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NONLINEAR FILTERING OF ITÔ-LÉVY STOCHASTIC DIFFERENTIAL EQUATIONS WITH CONTINUOUS OBSERVATIONS

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Abstract. We study the n-dimensional nonlinear filtering problem for jump-diffusion processes. The optimal filter is derived for the case when the observations are continuous. A proof of uniqueness is presented under fairly general circumstances.

1. Introduction

The problem of non-linear filtering can be described as follows: $X(t), t \geq 0$, called the signal or the state of the system, is a stochastic process, direct observation of which is not possible. We get information about $X(t), t \geq 0$ by observing a stochastic process $Y(t)$ which is a function of $X(t)$ corrupted by noise. Thus, at time t, the σ- algebra $\mathcal{Y}_t = \sigma\{Y(s); 0 < s < t\}$ provides the partial information available about $X(t)$, and our knowledge of $X(t)$ reduces, using the least square error criterion, to the conditional distribution of $X(t)$ given $\mathcal{Y}_t$ and represented by a measure-valued process P_t.

The primary goal of non-linear filtering theory is to characterize the evolution of the conditional distribution. This evolution is described in terms of an infinite-dimensional stochastic differential equation (or if $X(t)$ is a finite-dimensional diffusion, a stochastic partial differential equation). This is well known as the Kushner or the Fujisaki-Kallianpur-Kunita (FKK) equation (see Kushner [13], Fujisaki, Kalliapur and Kunita [7], Kallianpur [10]). Zakai [26] obtained an equivalent stochastic partial differential equation for a measure-valued process p_t which is easier to handle because it is linear. The process p_t is called the unnormalized conditional distribution of $X(t)$ given $\mathcal{Y}_t$. The filtering problem is said to be solvable if it can be shown that P_t (or p_t) is the unique solution of the FKK (respectively, Zakai) equation. This has been done in the case of finite dimensional signal using techniques from p.d.e. theory (see Baras, Blankenship and Hopkins [2], Sheu [21], Chaleyat-Maurel, Michel and Pardoux [6], Rozovskii [20], Bensoussan [4]) and using martingale problems (see Hijab [8], Kurtz and Ocone [12], Bhatt, Kalliapur and Karandikar [3]). Nonlinear filtering of stochastic PDEs is developed in Sritharan and Hobbs [23] and Sritharan [24].

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In this paper we study the characterization of the conditional distribution for non-linear filtering models in which the signal $X(t)$ is a $\Re^n$- valued Itô-Lévy process and the observation process $Y(t)$ is a $\Re^m$- valued stochastic process which in particular, it is a continuous function of the signal that satisfies a linear growth condition. We derive the Zakai and FKK equation for the filter and we show that, under suitable assumptions, they have a unique solution in the weak sense. We improve the uniqueness results of Kallianpur [10] by considering a priori estimates of the signal.

The paper is organized as follows. Section 1 contains a description of the model for the signal process. The main result of this paper, an apriori estimate of the signal process is derived in Section 2. Section 3 formulates the nonlinear filtering problem. The derivation of the Zakai and FKK equation are presented here also. Section 5 is devoted to the proof of the uniqueness of the FKK equation using Feller semigroups.

2. Jump-Diffusion Processes

Suppose we are given a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a σ-field filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We will focus on the following n-dimensional SDE with jumps, of the type considered by Øksendal and Sulem in [17]:

$$\begin{aligned} X^i(t) = X^i(0) &+ \int_0^t \alpha^i(X(s))ds + \sum_{j=1}^n \int_0^t \beta^i_j(X(s))dW_j(s) \\ &+ \sum_{j=1}^n \int_0^t \int_{\mathbb{R}} \gamma^i_j(X(s-), z_j)\tilde{N}_j(ds, dz_j); \quad 1 \leq i \leq n, \end{aligned} \tag{2.1}$$

where $W(t) = (W_1(t), \dots, W_n(t))$, $t \geq 0$, is an n-dimensional $\mathcal{F}_t$-adapted standard Brownian motion and $\tilde{N}_j(ds, dz_j)^T = \big(\tilde{N}_1(ds, dz_1), \dots, \tilde{N}_n(ds, dz_n)\big) = \big(N_1(ds, dz_1) - \chi_{|z_1|<R}\nu_1(dz_1)ds, \dots, N_n(ds, dz_n) - \chi_{|z_n|<R}\nu_n(dz_n)ds\big)$, where $\{N_j\}$ are independent Poisson random measures with Lévy measures ν_j coming from n independent Lévy processes $\eta_1, \dots, \eta_n$, and $R \in (0, \infty)$.

We also consider that W and $\tilde{N}$ are independent of $\mathcal{F}_0$ and that the initial condition of (2.1), $X(0) = X_0$ (a.s.), for which X_0 is $\mathcal{F}_0$-measurable.

We need to assume that the coefficients in (2.1) satisfy the following assumption: $\alpha : [0, \infty] \times \Re^n \times \Omega \to \Re^n$, $\beta : [0, \infty] \times \Re^n \times \Omega \to \Re^n \times \Re^n$, $\gamma : [0, \infty] \times \Re^n \times Z \times \Omega \to \Re^n \times \Re^n$ are assumed to be measurable. Further conditions on these mappings will follow later.

The results presented in this paper also hold for the type of SDEs with jumps considered by Applebaum in Chapter 6 of [1] where a single Poisson measure is used.

A process of the form (2.1) is a semimartingale (see Applebaum [1], Proposition 2.6.1). Therefore, it can be written as

$$X^i(t) = X^i(0) + \sum_{j=1}^n \int_0^t \eta^i_j(X(s))ds + \sum_{j=1}^n M^i_j(t), \tag{2.2}$$

where

$$\eta_j^i(X(t)) = \alpha^i(X(t)) + \int_{|z_j|\geq R} \gamma_j^i(X(t-), z_j) N_j(t, dz_j) \tag{2.3}$$

is an adapted process of finite variation and

$$M_j^i(t) = \int_0^t \beta_j^i(X(s)) dW_j(s) + \int_0^t \int_{|z_j|<R} \gamma_j^i(X(s-), z_j) \tilde{N}_j(ds, dz_j) \tag{2.4}$$

is a martingale with respect to the increasing σ-field family $\{\mathcal{F}_t\}_{t\geq 0}$ for every $1 \leq i, j \leq n$.

