m, n \le N$, $0 \le i, j \le k$, $0 \le p, q \le 2$ of the matrix representation of U is the basis $\mathcal{B}$. One has

$$\begin{aligned} U_{0,0,0}^{0,0,0} &= \langle \Omega, U\Omega \rangle = I - ihH_1 - \frac{1}{2}h \sum_{p=1}^{2}\sum_{n=1}^{N}\sum_{j=1}^{k} V_{n,j,p}^* V_{n,j,p} + o(h), \\ U_{n,j,p}^{0,0,0} &= \langle e_{n,j,p}, U\Omega \rangle = -i\sqrt{h} V_{n,j,p} + o(h),\ n,j,p \geq 1, \\ U_{0,0,0}^{n,j,p} &= \langle \Omega, U e_{n,j,p} \rangle = -i\sqrt{h} V_{n,j,p}^* + o(h),\ n,j,p \geq 1, \\ U_{n,j,p}^{m,i,q} &= \langle e_{n,j,p}, U e_{m,i,q} \rangle = \delta_{(m,i,q),(n,j,p)} I + O(h). \end{aligned}$$

Therefore, Theorem 17 pp. 32 in [2] implies that the solution $u_{[t/h]}$ of the discrete evolution equation (5.4) converges strongly to the solution U_t of the stochastic differential equation (5.3), where we use the notation $A_{n,j,p}^+(t) = a_{n,j,p}^{0,0,0}(t)$ and $A_{n,j,p}(t) = a_{0,0,0}^{n,j,p}(t)$.

Acknowledgment. The author gratefully acknowledges stimulating discussions with Rolando Rebolledo.

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