Let M and N be two real square integrable martingales with right continuous paths (defined on the same complete probability space $(\Omega, \mathcal{F}, P)$ with a σ-field filtration $\{\mathcal{F}_t\}_{t\geq 0}$). We recall the definition of the following processes $[M, N]$ and $\triangleleft M, N \triangleright$ called respectively the "mutual variation" and the Meyer process of the couple (M, N).

The process $[M, N]$ has the following two properties:

$$M(t)N(t) = M(0)N(0) + \int_0^t M(s-)dN(s) + \int_0^t N(s-)dM(s) + [M, N]_t \tag{2.5}$$

and

$$[M, N]_t = \lim prob_{n\to\infty} \sum_{t_i^n \in \Pi^n} \left(M_{t_{i+1}^n \wedge t} - M_{t_i^n \wedge t} \right) \left(N_{t_{i+1}^n \wedge t} - N_{t_i^n \wedge t} \right), \tag{2.6}$$

for any sequence $\{\Pi^n\}$ of subdivisions $\Pi^n = \{0 = t_0^n < t_1^n < \dots t_k^n < \dots\}$. Here

1. $\lim_{k\uparrow\infty} t_k^n = \infty$.
2. $\sup_k |t_{k+1}^n - t_k^n| \leq \frac{1}{n}$.

The process $\triangleleft M, N \triangleright$ is, by definition, the unique right continuous predictable process with paths of finite variation such that

$$M(t)N(t) - M(0)N(0) - \triangleleft M, N \triangleright \tag{2.7}$$

ia a martingale. When $M = N$, one usually notes $[M]$ instead of $[M, M]$ and $\triangleleft M \triangleright$ instead of $\triangleleft M, M \triangleright$. $[M]$ is called the quadratic variation of M and $\triangleleft M \triangleright$ the Meyer process of M. Both processes are increasing with $[M]_0 = \triangleleft M \triangleright_0 = 0$. If M and N are continuous, then $[M, N] = \triangleleft M, N \triangleright$, otherwise they are not.

Let us also note that (2.7) implies the property

$$E|M(T)|^2 = E[M]_T = E \triangleleft M \triangleright_T . \tag{2.8}$$

Let X(t) be a process whose dynamics are described by (2.1) and $X(t-) = \lim_{s\to t-} X(s)$. We define $\Delta X(t) = X(t) - X(t-)$, the jump at time t.

Lemma 2.1. *Let f be a C^2-class function on $\Re^n$. The following Itô formula holds:*

$$\begin{aligned} f(X(t)) &= f(X(0)) + \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(X(s-))dX^i(s) \\ &+ \frac{1}{2}\sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X(s-))d[X^i, X^j]_s \\ &+ \sum_{0<s\le t} \Big\{ f(X(s)) - f(X(s-)) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(s-))\Delta X^i(s) \\ &- \sum_{i,j=1}^{n} \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(X(s-))\Delta X^i(s)\Delta X^j(s) \Big\}. \end{aligned} \tag{2.9}$$

Proof. See Protter [19], p.79. □

3. A Priori Estimate and Existence of Solutions

We impose some conditions on the mappings η, β and γ that will enable us to solve equation (2.1). First, for each $f \in \Re^n$ and $g \in \Re^{n\times n}$ we introduce the following notation

$$\|f\|_1^2 = \sum_{i=1}^{n} |f^i|^2 \text{ and } \|g\|_2^2 = \sum_{i,j=1}^{n} |g_j^i|^2. \tag{3.1}$$

We impose the following two conditions:

(C1) **Lipschitz condition** There exists $K_1 > 0$ such that, for all $x, y \in \Re^n$,

$$\|\eta(x)-\eta(y)\|_2^2 + \|\beta(x)-\beta(y)\|_2^2 + \int_{|z|<R} \|\gamma(x,z)-\gamma(y,z)\|_2^2 \nu(dz) \le K_1\|x-y\|_1^2. \tag{3.2}$$

(C2) **Growth condition** There exists $K_2 > 0$ such that, for all $x \in \Re^n$,

$$\|\eta(x)\|_2^2 + \|\beta(x)\|_2^2 + \int_{|z|<R} \|\gamma(x,z)\|_2^2 \nu(dz) \le K_2(1+\|x\|_1^2). \tag{3.3}$$

Lemma 3.1. *Assume that $X(t)$ verifies (2.1) with $E|X(0)| < \infty$.*
Then, for any $0 < T < \infty$

$$E \sup_{0\le t\le T} |X(t)|^{2d} \le k_T < \infty, \text{ for any } d \ge 1. \tag{3.4}$$

Proof. To prove the inequality (3.4), we will use the finite form of Jensen's inequality for $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$.

$$\varphi\left(\frac{\sum |x_i|}{m}\right) \le \frac{\sum \varphi(|x_i|)}{m}. \tag{3.5}$$

Inequality (3.5) applied for the convex function $\varphi(x) = x^{2d}$ gives that

$$|X(t)|^{2d} \leq 4^{2d-1} |X(0)|^{2d} + 4^{2d-1} \left| \int_0^t \eta(X(s))ds \right|^{2d} + 4^{2d-1} \left| \int_0^t \beta(X(s))dW(s) \right|^{2d} + 4^{2d-1} \left| \int_0^t \int_{|z|<R} \gamma(X(s),z)\tilde{N}(dz,ds) \right|^{2d}. \tag{3.6}$$

Taking supremum over all $t \in [0,T]$ and expectation, we obtain

$$\begin{aligned} E \sup_{0\leq t\leq T} |X(t)|^{2d} \leq 4^{2d-1} |X(0)|^{2d} &+ 4^{2d-1} E \sup_{0\leq t\leq T} \left| \int_0^t \eta(X(s))ds \right|^{2d} \\ &+ 4^{2d-1} E \sup_{0\leq t\leq T} \left| \int_0^t \beta(X(s))dW(s) \right|^{2d} \\ &+ 4^{2d-1} E \sup_{0\leq t\leq T} \left| \int_0^t \int_{|z|<R} \gamma(X(s),z)\tilde{N}(dz,ds) \right|^{2d}. \end{aligned} \tag{3.7}$$

Using Hölder's inequality

$$\left| \int_0^t \eta(X(s))ds \right|^{2d} \leq t^{2d-1} \int_0^t |\eta(X(s))|^{2d} ds \tag{3.8}$$

and the Burkholder-Davis-Gundy inequality applied to the martingale terms $\int_0^t \beta(X(s))dW(s)$ and $\int_0^t \int_{|z|<R} \gamma(X(s),z)\tilde{N}(dz,ds)$,

$$\begin{aligned} 4^{2d-1} E \sup_{0\leq t\leq T} \left| \int_0^t \beta(X(s))dW(s) \right|^{2d} &\leq C_1^{(d)} E \left[\int_0^T |\beta(X(s))|^2 ds \right]^d \\ &\leq C_1^{(d)} E \int_0^T |\beta(X(s))|^{2d} ds \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &4^{2d-1} E \sup_{0\leq t\leq T} \left| \int_0^T \int_{|z|<R} \gamma(X(s),z)\tilde{N}(dz,ds) \right|^{2d} \\ &\leq C_2^{(d)} E \left[\int_0^T \int_{|z|<R} |\gamma(X(s),z)|^2 \nu(dz)ds \right]^d \\ &\leq C_2^{(d)} E \int_0^T \int_{|z|<R} |\gamma(X(s),z)|^{2d} \nu(dz)ds, \end{aligned} \tag{3.10}$$

we obtain

$$
\begin{aligned}
&E \sup_{0\le t\le T} |X(t)|^{2d} \\
&\le 4^{2d-1}E|X(0)|^{2d} + C^{(d)}(T)\Big[\int_0^T E|\eta(X(s))|^{2d}ds + \int_0^T E|\beta(X(s))|^{2d}ds \\
&+ \int_0^T \int_{|z|<R} E|\gamma(X(s),z)|^{2d}\nu(dz)ds\Big] \\
&\le 4^{2d-1}E|X(0)|^{2d} + C(T)\int_0^T E\sup_{r\le s}|X(r)|^{2d}ds.
\end{aligned}
\tag{3.11}
$$

On applying the Gronwall's inequality, we can finally deduce that

$$E \sup_{0\le t\le T} |X(t)|^{2d} \le 4^{2d-1}E|X(0)|^{2d}e^{C(T)T} = k_T < \infty. \tag{3.12}$$

□

Theorem 3.2. *Assume the Lipschitz and growth conditions (3.2) and (3.3). There exists a unique solution $X = (X(t), t \ge 0)$ to the SDE (2.1) with the standard initial condition. The process X is adapted and càdlàg.*

Proof. See Applebaum [1], p.304-310. □

4. The Non-linear Filtering Equations

We are interested in the estimation of the $\Re^n$-valued stochastic process $X = (X(t), t \ge 0)$ whose dynamics is described by equation (2.1) which cannot be observed directly. Instead, we have an observation process $Y = (Y(t), t \ge 0)$ which is related to $X(t)$ as follows:

$$Y(t) = Y(0) + B(t) + \int_0^t h(X(s))ds, \tag{4.1}$$

where

- (A1) the observation process $Y(t)$ is assumed to be a $\Re^m$-valued stochastic process $(m < n)$.
- (A2) $B(t)$ is a m-dimensional $\mathcal{F}_t$-adapted standard Brownian motion
- (A3) $h(X(t))$ is a (t,ω)-measurable process which satisfies the quadratic growth condition $|h(x)| \le C(1 + |x|^2)$, for all $t \in [0,T]$.
- (A4) For each $s \in [0,T]$, the σ-fields $\sigma\{h(X(u)), B(u); 0 < u < s\}$ and $\sigma\{B(v) - B(u); s < u < v < T\}$ are independent.

Let $\mathcal{Y}_t = \sigma\{Y(s); 0 < s < t\}$. The monotone family $(\mathcal{Y}_t)$ represents all the statistical data or information concerning the signal process $X(t)$.

Let f be a measurable function on $\Re^n$ such that f satisfies the quadratic growth condition $|f(x)| \le C(1 + |x|^2)$, for all $t \in [0,T]$. The filtering problem of interest is the computation of the least squares estimate of $f(X(t))$ given the observation information $\mathcal{Y}_t$: $E[f(X(t))|\mathcal{Y}_t]$. One of the methods known to solve this problem is the change of measure method. By a version of Girsanov's change of measure theorem, we can define a new measure P_0, such that the observations $Y(t)$ become P_0-independent of the signal variable $X(t)$. This can be done by choosing, for each $t > 0$:

$$\frac{dP_0}{dP} = \exp\left[-\sum_{k=1}^{m}\int_0^t h_k(X(s))dB^k(s) - \frac{1}{2}\int_0^t |h(X(s))|^2 ds\right]. \tag{4.2}$$

Most of the results available in the literature assume the following Novikov condition to guarantee that $\frac{dP_0}{dP}$ is a martingale (w.r.t. $\mathcal{F}_t$ and P):

$$E\left[\exp\left(\frac{1}{2}\int_0^t |h(X(s))|^2 ds\right)\right] < \infty, \tag{4.3}$$

where E is the expectation w.r.t. P.

However, we assume here a much weaker condition (see Thorem 3.1, Mikulevicious, Rozozvskii [16]) which ensures that (4.2) is a martingale:

$$P\left[\int_0^\tau |h(X(s))|^2 ds < \infty\right] = 1, \tag{4.4}$$

where τ is a stopping time.

Define

$$\Lambda_t = E^0\left[\frac{dP}{dP_0}\bigg|\mathcal{Y}_t\right], \tag{4.5}$$

where E^0 denotes expectation under the reference measure P_0.

Then we have the representation:

$$E[f(X(t))|\mathcal{Y}_t] = \frac{E^0[f(X(t))\Lambda_t|\mathcal{Y}_t]}{E^0[\Lambda_t|\mathcal{Y}_t]}, \tag{4.6}$$

which is the Kallianpur-Striebel's formula (Theorem 3, Kallianpur, Striebel [11]).

Lemma 4.1. *With the equivalent measure $P \sim P_0$ defined in (4.2), we have the following:*

1. *Under P_0, $Y(t)$ is a standard Brownian Motion.*
2. $$\Lambda_t = 1 + \sum_{k=1}^{m}\int_0^t \Lambda_{s-}h_k(X(s-))dY^k(s). \tag{4.7}$$
3. *Under P_0, the process $X(t)$ and $Y(t)$ are independent.*
4. $E^0[\Lambda_t|\mathcal{F}_t] = 1$.
5. *The restrictions of P_0 and P to the $\mathcal{F}_t$ are the same.*

Proof. 1. and 3. follow from Bensoussan [4] p.77-81, while 2. follows from Itô's formula. 4. follows from 3. since the σ-algebras $\mathcal{F}_t$ and $\mathcal{Y}_t$ are independent under P_0, which means that Λ_t is an P_0-martingale for every fixed trajectory of $X(t)$. 5. is a consequence of equation (4.2). □

Definition 4.2. Let $X(t) \in \Re^n$ be a jump diffusion. Then the *generator* of X is defined on functions $f : \Re^n \to \Re$ by

$$Af(x) = \lim_{t \to 0+} \frac{1}{t} \left\{ E^x[f(X(t))] - f(x) \right\} \qquad \text{(if the limit exists)}, \tag{4.8}$$

where $E^x[f(X(t))] = E[f(X^{(x)}(t))]$, $X^{(x)}(0) = x$.

Theorem 4.3. *Suppose $f \in C_0^2(\Re^n)$ (C^2-class function that vanishes at infinity) and $X(t)$ is given by (2.1). Then $Af(x)$ exists and is given by*

$$\begin{aligned} Af(x) = &\sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}(x)\eta_j^i(x) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \sum_{l=1}^{n} \beta_l^i(x)\beta_l^j(x) \\ &+ \int_{\mathbb{R}^n} \sum_{i=1}^{n} \left\{ f\left(x + \gamma^{(i)}(x,z)\right) - f(x) - \nabla f(x)\gamma^{(i)}(x,z) 1_{\{|z|<R\}}(z) \right\} \nu_i(dz_i). \end{aligned} \tag{4.9}$$

Proof. See Skorokhod [22], p.170. □

Definition 4.4. The *unnormalized conditional distribution* of $f(X(t))$, given Y_t is defined as $p_t(f) = E^0[f(X(t)\Lambda_t|\mathcal{Y}_t]$.

Theorem 4.5. *Assume (2.1) and (4.1), conditions $(A)_1$, $(A)_2$, (C1), (C2) and conditions (A1) to (A3). If f is a function belonging to the domain of the operator A defined in (4.8) we have:*

$$p_t(f) = p_0(f) + \int_0^t p_{s-}(Af)ds + \sum_{k=1}^{m} \int_0^t p_{s-}(fh_k)dY^k(s). \tag{4.10}$$

Proof. First we determine the form of $f(X(t))\Lambda_t$. We use the Itô product rule for semimartingales and the fact that the processes $X(t)$ and $Y(t)$ are independent under P_0, to obtain:

$$f(X(t))\Lambda_t = \int_0^t f(X(s-))d\Lambda_s + \int_0^t \Lambda_{s-} df(X(s)) + f(x(0)). \tag{4.11}$$

We take now conditional expectation in equation (4.11) to obtain:

$$\begin{aligned} E^0\left[f(X(t))\Lambda_t|\mathcal{Y}_t\right] = &E^0\left[\int_0^t f(X(s-))d\Lambda_s \middle| \mathcal{Y}_t\right] + \\ &+ E^0\left[\int_0^t \Lambda_{s-} df(X(s)) \middle| \mathcal{Y}_t\right] + E^0\left[f(X(0)) \middle| \mathcal{Y}_t\right]. \end{aligned} \tag{4.12}$$

We compute the conditional expectations on the right-hand side of equation (4.12) in their order of appearance:

$$
\begin{aligned}
E^0\left[\int_0^t f(X(s-))d\Lambda_s\Big|\mathcal{Y}_t\right] &= E^0\left[\sum_{k=1}^m\int_0^t f(X(s-))h_k(X(s-))\Lambda_{s-}dY^k(s)\Big|\mathcal{Y}_t\right]\\
&= \sum_{k=1}^m\int_0^t E^0\left[f(X(s-))h_k(X(s-))\Lambda_{s-}|\mathcal{Y}_s\right]dY^k(s)\\
&= \sum_{k=1}^m\int_0^t p_{s-}(fh_k)dY^k(s).
\end{aligned}
\tag{4.13}
$$

For the calculation of the second term in (4.12), we will first evaluate the following quantity using equation (2.9):

$$
\begin{aligned}
df(X(t)) = &\sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(t-))\Bigg[\alpha^i(X(t))dt + \sum_{j=1}^n \beta^i_j(X(t))dW^j(t)\\
&+ \sum_{j=1}^n\int_{\mathbb{R}}\gamma^i_j(X(t),z_j)\tilde{N}_j(t,dz_j)\Bigg]\\
&+ \frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2 f}{\partial x_i\partial x_j}(X(t-))\sum_{l=1}^n\beta^i_l(X(t))\beta^j_l(X(t))dt\\
&+ \sum_{0<s\le t}\left[f(X(s))-f(X(s-))-\sum_{i=1}^n\frac{\partial f}{\partial x_i}(X(s-))\left(X^i(s)-X^i(s-)\right)\right].
\end{aligned}
\tag{4.14}
$$

Hence

$$
\begin{aligned}
&E^0\left[\int_0^t \Lambda_{s-}df(X(s))\Big|\mathcal{Y}_t\right]\\
&= E^0\Bigg\{\int_0^t\Lambda_{s-}\Bigg[\Bigg(\sum_{i=1}^n\frac{\partial f}{\partial x_i}(X(s-))\alpha^i(X(s))\\
&+ \frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2 f}{\partial x_i\partial x_j}(X(s-))\sum_{l=1}^n\beta^i_l(X(s))\beta^j_l(X(s))dt\Bigg)ds\\
&+ \sum_{i,j=1}^n\int_{|z_j|\ge R}\frac{\partial f}{\partial x_i}(X(s-))\gamma^i_j(X(s),z_j)N_j(ds,dz_j)\\
&+ \sum_{0<s\le t}\left[f(X(s))-f(X(s-))-\sum_{i=1}^n\frac{\partial f}{\partial x_i}(X(s-))\left(X^i(s)-X^i(s-)\right)\right]\Bigg]\Bigg|\mathcal{Y}_t\Bigg\}\\
&+ E^0\Bigg[\int_0^t\Lambda_{s-}\sum_{i,j=1}^n\frac{\partial f}{\partial x_i}(X(s-))\beta^i_j(X(s))dW_j(s)\Big|\mathcal{Y}_t\Bigg]
\end{aligned}
$$

$$+ E^0\left[\int_0^t \Lambda_{s-} \sum_{i,j=1}^n \int_{|z_j|<R} \frac{\partial f}{\partial x_i}(X(s-))\gamma_j^i(X(s), z_j) \left(\widetilde{N}_j(ds, dz_j) - \nu_j(ds, dz_j)\right)\bigg|\mathcal{Y}_t\right]. \quad (4.15)$$

Under the reference measure P_0 the process $X(t)$ is independent of $\mathcal{Y}_t$ and the standard Brownian motion $W(t)$ is also independent of $\mathcal{Y}_t$ so it follows

$$E^0\left[\int_0^t \Lambda_s \sum_{i,j=1}^n \frac{\partial f}{\partial x_i}(X(s-))\beta_j^i(X(s))dW_j(s)\bigg|\mathcal{Y}_t\right] = 0. \quad (4.16)$$

Similar arguments explain the next result

$$E^0\left[\int_0^t \Lambda_{s-} \sum_{i,j=1}^n \int_{|z_j|<R} \frac{\partial f}{\partial x_i}(X(s-))\gamma_j^i(X(s), z_j) \left(\widetilde{N}_j(ds, dz_j) - \nu_j(ds, dz_j)\right)\bigg|\mathcal{Y}_t\right] = 0. \quad (4.17)$$

With Theorem 1 and equations (4.16) and (4.17), equation (4.15) becomes

$$\begin{aligned} E^0\left[\int_0^t \Lambda_{s-} df(X(s))\bigg|\mathcal{Y}_t\right] &= E^0\left[\int_0^t Af(X(s))\Lambda_{s-}ds\bigg|\mathcal{Y}_t\right] \\ &= \int_0^t E^0\left[Af(X(s))\Lambda_{s-}|\mathcal{Y}_s\right]ds \quad (4.18) \\ &= \int_0^t p_{s-}(Af)dY(s). \end{aligned}$$

Obviously

$$E^0\left[f(X(0))|\mathcal{Y}_t\right] = p_0(f). \quad (4.19)$$

Collecting all the terms (4.13)-(4.19), equation (4.12) becomes the stochastic integral equation (4.10). □

Definition 4.6. The *conditional measure* P_t of $f(X_t)$ given $\mathcal{Y}_t$ is defined by $P_t(f) = E[f(X(t))|\mathcal{Y}_t]$.

Corollary 4.7. *Let conditions A1-A3 hold. If* $f \in D(A)$ *then the filter* $P_t(f)$ *satisfies the stochastic equation*

$$P_t(f) = P_0(f) + \int_0^t P_s(Af)ds + \sum_{k=1}^n \int_0^t \left[P_s(fh_k) - P_s(f)P_s(h_k)\right]d\nu_s^k \quad (4.20)$$

with $\nu_t^k = Y^k(t) - \int_0^t P_s(h_k)ds$ *being the innovation process.*

Proof. We will first show that $E\int_0^t P_s(h_k)dY(s) = 0$, for every $k = 1, \ldots, n$. This is equivalent with proving that $\int_0^t E|P_s(h_k)|^2ds < \infty$. Using Jensen's inequality for conditional expectation (see Øksendal [17]), we obtain that

$$\begin{aligned}\int_0^t E|P_s(h_k)|^2 ds &= \int_0^t E|E\left[h_k(X(s))|\mathcal{Y}_s\right]|^2 ds \\ &\le \int_0^t E\left[E|h_k(X(s))|^2|\mathcal{Y}_s\right] ds \\ &= \int_0^t E|h_k(X(s))|^2 ds,\end{aligned} \tag{4.21}$$

which, according to Lemma 2, is finite since h has quadratic growth. Hence, using the Itô isometry, $E\int_0^t P_s(h_k)dY^k(s) = 0$, for every $k = 1, \dots, n$.

Secondly, we will prove that $E\int_0^t |P_s(Af)|ds < \infty$. With equation (4.9), we have that

$$\begin{aligned}E\int_0^t |Af(X(s))|\, ds \le& \sum_{i,j=1}^n \int_0^t E\left|\frac{\partial f}{\partial x_i}(X(s))\eta_j^i(X(s))\right| ds \\ &+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t E\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(X(s))\sum_{l=1}^n \beta_l^i(X(s))\beta_l^j(X(s))\right| ds \\ &+ \int_0^t \left|\int_{\mathbb{R}^n}\sum_{i=1}^n \left\{ f\left(X(s)+\gamma^{(i)}(X(s),z)\right) - f(X(s)) \right.\right. \\ &\left.\left. - \nabla f(X(s))\gamma^{(i)}(X(s),z)1_{\{|z|<R\}}(z)\right\}\nu_i(dz_i)\right| ds.\end{aligned} \tag{4.22}$$

Using a second-order Taylor expansion for f we obtain

$$\begin{aligned}E\int_0^t |Af(X(s))|\, ds \le& \sum_{i,j=1}^n \int_0^t E\left|\frac{\partial f}{\partial x_i}(X(s))\eta_j^i(X(s))\right| ds \\ &+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t E\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(X(s))\sum_{l=1}^n \beta_l^i(X(s))\beta_l^j(X(s))\right| ds \\ &+ \sum_{i,j=1}^n \int_0^t \left|\int_{|z|<R} \frac{1}{2!}\frac{\partial^2 f}{\partial x_i \partial x_j}(X(s))\gamma^{(i)}(X(s),z)\gamma^{(j)}(X(s),z)\right. \\ &\left. + \mathcal{R}_2\left(X(s)+\gamma^{(i)}(X(s),z)\right)\right|\nu_i(dz_i)ds.\end{aligned} \tag{4.23}$$

Young's inequality applied to each of the terms in the above sum gives that

$$\begin{aligned}&E\int_0^t |Af(X(s))|\, ds \\ &\le \sum_{i,j=1}^n \int_0^t E \quad \frac{3}{4}\left|\frac{\partial f}{\partial x_i}(X(s))\right|^{4/3} + \frac{1}{2}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(X(s))\right|^2\Bigg) ds\end{aligned}$$

$$+\frac{1}{4}\sum_{i,j=1}^{n}\int_0^t E\bigg(\left|\eta_j^i(X(s))\right|^4+\left|\sum_{l=1}^{n}\beta_l^i(X(s))\beta_l^j(X(s))\right|^2$$
$$+\int_{|z|<R}\left|\gamma^{(i)}(X(s),z)\gamma^{(j)}(X(s),z)\right|^2\nu_i(dz_i)\bigg)ds \tag{4.24}$$
$$+\sum_{i=1}^{n}\int_0^t\int_{|z|<R}\left|\mathcal{R}_2\left(X(s)+\gamma^{(i)}(X(s),z)\right)\right|\nu_i(dz_i)ds.$$

We started with the assumption that f is a C^2-class function in $\mathbb{R}^n$. It follows that the first integral term in the right hand side of the inequality (4.24) is bounded. The remainder $\mathcal{R}_2$ is also bounded since $f \sim |x|^2$. We can now use the growth condition C2 to obtain that

$$E\int_0^t |Af(X(s))|\,ds \le C+\int_0^t\left(1+2E\sup_{r\le s}|X(r)|^2+E\sup_{r\le s}|X(r)|^4\right)<\infty \tag{4.25}$$

according to Lemma 2. We obtained that $E\int_0^t |Af(X(s))|ds < \infty$ which, using similar arguments as for equation (4.21), implies that $E\int_0^t |P_s(Af)| < \infty$.

We will proceed now to the proof of the main part of the theorem. From (4.6), we have that

$$P_t(f)=p_t(f)\left(p_t(1)\right)^{-1}. \tag{4.26}$$

Using Theorem 2 and equation (4.6) we have that

$$p_t(1)=1+\sum_{k=1}^{m}\int_0^t p_s(h_k)dY^k(s)=1+\sum_{k=1}^{m}\int_0^t p_s(1)P_s(h_k)dY^k(s). \tag{4.27}$$

Hence, with Itô's formula applied to equation (4.26),

$$P_t(f)=P_0(f)+\int_0^t\frac{1}{p_s(1)}dp_s(f)-\int_0^t\frac{p_s(f)}{p_t(1)^2}dp_t(1)$$
$$-\frac{1}{2}(-2)\int_0^t\frac{p_s(f)}{p_s(1)^3}dp_s(1)^2-\int_0^t\frac{1}{p_s(1)^2}dp_s(f)dp_s(1)$$

$$=P_0(f)+\int_0^t\frac{p_s(Af)}{p_s(1)}ds-\sum_{k=1}^{m}\int_0^t\frac{p_s(f)P_s(h_k)p_s(1)}{p_s(1)^2}dY^k(s) \tag{4.28}$$
$$+\sum_{k=1}^{m}\int_0^t\frac{p_s(f)P_s(h_k)^2p_s(1)^2}{p_s(1)^3}ds-\sum_{k=1}^{m}\int_0^t\frac{p_s(fh_k)P_s(h_k)p_s(1)}{p_s(1)^2}ds$$
$$+\sum_{k=1}^{m}\int_0^t\frac{p_s(fh_k)}{p_s(1)}dY^k(s)$$

Hence

$$\begin{aligned} P_t(f) &= P_0(f) + \int_0^t P_s(Af)ds - \sum_{k=1}^m \int_0^t P_s(f)P_s(h_k)dY^k(s) \\ &+ \sum_{k=1}^m \int_0^t P_s(h_k)^2 P_s(f)ds - \sum_{k=1}^m \int_0^t P_s(fh_k)P_s(h_k)d(s) \\ &+ \sum_{k=1}^m \int_0^t P_s(fh_k)dY^k(s) \\ &= P_0(f) + \int_0^t P_s(Af)ds - \sum_{k=1}^m \int_0^t P_s(h_k)P_s(f)[dY^k(s) - P_s(h_k)ds] \\ &+ \sum_{k=1}^m \int_0^t P_s(fh_k)[dY^k(s) - P_s(h_k)ds] \\ &= P_0(f) + \int_0^t P_s(Af)ds + \sum_{k=1}^m \int_0^t [P_s(fh_k) - P_s(f)P_s(h_k)]d\nu_s^k. \end{aligned} \tag{4.29}$$

with $\nu_t^k = Y^k(t) - \int_0^t P_s(h_k)ds$, the innovation process. □

5. Uniqueness Results

In this section we shall prove that the optimal filter $P_t(f)$ is the unique solution of a stochastic equation related to (4.20). Let the semigroup $T_t(t \geq 0)$ associated with the transitional probabilities $T_t(x, E)$ be a Feller semigroup, that is

$$T_t f(x) = E\left(f(X(t))|X(0) = x\right). \tag{5.1}$$

The Feller semigroup maps $C_0(\Re^n)$ (the space of all continuous functions on $\Re^n$ that vanish at infinity) into itself for all $t \geq 0$ and satisfies $\lim_{t\to 0} \|T_t f - f\| = 0$ for all $f \in C_0(\Re^n)$ (see Applebaum [1], p.126). Let A denote the infinitesimal generator defined in equation (4.9) and D(A) the domain of A.
The stochastic differential equation (4.20) is satisfied for all $f \in D(A)$. It follows that for each f in D(A), $P_t(f)$ has a sample continuous version (see Kallianpur [10], p.290). Since (T_t) is a Feller semigroup, D(A) is dense in $C_0(\Re^n)$ (for a complete proof of this result, see Theorem 3.2.6, Applebaum [1]). Using this fact, there exist a sample-continuous version of $P_t(f)$ for each $f \in C_0(\Re^n)$.

Lemma 5.1. *For each $f \in D(A)$, $P_t(f)$ satisfies the equation*

$$P_t(f) = P_0\left(T_t f\right) + \sum_{k=1}^m \int_0^t \left[P_s\left(\left(T_{t-s}f\right)h_k\right) - P_s\left(T_{t-s}f\right)P_s(h_k)\right]d\nu_s^k. \tag{5.2}$$

Proof. This result can be found in Kallianpur [10], p.290. Note that $E[f(X_t)|\mathcal{F}_s] = T_{t-s}f(X_s)$. The proof is then reduced to proving that

$$E\left([P_t(f) - P_0\left(T_t f\right)]\left(Y_t - Y_0\right)\right) = \sum_{k=1}^{m}\int_0^t E\left[P_s\left(\left(T_{t-s}f\right)h_k\right) - P_s\left(T_{t-s}f\right)P_s(h_k)\right]\phi_s^k ds \quad (5.3)$$

for all square-integrable martingales Y_t of the form

$$Y_t - Y_0 = \sum_{k=1}^{m}\int_0^t \phi_s^k d\nu_s^k \quad (5.4)$$

since such Y_t is dense in $L^2(\mathcal{Y}_t, P)$ (up to some constants) and $\phi_s = \left(\phi_s^1, \dots, \phi_s^n\right)$ is (s,ω)-measurable, $\mathcal{Y}_s$-adapted and $E\int_0^t \phi_s^2 ds < \infty$ for each t (by virtue of Theorem 2.1, p. 213 in Kunita, Watanabe [14]). Since for each $k = 1, \dots, n$ we have that

$$\nu_t^k = Y^k(t) - \int_0^t P_s(h_k)ds = B^k(t) + \int_0^t \left[h_k(X_s) - P_s(h_k(X_s))\right]ds, \quad (5.5)$$

the left-hand side of (5.3) equals

$$\begin{aligned} E\left[P_t(f)\left(Y_t - Y_0\right)\right] &= E\left[E\left(f(X_t)|\mathcal{Y}_t\right)\left(Y_t - Y_0\right)\right] \\ &= E\left[E\left(f(X_t)\left(Y_t - Y_0\right)|\mathcal{Y}_t\right)\right] \\ &= E\left[f(X_t)\left(Y_t - Y_0\right)\right] \\ &= \sum_{k=1}^{m} E\left[f(X_t)\int_0^t \phi_s^k dB^k(s)\right] \\ &+ \sum_{k=1}^{m} E\left[f(X_t)\int_0^t \phi_s^k\left(h_k(X_s) - P_s(h_k(X_s))\right)ds\right]. \end{aligned} \quad (5.6)$$

The first member on the right-hand side is 0 since $X(t)$ and $B(t)$ are independent (according to Lemma 2). The second member in (5.6) can be written as

$$\begin{aligned} &\sum_{k=1}^{m}\int_0^t E\left\{\left[E\left(f(X_t)|\mathcal{Y}_s\right)h_k(X_s) - E\left(f(X_t)|\mathcal{Y}_s\right)P_s(h_k)\right]\phi_s^k\right\}ds \\ &= \sum_{k=1}^{m}\int_0^t E\left\{\left[\left(T_{t-s}f(X_s)\right)h_k(X_s) - T_{t-s}f(X_s)P_s\left(h_k(X_s)\right)\right]\phi_s^k\right\}ds. \end{aligned} \quad (5.7)$$

Since ϕ_s is $\mathcal{Y}_s$-adapted, the last member coincides with the right-hand side of (5.3). □

Let $\beta(t), t \in [0,T]$ be an m-dimensional Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with a σ-field filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and let $\pi_0(f) = E(f(x(0)))$. A measure valued stochastic process π_t is said to be a solution of the equation

$$\pi_t(f) = \pi_0(T_t f) + \sum_{k=1}^{m}\int_0^t \left[\pi_s((T_{t-s}f)h_k) - \pi_s(T_{t-s}f)\pi_s(h_k)\right]d\beta_s^k \quad (5.8)$$

if it satisfies the following conditions:

1. $\pi_t(f)(\omega)$ is (t,ω)-measurable for each $f \in C_0(\Re^n)$;
2. π_s and $\sigma\{\beta(v)-\beta(u); s \le u < v \le s'\}$ are independent for every s, s' $(0 \le s \le s')$;
3. $\pi_t(f)(\omega)$ satisfies (5.8) a.s. for each t.

Theorem 5.2. *Let $X(t) \in \mathbb{R}^n$ be a signal process whose dynamics is described by equations (2.2), (2.3) and (2.4), with the mappings η, β and γ satisfying the Lipschitz and Growth conditions C1 and C2 and with $E|X(0)| < \infty$. Let the model of the observation process be given by equation (4.1) where conditions (A1)-(A3) hold. Let $f \in D(A)$ be a measurable function on $\mathbb{R}^n$, such that f satisfies the quadratic growth condition $|f(x)| \le C(1+|x|^2)$, for all $t \in [0,T]$. Then equation (5.8) has a unique solution within the class of measures $\{\pi_t\}$ that satisfy the estimate:*

$$E|\pi_t(f)|^2 \le CE|f(X_t)|^2. \tag{5.9}$$

Proof. The measure valued optimal filter defined in (4.20) is given by

$$E[f(X(t))|\mathcal{Y}_t] = \int_{\mathbb{R}^m} f(\xi)\pi_t(d\xi). \tag{5.10}$$

Our goal is to prove that the conditional expectation defined in equation (5.10) is the unique solution of (5.8). Suppose that π_t and π'_t are two measures that satisfy the inequality

$$E|\pi_t(f)|^2 \le CE|f(X_t)|^2 \text{ and } E|\pi'_t(f)|^2 \le C'E|f(X_t)|^2, \tag{5.11}$$

where C and C' are two constants and let $\mathcal{C} = \max\{C, C'\}$.

Let $\theta_t(f) = E\left(|\pi_t(f) - \pi'_t(f)|^2\right)$. We will show that $\theta_t(f) = 0$ for all $t > 0$ and each $f \in D(A)$. Using the triangle inequlity, we obtain that

$$\theta_t(f) \le 2\left(E|\pi_t(f)|^2 + E|\pi'_t(f)|^2\right). \tag{5.12}$$

Hence equations (5.12) and (5.9) give that

$$\theta_t(f) \le 4\mathcal{C}E|f(X_t)|^2 < \infty. \tag{5.13}$$

Let $\tau_1 = \inf\{t \in [0,T]; |h(X_t)| > N\}$, $\tau_2 = \inf\{t \in [0,T]; |\pi(h(X_t))| > N\}$, $\tau_3 = \inf\{t \in [0,T]; |\pi(T_{t-s}f)| > N\}$ and $\tau = \min\{\tau_1, \tau_2, \tau_3\}$.

Then with Itô's isometry and the triangle inequality (see Applebaum [1], page 304), we get:

$$\begin{aligned}\theta_{t\wedge\tau}(f) &= E\bigg|\pi_0(T_{t\wedge\tau}f) + \int_0^{t\wedge\tau}[\pi_s(hT_{t-s}f) - \pi_s(T_{t-s}f)\pi_s(h)]\,d\beta_s \\ &\quad - \pi_0'(T_{t\wedge\tau}f) - \int_0^{t\wedge\tau}[\pi_s'(hT_{t-s}f) - \pi_s'(T_{t-s}f)\pi_s'(h)]\,d\beta_s\bigg|^2 \\ &\le 3\int_0^{t\wedge\tau} E\Big\{|\pi_s(hT_{t\wedge\tau-s}f) - \pi_s'(hT_{t\wedge\tau-s}f)|^2 \\ &\quad + |\pi_s(h)|^2|\pi_s(T_{t\wedge\tau-s}f) - \pi_s'(T_{t-s}f)|^2 \\ &\quad + |\pi_s'(T_{t\wedge\tau-s}f)|^2|\pi_s(h) - \pi_s'(h)|^2\Big\}ds.\end{aligned} \tag{5.14}$$

Hence

$$\begin{aligned}\theta_{t\wedge\tau}(f) &\le 3\int_0^{t\wedge\tau}\Big\{E|\pi_s(hT_{t\wedge\tau-s}f) - \pi_s'(hT_{t\wedge\tau-s}f)|^2 \\ &\quad + N^2E|\pi_s(T_{t\wedge\tau-s}f) - \pi_s'(T_{t\wedge\tau-s}f)|^2 + 4N^2E|\pi_s'(T_{t\wedge\tau-s}f)|^2\Big\}ds \\ &\le 3\int_0^{t\wedge\tau}\Big\{\theta_s\left(hT_{t\wedge\tau-s}f\right) + N^2\theta_s\left(T_{t\wedge\tau-s}f\right) + 4N^2E|\pi_s'(T_{t\wedge\tau-s}f)|^2\Big\}ds \\ &\le 3\cdot 4\cdot C\int_0^{t\wedge\tau}\Big\{E|h\cdot T_{t\wedge\tau-s}f|^2 + N^2E|T_{t\wedge\tau-s}|^2 + N^2E|T_{t\wedge\tau-s}|^2\Big\}ds \\ &\le 3\cdot 4\cdot C\cdot N^2\cdot E|f(X_{t\wedge\tau})|^2\cdot(t\wedge\tau).\end{aligned} \tag{5.15}$$

Lemma 3.1 gives that $E|f(X_t)|^2 \le M < \infty$ and $E|h(X_t)|^2 \le \tilde{M} < \infty$. Let $\bar{M} = \max\{M, \tilde{M}\}$. Hence inequality (5.15) becomes

$$\theta_{t\wedge\tau}(f) \le 3\cdot 4\cdot C\cdot N^2\cdot\bar{M}\cdot(t\wedge\tau) \tag{5.16}$$

Inequality (5.14) gives also that

$$\begin{aligned}\theta_{t\wedge\tau}(f) &\le 3\int_0^{t\wedge\tau}\Big\{E|\pi_s(hT_{t\wedge\tau-s}f) - \pi_s'(hT_{t\wedge\tau-s}f)|^2 \\ &\quad + N^2E|\pi_s(T_{t\wedge\tau-s}f) - \pi_s'(T_{t\wedge\tau-s}f)|^2 + N^2E|\pi_s(h) - \pi_s'(h)|^2\Big\}ds \qquad (5.17) \\ &\le 3\int_0^{t\wedge\tau}\Big\{\theta_s\left(hT_{t\wedge\tau-s}f\right) + N^2\theta_s\left(T_{t\wedge\tau-s}f\right) + N^2\theta_s(h)\Big\}ds\end{aligned}$$

which, with the inequality (5.16) applied recursively n times gives

$$\theta_{t\wedge\tau}(f) \leq 3 \cdot 4 \cdot C \cdot N^{2n} \cdot \bar{M} \cdot \frac{(t\wedge\tau)^n}{n!} \tag{5.18}$$

Taking infinite sum in both sides of the inequality (5.18), we obtain

$$\sum_{n=1}^{\infty} \theta_{t\wedge\tau}(f) \leq 4 \cdot C \cdot \bar{M} \cdot e^{3^2 N^2 (t\wedge\tau)}. \tag{5.19}$$

The right hand side of the inequality (5.19) is finite, hence the left hand side has to be finite and this is true only when $\theta_{t\wedge\tau}(f) = 0$ for all stopping times τ and $t \leq T$.

We can use arguments similar to those described in Section 5.4 of Sritharan [25] to show that uniqueness holds in the whole interval with the help of the moments estimates (Lemma 3.1) and Chebychev inequality. □

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